ON A RELAXATION APPROXIMATION OF THE INCOMPRESSIBLE NAVIER-STOKES EQUATIONS

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Abstract. We consider an hyperbolic singular perturbation of the incompressible Navier Stokes equations in two space dimensions. The approximating system under consideration, arises as a diffusive rescaled version of a standard relaxation approximation for the incompressible Euler equations. The aim of this work is to give a rigorous justification of its asymptotic limit toward the Navier Stokes equations using the modulated energy method.

1. Introduction

Let us consider the incompressible Euler equations, namely

$$\begin{cases} \partial_t u + \nabla \cdot (u \otimes u) = \nabla \phi, \\ \nabla \cdot u = 0, \\ u(0,x) = u_0(x), \end{cases} \tag{1.1}$$

for $(t,x) \in [0,T] \times \mathbb{T}^2$, where $\mathbb{T}^2$ is the unit periodic square $\mathbb{R}^2/\mathbb{Z}^2$. This system describes a perfect incompressible fluid, the unknowns u and ϕ corresponding respectively to the velocity, which is valued in $\mathbb{R}^2$, and to the pressure of the fluid.

To approximate these equations, most in the spirit of [14], we introduce its relaxed version, which is obtained by a singular perturbation of the nonlinear term $(u \otimes u)$, through a supplementary matrix valued variable $V : \mathbb{T}^2 \to \mathbb{R}^4$. This leads to the following system

$$\begin{cases} \partial_t u + \nabla \cdot (V) = \nabla \phi, \\ \partial_t V + a \nabla u = -\dfrac{1}{\eta}(V - u \otimes u), \\ \nabla \cdot u = 0, \\ u(0,x) = u_0(x), V(0,x) = V_0(x). \end{cases} \tag{1.2}$$

Let us notice that, as η goes to zero, we formally recover system (1.1).

1991 *Mathematics Subject Classification.* Primary: 35Q30; Secondary: 76D05.

Key words and phrases. Incompressible Navier-Stokes equations, relaxation approximations, hyperbolic singular perturbations, modulated energy method.

Work partially supported by European TMR projects NPPDE # ERB FMRX CT98 0201 and CNR Short Term Visiting program.

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Let us consider now a diffusive scaling, namely, for $\varepsilon > 0$, we set

$$\begin{cases} u^\varepsilon(t,x) & := \dfrac{1}{\sqrt{\varepsilon}} u\left(\dfrac{x}{\sqrt{\varepsilon}}, \dfrac{t}{\varepsilon}\right), \\ V^\varepsilon(x,t) & := \dfrac{1}{\varepsilon} V\left(\dfrac{x}{\sqrt{\varepsilon}}, \dfrac{t}{\varepsilon}\right), \\ \phi^\varepsilon(x,t) & := \dfrac{1}{\varepsilon} \phi\left(\dfrac{x}{\sqrt{\varepsilon}}, \dfrac{t}{\varepsilon}\right). \end{cases} \tag{1.3}$$

Therefore system (1.2) becomes, setting from now $\eta = 1$,

$$\begin{cases} \partial_t u^\varepsilon + \nabla \cdot (V^\varepsilon) = \nabla \phi^\varepsilon, \\ \sqrt{\varepsilon}\partial_t V^\varepsilon + \dfrac{a}{\sqrt{\varepsilon}} \nabla u^\varepsilon = -\dfrac{1}{\sqrt{\varepsilon}}(V^\varepsilon - u^\varepsilon \otimes u^\varepsilon), \\ \nabla \cdot u^\varepsilon = 0, \\ u^\varepsilon(0,x) = u_0^\varepsilon(x), V^\varepsilon(0,x) = V_0^\varepsilon(x). \end{cases} \tag{1.4}$$

In this paper we shall prove that, under some suitable assumptions, the solutions to (1.4) converge, when ε goes to 0, to the (smooth) solutions of the incompressible Navier Stokes equations

$$\begin{cases} \partial_t U + \nabla \cdot (U \otimes U) - a\Delta U = \nabla \phi, \\ \nabla \cdot U = 0, \\ U(0,x) = U_0(x). \end{cases} \tag{1.5}$$

This result could be promptly recovered, at least at a formal level, if we assume that, in some (weak) topologies, not only $u^\varepsilon \to U$, but also $\varepsilon V^\varepsilon \to 0$ and $u^\varepsilon \otimes u^\varepsilon \to U \otimes U$. The aim of this paper is to show how to obtain this result in a different (and simpler) way by using the modulated energy method [3], leading to a direct error estimate in the strong $L^\infty([0,T], L^2(\mathbb{T}^2))$ norm, for all finite positive T.

Let us recall that the diffusive scaling $(\frac{x}{\sqrt{\varepsilon}}, \frac{t}{\varepsilon})$ has been largely investigated in the framework of hydrodynamic limits of the Boltzmann equations, see for instance [8] and references therein. Starting from the works about the diffusive limit of the Carleman equations by Kurtz [11] and McKean [20], this scaling has also been systematically used in the analysis of hyperbolic-parabolic relaxation limits for weak solutions of hyperbolic systems of balance laws with strongly diffusive source terms by means of compensated compactness techniques by Marcati and collaborators [18, 17, 19, 7]. For other diffusive kinetic models and approximations, we refer to [15, 13, 12]. A general class of kinetic approximations for (possibly degenerate) parabolic equations in multi-D has been considered in [4, 1]. Let us also point out that the same scaling was used in [16] to analyze the time-asymptotic limit of the Jin-Xin relaxation model [14], towards the fundamental solution of the diffusive Burgers equation.

Finally let us remark that our scaling can be considered as a hyperbolic perturbation of the Navier Stokes equations, which is similar to the Cattaneo *hyperbolic heat equation* [6], just eliminating the unknown V in equations (1.4)

$$\begin{cases} \partial_t u^\varepsilon + P(\nabla \cdot (u^\varepsilon \otimes u^\varepsilon)) - a\Delta u^\varepsilon + \varepsilon \partial_{tt} u^\varepsilon = 0, \\ \nabla \cdot u^\varepsilon = 0, \end{cases} \tag{1.6}$$

where P represents the projection on the divergence free vectors. In this regard, we mention that some quite different hyperbolic perturbations of the Navier Stokes equations has been investigated in [21], by considering incompressible viscoelastic fluids of Oldroyd type. We also point out that a similar approximation has been also recently proposed in [2] for numerical purposes, as a reduced kinetic model.

Concerning the method of the modulated energy, let us recall that it has been used by Brenier in [3] to prove the convergence in a quasi-neutral limit of the current involved in the Vlasov-Poisson system toward a dissipative solution of the incompressible Euler equations. The method consists in estimating, through its time derivative, a suitable modification of the the standard energy functional, which is obtained by introducing in the energy a modulation by a well-adapted test function, in practice the (smooth) solution to the limit equation. This method has connections with the relative entropy method used by Yau [23], and the modulated hamiltonian method introduced by Grenier [9] to solve boundary layer problems. Here we can use some special energy functionals, most in the spirit of Tzavaras estimates for the Jin–Xin relaxation model [22].

The paper is organized as follows. In Section 2 we give some analytical backgrounds and state our main result. Estimates and proofs are given in Section 3.

2. Analytical backgrounds and statements

First we shall state the existence of smooth local solutions for system (1.4).

Theorem 2.1. *Suppose the initial data $(u_0^\varepsilon(x), V_0^\varepsilon(x))$ are smooth functions belonging to H^s for $s \geq 2$. Then, there exists a positive time T^ε, which depends only on the initial data, and a solution $(u^\varepsilon, V^\varepsilon, \phi^\varepsilon) \in C([0,T]; (H^s)^3)$ to system (1.4). Moreover, if $T^\varepsilon < \infty$, then*

$$\lim_{t \to T^\varepsilon_-} ||(u^\varepsilon, V^\varepsilon)||_{H^2} \to \infty. \tag{2.1}$$

The proof follows easily by arguing as for the classical wave equation, by using energy estimates and the Gagliardo–Nirenberg inequalities, see for instance [10], and it is omitted.

In the following we shall use the norm

$$|u|_{H^2(\mathbb{T}^2)} = ||u||_{L^2(\mathbb{T}^2)} + ||\text{curl } u||_{L^2(\mathbb{T}^2)} + ||\nabla(\text{curl } u)||_{L^2(\mathbb{T}^2)}.$$

Let us recall that, since $\nabla \cdot u = 0$, this norm is equivalent to the H^2 norm. Moreover we shall denote by C_0 a given positive constant such that $C_0 < \sqrt{a}$. Finally K_s is the constant which appears in the Sobolev inequality in two space dimensions, under the norm $| \cdot |_{H^2(\mathbb{T}^2)}$.

The study of the asymptotic behavior of the sequence u^ε, as ε goes to zero, leads to the statement of our main result.

Theorem 2.2. *Let $T \geq 0$ and U^0 be a smooth divergence free vector field on $\mathbb{T}^2$. Let also $(u_0^\varepsilon, V_0^\varepsilon)$ be a sequence of smooth initial data on $\mathbb{T}^2$ for problem (1.4). Assume moreover that there exists a constant C independent of ε such that*

$$||u_0^\varepsilon||_{H^1(\mathbb{T}^2)} \leq C \tag{2.2}$$

$$||V_0^\varepsilon||_{H^2(\mathbb{T}^2)} \leq \frac{C}{\sqrt{\varepsilon}} \tag{2.3}$$

$$|u_0^\varepsilon|_{H^2(\mathbb{T}^2)} < \frac{C_0}{K_s\sqrt{\varepsilon}} \tag{2.4}$$

$$\int_{\mathbb{T}^2} |u_0^\varepsilon(x) - U^0(x)|^2 dx \leq C\sqrt{\varepsilon}. \tag{2.5}$$

Then, u^ε is a global solution of the relaxed system (1.4) and converges, as $\varepsilon \to 0$, in $L^\infty([0,T], L^2(\mathbb{T}^2))$ towards the (unique smooth) solution U of the incompressible Navier Stokes equations (1.5) with U^0 as initial data. In addition

$$\sup_{t\in[0,T]} \int_{\mathbb{T}^2} |u^\varepsilon - U|^2 dx \le C_T\sqrt{\varepsilon},$$

where C_T depends only on T, U, C and C_0.

3. Proof of the theorem

3.1. **Preliminaries.** First, we shall prove some energy estimates under an a priori assumption on the L^∞ norm of u^ε. Therefore we shall verify that this assumption holds actually true.

3.1.1. *The energy estimate.* Let us give our basic energy estimate.

Proposition 3.1. *Assume that there exists $T > 0$ such that $||u^\varepsilon||_{L^\infty} \le \sqrt{\frac{a}{\varepsilon}}$ for all $t \le T$. Then, setting $w^\varepsilon := curl\ u^\varepsilon$, we have the following estimates*

$$\frac{d}{dt}\int (\frac{1}{2}|u^\varepsilon + \varepsilon\partial_t u^\varepsilon|^2 + \varepsilon^2|\partial_t u^\varepsilon|^2 + \varepsilon a|\nabla u^\varepsilon|^2)dx \le 0, \tag{3.1}$$

and

$$\frac{d}{dt}\int (\frac{1}{2}|w^\varepsilon + \varepsilon\partial_t w^\varepsilon|^2 + \varepsilon^2|\partial_t w^\varepsilon|^2 + \varepsilon a|\nabla w^\varepsilon|^2)dx \le 0, \tag{3.2}$$

for all $t \le T$.

Proof. Let us multiply equation (1.6) by $(u^\varepsilon + 2\varepsilon\partial_t u^\varepsilon)$ to obtain, after integration by parts in space and writing $\partial_t u \partial_{tt} u = \partial_t(u\partial_t u) - (\partial_t u)^2$,

$$\begin{aligned}&\frac{d}{dt}\int (\frac{1}{2}|u^\varepsilon + \varepsilon\partial_t u^\varepsilon|^2 + \varepsilon^2|\partial_t u^\varepsilon|^2 + \varepsilon a|\nabla u^\varepsilon|^2)dx \\ &+\varepsilon\int |\partial_t u^\varepsilon + \nabla\cdot(u^\varepsilon\otimes u^\varepsilon)|^2 dx + \int \left(a|\nabla u^\varepsilon|^2 - \varepsilon|\nabla\cdot(u^\varepsilon\otimes u^\varepsilon)|^2\right)dx = 0.\end{aligned} \tag{3.3}$$

Then, since $||u^\varepsilon||_{L^\infty} \le \sqrt{\frac{a}{\varepsilon}}$, we obtain (3.1).

For the second estimate, we consider the equation satisfied by w^ε. Since in two space dimensions we have $w = \partial_2 u_1 - \partial_1 u_2$, then

$$\partial_t w^\varepsilon + u^\varepsilon\cdot\nabla w^\varepsilon - a\Delta w^\varepsilon + \varepsilon\partial_{tt} w^\varepsilon = 0. \tag{3.4}$$

If we multiply this equation by $(w^\varepsilon + 2\varepsilon\partial_t w^\varepsilon)$, we obtain

$$\begin{aligned}&\frac{d}{dt}\int (\frac{1}{2}|w^\varepsilon + \varepsilon\partial_t w^\varepsilon|^2 + \varepsilon^2|\partial_t w^\varepsilon|^2 + \varepsilon a|\nabla w^\varepsilon|^2)dx \\ &+\varepsilon\int |\partial_t w^\varepsilon + u^\varepsilon\cdot\nabla w^\varepsilon|^2 + \int (a|\nabla w^\varepsilon|^2 - \varepsilon|u^\varepsilon\cdot\nabla w^\varepsilon|^2) = 0.\end{aligned} \tag{3.5}$$

The conclusion follows as previously. □

3.1.2. *L^∞ bounds.* Let us prove a uniform L^∞ bound for u^ε , which implies the assumption made in in the previous statement.

Proposition 3.2. *Under the assumptions of Theorem 2.2, if*

$$|u_0^\varepsilon|_{H^2(\mathbb{T}^2)} < \frac{C_0}{K_s\sqrt{\varepsilon}},$$

where C_0 is a given positive constant such that $C_0 < \sqrt{a}$, then the solution u^ε verifies the following estimate

$$||u^\varepsilon||_{L^\infty} \leq \frac{C_0}{\sqrt{\varepsilon}}, \tag{3.6}$$

for all positive t and, therefore is global.

Proof. Take a positive constant δ such that $\delta < \sqrt{a} - C_0$ and set

$$T^\delta = \sup\{0 \leq t \leq T; \sup_{0\leq\tau\leq t} ||u^\varepsilon(\tau)||_{L^\infty} \leq \frac{C_0+\delta}{\sqrt{\varepsilon}}\}.$$

Since $|u_0^\varepsilon|_{H^2(\mathbb{T}^2)} < \frac{C_0}{K_s\sqrt{\varepsilon}}$, we have that $||u_0^\varepsilon||_{L^\infty} < \frac{C_0+\delta}{\sqrt{\varepsilon}}$, thanks to the the Sobolev inequalities. Since $u^\varepsilon \in C^0([0,T], L^\infty(\mathbb{T}^2))$, we have that $T^\delta > 0$. Let us prove now that $T^\delta = T$. If $T^\delta < T$, we have

$$||u^\varepsilon(T^\delta)||_{L^\infty} = \frac{C_0+\delta}{\sqrt{\varepsilon}} < \sqrt{\frac{a}{\varepsilon}}.$$

Then, there exists $\mu > 0$ such that for all $t \leq T^\delta + \mu$, $||u^\varepsilon(t)||_{L^\infty} \leq \sqrt{\frac{a}{\varepsilon}}$. On the other hand, for all $t \leq T^\delta + \mu$, the estimates (3.1) and (3.2) hold true. This implies that

$$||u^\epsilon(T^\delta)||_{L^2} + ||w^\epsilon(T^\delta)||_{L^2} \leq C,$$
$$|u^\epsilon(T^\delta)|_{H^2} \leq \frac{C}{\sqrt{\epsilon a}}.$$

By standard elliptic regularity, the L^2 norm of the curl w of a divergence free vector field u is equivalent to the H^1 semi-norm of u. Therefore, by the Brezis-Gallouet inequality [5], we have that

$$||u^\epsilon(T^\delta)||_{L^\infty} \leq C(1 + log^+(\epsilon a)),$$

which yields a contradiction. □

3.2. **Convergence.** Let U be a smooth solution of the Navier Stokes equations with U^0 as initial data. We shall prove here that $\frac{1}{2}\int |u^\varepsilon - U|^2 dx \leq C_T\sqrt{\varepsilon}$. To prove that, we shall define a specific modulated energy which control this quantity.

3.2.1. *Definition and properties of the modulated energy.* Let us define the energy in the following way

$$E^\varepsilon(t) = \int (\frac{1}{2}|u^\varepsilon + \varepsilon\partial_t u^\varepsilon|^2 + \varepsilon^2|\partial_t u^\varepsilon|^2 + \varepsilon a|\nabla u^\varepsilon|^2)dx. \tag{3.7}$$

For all smooth divergence free v, we introduce the modulated energy

$$E_v^\varepsilon(t) = \int (\frac{1}{2}|u^\varepsilon - v(t,x) + \varepsilon\partial_t u^\varepsilon|^2 + \varepsilon^2|\partial_t u^\varepsilon|^2 + \varepsilon a|\nabla u^\varepsilon|^2)dx. \tag{3.8}$$

Let us prove now a useful identity.

Proposition 3.3. *The modulated energy satisfies the identity*

$$\begin{aligned} \frac{d}{dt}E_v^\varepsilon(t) \quad = \quad & \int v\cdot\nabla : (u^\varepsilon - v)\otimes(u^\varepsilon - v) + \int (\partial_t v + v\cdot\nabla v - a\Delta v)(v - u^\varepsilon) \\ & -\varepsilon\int |\partial_t u^\varepsilon + \nabla\cdot(u^\varepsilon\otimes u^\varepsilon)|^2 - \varepsilon\int \partial_t v\cdot\partial_t u^\varepsilon \\ & -a\int |\nabla(u^\varepsilon - v)|^2 + \varepsilon\int |\nabla(u^\varepsilon\otimes u^\varepsilon)|^2. \end{aligned} \tag{3.9}$$

Proof. We have

$$\frac{d}{dt}E_v^\varepsilon(t) = \frac{d}{dt}E(t) - \int v\cdot\partial_t u^\varepsilon - \int \partial_t v\cdot u^\varepsilon - \varepsilon\int \partial_t v\cdot\partial_t u^\varepsilon - \varepsilon\int v\cdot\partial_{tt}u^\varepsilon + \int v\partial_t v.$$

Then, using (1.6), (3.3) and the equality

$$\begin{aligned}\int v\cdot\nabla : (u^\varepsilon\otimes u^\varepsilon) &= \int v\cdot\nabla : (u^\varepsilon - v)\otimes(u^\varepsilon - v) + \int v\cdot\nabla : (u^\varepsilon\otimes v)\\ &+ \int v\cdot\nabla : (v\otimes u^\varepsilon) - \int v\cdot\nabla : (v\otimes v)\end{aligned}$$

we obtain

$$\begin{aligned}\frac{d}{dt}E_v^\varepsilon(t) &= \int v\cdot\nabla : (u^\varepsilon - v)\otimes(u^\varepsilon - v) + \int(\partial_t v + v\cdot\nabla v)(v - u^\varepsilon)\\ &-\varepsilon\int|\partial_t u^\varepsilon + \nabla\cdot(u^\varepsilon\otimes u^\varepsilon)|^2 - \varepsilon\int\partial_t v\cdot\partial_t u^\varepsilon\\ &+a\int\Delta u^\varepsilon(u^\varepsilon - v) + \varepsilon\int|\nabla(u^\varepsilon\otimes u^\varepsilon)|^2.\end{aligned}\tag{3.10}$$

Therefore, since

$$a\int\Delta u^\varepsilon(u^\varepsilon - v) = a\int\Delta(u^\varepsilon - v)(u^\varepsilon - v) + a\int\Delta v(u^\varepsilon - v),$$

we have (3.9). □

3.2.2. *Proof of Theorem 2.2.* Thanks to the assumptions on the initial data, we have that

$$\int|u^\varepsilon|^2 \le CE^\varepsilon(t) \le CE^\varepsilon(0) \le C.$$

Moreover, we have the inequality

$$\int|u^\varepsilon - v|^2dx \le CE_v^\varepsilon(t).$$

Now, we assume $v = U$, where U is a smooth solution to the incompressible Navier-Stokes equations (1.5), with U^0 as initial data, which has a globally bounded spatial gradient. From (3.9), we obtain

$$\tfrac{d}{dt}E_v^\varepsilon(t) \le CE_v^\varepsilon(t) - \varepsilon\int\partial_t v\cdot\partial_t u^\varepsilon - a\int|\nabla(u^\varepsilon - v)|^2 + \varepsilon|u^\varepsilon\cdot\nabla u^\varepsilon|^2.$$

We have used, in the right-hand side of (3.9), i) that v is smooth in order to bound the first term by CE_v^ε, ii) that v is a solution to the Navier-Stokes equations to cancel the second term. We see that

$$-\varepsilon\int\partial_t v\cdot\partial_t u^\varepsilon = -\varepsilon\frac{d}{dt}\int\partial_t v\cdot u^\varepsilon + \varepsilon\int\partial_{tt}v\cdot u^\varepsilon,$$

which is of order ε. We want to prove now that the term

$$A^\varepsilon = -a\int|\nabla(u^\varepsilon - v)|^2 + \varepsilon\int|u^\varepsilon\cdot\nabla u^\varepsilon|^2$$

goes to zero, as $\varepsilon\to 0$. In this regard, let us write

$$\varepsilon\int|u^\varepsilon\cdot\nabla u^\varepsilon|^2 \le \varepsilon(1+\theta)\int|u^\varepsilon\cdot\nabla(u^\varepsilon - v)|^2 + \varepsilon(1+\frac{1}{\theta})\int|u^\varepsilon\cdot\nabla v|^2.$$

Then, since $||u^\varepsilon||_{L^\infty} \le \frac{\sqrt{a}}{\sqrt{\varepsilon}}$, we have the inequality

$$A^\varepsilon \le \theta a\int|\nabla(u^\varepsilon - v)|^2 + \varepsilon(1+\frac{1}{\theta})\int|u^\varepsilon\cdot\nabla v|^2.$$

This yields

$$A^\varepsilon \leq \theta a||\nabla u^\varepsilon||^2_{L^2(\mathbb{T}^2)} + \varepsilon(1+\frac{1}{\theta})c||u^\varepsilon||^2_{L^2(\mathbb{T}^2)}.$$

Since, thanks to the estimates (3.1) and (3.2), we have

$$\int |u^\varepsilon|^2 + \int |\nabla(u^\varepsilon - v)|^2 \leq C.$$

If we take $\theta = \sqrt{\varepsilon}$, we obtain that $A^\varepsilon = O(\sqrt{\varepsilon})$, when ε goes to zero. Thus, we have obtained

$$\frac{d}{dt}(E_v(t) + O(\varepsilon)) \leq CE_v(t) + O(\sqrt{\varepsilon}).$$

The assumptions that we have made on the initial data imply that

$$E_v(0) = O(\sqrt{\varepsilon}).$$

We conclude that

$$\sup_{t\in[0,T]} \int |u^\varepsilon - v|^2 dx \leq CE^\varepsilon_v \leq C_T\sqrt{\varepsilon},$$

where C_T depends only on T, v and the initial conditions. □

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