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par

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TITRE :

# Contributions à l'étude des équations de la physique des plasmas

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devant le jury composé de

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à la mémoire de Naoufel Ben Abdallah

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# Première partie

# Mémoire

Ce travail s'articule autour de deux thèmes principaux. Le premier concerne les équations cinétiques modélisant le mouvement de particules chargées dans les plasmas, le deuxième, le transport relativiste en relation avec le transport optimal.

L'idée commune aux travaux présentés dans la première partie de ce mémoire est l'élaboration de liens entre différentes équations, le but étant le plus souvent d'approcher les solutions d'une équation par celles d'une équation plus simple. En effet, depuis le développement des simulations numériques, notamment dans le cadre de l'industrie nucléaire, on cherche à justifier rigoureusement que dans certains régimes physiques, il est raisonnable de remplacer des équations microscopiques par des équations macroscopiques. Ainsi, la dérivation d'équations macroscopiques telles que les équations de diffusion ou les équations hydrodynamiques est un sujet qui a reçu beaucoup d'attention ces dernières années (voir [56], [63], [49], [50], [65], [83], [84], [85], [100], [72], [73]). Par exemple, les équations de transport cinétiques peuvent dans certains cas être approchées par des équations de la mécanique des fluides dont l'implémentation numérique est bien moins coûteuse.

Le premier modèle que nous considérons est un système décrivant le mouvement de particules chargées du type électron dans un environnement caractérisé par la présence d'un autre type de particules chargées immobiles très lourdes, les ions. Nous prenons en compte l'effet sur les électrons du champ électro-magnétique $((E,B))$ induit par leur propre déplacement en écrivant, via le système de Vlasov-Maxwell, que l'accélération des particules est due à la force électromagnétique de Lorenz

$$\begin{aligned}
&\partial_t f + V(\xi)\cdot\nabla_x f - (E + V(\xi)\wedge B)\cdot\nabla_\xi f = 0,\\
&\mathrm{div}_x B = 0, \quad \mathrm{div}_x E = n - \int f d\xi,\\
&\partial_t B + \ \mathrm{curl}_x E = 0, \quad -\partial_t E + \ \mathrm{curl}_x B = -\int V(\xi) f d\xi.
\end{aligned}$$

C'est un modèle cinétique faisant partie de la mécanique statistique, c'est à dire que les particules sont décrites par la probabilité $f$ de se trouver à un temps donné $t$, à une position donnée $x$ avec une quantité de mouvement donnée $\xi$. Lorsque le plasma est quasi-neutre, c'est à dire lorsque la densité des ions est très proche de la densité des électrons ($\varepsilon \to 0$), dans une limite non relativiste, supposant que l'ordre de grandeur des vitesses en jeu est très petit par rapport à la vitesse de la lumière ($V(\xi) \to \xi$), la solution du système de Vlasov-Maxwell correspond à des grandeurs macroscopiques (densité $n$, vitesse $v$) qui satisfont une équation fluide du type Euler-Magnéto-Hydrodynamique

$$\left\{\begin{array}{r}
\partial_t v + v\cdot\nabla v + \underline{e} + \alpha v\wedge b = 0\\
\partial_t b + \dfrac{1}{\alpha}\mathrm{curl}\underline{e} = 0\\
nv + \dfrac{1}{\alpha}\ \mathrm{curl} b = 0
\end{array}\right. \tag{1}$$

où $b$ est la limite du champ magnétique et $\underline{e}$ joue le rôle du multiplicateur de Lagrange correspondant à la troisième équation. Pour cela, nous nous inspirons de l'article de Brenier [6] sur la limite quasi-neutre du système de Vlasov-Poisson dans lequel il introduit la méthode de l'énergie modulée. Cette méthode consiste à introduire dans l'expression de l'énergie conservée par le système une modulation par une solution régulière du système limite. Un contrôle de cette quantité permet de montrer une convergence forte dans le cas de données bien préparées,

c'est à dire monocinétiques centrées sur une condition initiale compatible avec le système limite.

Cette méthode est à rapprocher des méthodes d'entropie relative introduites par Yau dans [36] et de la méthode du Hamiltonien modulé introduite par Grenier pour résoudre des problèmes de couches limites [16] .

Dans un premier article en collaboration avec Y. Brenier et N. Mauser [BMP] [1], on obtient un résultat de convergence forte pour des données bien préparées dans le cas du système de Vlasov non relativiste -Maxwell ($V = \xi$) en dimension deux.

Dans [PSR] écrit en collaboration avec L. Saint-Raymond, nous avons traité le cas Vlasov relativiste - Maxwell ($V = \dfrac{\xi}{\sqrt{1+\gamma^2\xi^2}}$) qui fait intervenir un petit paramètre supplémentaire $\gamma$ au niveau de l'équation de Vlasov. Cette nouvelle asymptotique complique techniquement le raisonnement mais la méthode de l'énergie modulée reste efficiente. De plus nous avons considéré le cas de la dimension trois ce qui signifie que nous avons dû utiliser la notion de solutions faibles globales du système de Vlasov-Maxwell introduite par Di Perna et Lions pour pouvoir travailler avec des temps d'existence indépendants des petits paramètres. La nouvelle difficulté induite par cette notion réside dans le fait que les équations de conservation ne peuvent être écrites directement. Il faut introduire à chaque étape des mesures de défaut dont les propriétés sont, nous le verrons, bien adaptées aux estimations nécessaires au raisonnement. Enfin, nous énonçons un résultat plus précis que dans [BMP] car nous construisons dans le cas de données plus générales (données mal préparées) une solution approchée du système limite. Pour cela, nous exhibons un système singulier dont nous savons grâce aux estimations sur l'énergie modulée que le comportement asymptotique approche le système limite. En utilisant des méthodes inspirées du travail de Golse et Saint-Raymond [13], nous construisons à partir de ce système singulier une solution qui oscille autour d'un champ moyen lui-même solution du système limite.

Restent les hypothèses sur la densité des ions ($n = 1$) et sur la donnée initiale qui doit être monocinétique à $\varepsilon$ fixé, c'est à dire $f^\varepsilon = \delta_{\xi - v_0^\varepsilon(x)}$. L'hypothèse sur la densité des ions permet de déterminer le noyau de la singularité qui serait, sinon, bien plus petit et bien plus difficile à caractériser. L'hypothèse sur la donnée initiale monocinétique semble en accord avec la moyennisation étudiée ici et la description d'un modèle cinétique par des équations fluides.

C'est aussi grâce à la méthode de l'énergie modulée que nous traitons une perturbation de Navier Stokes introduite dans un court papier écrit en collaboration avec Y. Brenier et R. Natalini [BNP].

Dans une série de papiers écrits notamment avec N. Ben Abdallah, G. Bal, A. Mellet et M. Vogelius, nous avons étudié un deuxième modèle cinétique, l'équation de Boltzmann linéaire qui représente cette fois des plasmas plus denses où les collisions entre les particules constituent le phénomène prépondérant dans la détermination de leurs trajectoires. Dans ce cas, l'accélération des particules est balancée par l'action des collisions décrites par un opérateur intégral en vitesse $Q$ composé d'un terme de gain, qui comptabilise les particules qui ont pour vitesse $v$ après la collision et d'un terme de perte comptant les particules qui ne se déplacent plus à la vitesse $v$ après la collision

$$\begin{cases} \varepsilon^\delta \partial_t f^\varepsilon + \varepsilon v \cdot \nabla_x f^\varepsilon = Q(f^\varepsilon), & \text{pour tout } (x,v) \in \mathbf{R}^N \times \mathbf{R}^N \text{ et } t > 0, \\ f^\varepsilon(x,v,0) = f_0(x,v), & \text{pour tout } (x,v) \in \mathbf{R}^N \times \mathbf{R}^N, \end{cases} \tag{2}$$

1. Les publications décrites dans ce mémoire sont listées p.13 et sont repérées par des lettres.

où

$$Q(f) := \int_{\mathbf{R}^N} \tilde{\sigma}(x,v,v')[f(v')M(v) - f(v)M(v')]\,dv'. \tag{3}$$

Dans cette expression, la fonction d'équilibre thermodynamique $M(v)$ est donnée et correspond à un profil en vitesse qui ne voit pas l'opérateur de collision. On se référera souvent à $M$ comme à une fonction Maxwellienne par allusion au cas où $M$ est une Gaussienne. Le paramètre $\varepsilon$ dans (2) est le nombre de Knudsen, ratio entre le libre parcours moyen et la longueur de référence. Il est bien connu que lorsque les collisions sont très nombreuses ($\varepsilon \ll 1$) et que le temps d'observation est très grand ($t \sim \varepsilon^{-\delta}$), la densité de présence des particules peut être approchée par une fonction d'équilibre, correspondant à une densité qui satisfait une équation de diffusion. En effet, lorsque $\delta = 2$ (voir [65] par exemple), la fonction de distribution $f^\varepsilon$ converge, lorsque $\varepsilon \to 0$, vers $n(x,t)M(v)$ où $n$ est la solution de l'équation de diffusion

$$\partial_t n - \ \text{div}\ (D\nabla n) = 0,$$

où

$$D \sim \int v \otimes v \frac{M(v)}{\nu(v)}\,dv \quad \text{avec } \nu(v) = \int \tilde{\sigma}(v,v')M(v')dv', \tag{4}$$

au moins lorsque $\sigma$ ne dépend pas de $x$.

Dans un travail en collaboration avec N. Ben Abdallah et M. Vogelius [NBAPV], nous nous plaçons par exemple dans le cas où le milieu considéré est périodique de période petite et nous étudions l'influence de l'homogénéisation, c'est à dire de la moyennisation des coefficients, sur le procédé d'approximation de diffusion. L'équation devient alors

$$\partial_t f + \frac{1}{\varepsilon} v \cdot \nabla f + \frac{1}{\varepsilon^2}\nu(\frac{x}{\alpha}, v)f = \frac{1}{\varepsilon^2}\int \sigma(\frac{x}{\alpha}, v, v')f(x,v')dv' \tag{5}$$

avec $\sigma(x,v,v') = \tilde{\sigma}(x,v,v')M(v)$, et les cœfficients intervenant dans le noyau de collision sont périodiques de petite période. Le cas où le libre parcours moyen tend vers zéro en restant du même ordre que la taille de la cellule avait été étudié entre autres par T. Goudon et A. Mellet [72] ou par G. Bal [48]. D'un point de vue modélisation, ces deux paramètres correspondent à deux phénomènes différents et pourraient donc être découplés. En effet, lorsque la taille de la cellule est égale à la taille du libre parcours moyen, les deux phénomènes interfèrent. L'équation de cellule donnant le profil d'équilibre s'écrit

$$v \cdot \nabla F + Q(F) = 0$$

et la densité satisfait une équation de diffusion dont le coefficient est donné par

$$D = \int \mathbf{X}^* \otimes vF dvdy \quad \text{avec} \quad v \cdot \nabla_y \mathbf{X}^* - Q^*(\mathbf{X}^*) = v.$$

Lorsque ces deux paramètres ne sont pas du même ordre, on introduit un nouvel ordre de grandeur, le rapport entre le libre parcours moyen et la taille de la cellule dont la taille est soit très petite soit très grande selon les hypothèses faites sur le milieu dans lequel évoluent les particules. Nous nous intéressons au cas où le libre parcours moyen est très petit devant la taille de la cellule, c'est à dire lorsque $\eta = \dfrac{\varepsilon}{\alpha} \to 0$. Ainsi, les équations deviennent

$$\eta v \cdot \nabla F^\eta + Q(F^\eta) = 0$$

et

$$D^\eta = \int X^{*\eta} \otimes vF^\eta dvdy \quad \text{avec} \quad \eta v \cdot \nabla_y \mathbf{X}^{*\eta} - Q^*(\mathbf{X}^{*\eta}) = v,$$

$y$ désignant la variable rapide $\frac{x}{\alpha}$.

Le point clef de ce travail réside dans le développement selon $\eta$ des quantités $F^\eta$ et $\mathbf{X}^{*\eta}$ impliquées dans ces dernières équations. Ainsi, nous obtenons dans [NBAPV], écrit en collaboration avec N. Ben Abdallah et M. Vogelius, un premier résultat de convergence faible deux échelles obtenu par la méthode des moments en utilisant des fonctions test oscillantes.

Dans un article plus ancien [102], R. Sentis traite le cas où la Maxwellienne ne dépend pas de la variable d'espace par une méthode entièrement basée sur les développements de Hilbert nécessitant une loi liant les deux paramètres $\alpha$ et $\varepsilon$. Nous pensions a priori que sans cette hypothèse plus forte liant les deux paramètres $\alpha$ et $\varepsilon$, la méthode basée sur les développements de Hilbert ne pouvait être appliquée. Il nous manquait une estimation plus précise que nous avons obtenue dans [NBABP] écrit en collaboration avec G. Bal. Nous montrons un résultat de convergence forte et nous étendons le résultat de [NBAPV] à la recherche des correcteurs : les termes d'ordre suivant dans le développement asymptotique de la fonction de distribution. Notons que dans le cas de la limite asymptotique alternative, c'est à dire lorsque le libre parcours moyen est très grand par rapport à la taille de la cellule, nous nous attendons à ce que l'équation limite soit la limite de diffusion de l'équation de transport moyennée obtenue dans l'article de Dumas et Golse [66].

Toujours dans le cadre de l'approximation diffusion, il est des cas où le coefficient de diffusion n'est pas défini : par exemple lorsque le moment d'ordre deux de la fonction d'équilibre est infini ou lorsque la fréquence de collision est dégénérée en zéro. Dans ces cas, après une remise à l'échelle en temps, on obtient encore une équation macroscopique. On parle alors de diffusion anormale. En collaboration avec N. Ben Abdallah et A. Mellet [NBAMP1], nous avons revisité l'approximation de diffusion de l'équation de Boltzmann linéaire dans le cas où la Maxwellienne n'est plus une Gaussienne mais a un profil de type puissance au voisinage des grandes vitesses. Dans [92] ou [91], on voit que l'équation du type diffusion obtenue lorsque le libre parcours moyen tend vers zéro est alors une équation de diffusion fractionnaire. Le résultat de [92] est basé sur l'utilisation de la transformée de Fourier et est spécifique aux noyaux de collision indépendants de la variable d'espace. Ce résultat a été généralisé dans [91] aux noyaux de collision inhomogènes en utilisant la méthode des moments. Nous proposons une méthode de type développement de Hilbert dans la perspective de pouvoir traiter le cas non-linéaire. Il faut pour cela procéder à un réarrangement des termes du développement prenant en compte le fait que pour de grandes vitesses, la partie correspondant à l'advection est d'ordre supérieur à la partie gain de l'opérateur de collision.

Dans un second travail en collaboration avec N. Ben Abdallah et A. Mellet [NBAMP2], nous traitons, grâce à la méthode des moments, le cas où le coefficient de diffusion classique n'est plus défini car la fréquence de collision est dégénérée.

Nous présentons ensuite [CGPS], écrit avec J.-F. Clouët, F. Golse et R. Sentis, où nous considérons des particules chargées évoluant dans un milieu pollué. Ces particules sont ralenties de façon hétérogène par ce milieu et nous souhaitons évaluer la proportion de particules qui stoppe dans le polluant et qui est par là même perdue pour les réactions de fusion. Nous établissons une formule analytique construite à partir d'une analyse précise de deux cas extrêmes ainsi qu'une adéquation avec des résultats numériques implémentés dans le cas d'une géométrie simple.

Enfin, dans un travail en collaboration avec G. Allaire et Y. Capdeboscq [ACP], nous étudions l'homogénéisation d'un problème aux valeurs propres pour un opérateur elliptique avec conditions de Neumann. Le cas de conditions de Dirichlet est bien connu depuis plusieurs années. Dans le cas des conditions au bord de Neumann, plusieurs phénomènes sont possibles et leur apparition dépend de la taille de la fraction de cellule résiduelle résultant du découpage

de l'intervalle de départ en petites cellules.

Nous identifions les différentes asymptotiques possibles correspondant soit à un phénomène d'homogénéisation rejoignant les résultats obtenus dans le cas de conditions de bord de Dirichlet, soit des phénomènes de localisation de la solution aux bords de l'intervalle. La description de ces comportements repose sur l'introduction de fonctions exponentielles périodiques déjà présentes dans le cas de conditions limites de Dirichlet avec dérive.

La deuxième partie de ce mémoire présente deux articles traitant du transport relativiste de densité par des méthodes de transport optimal. La théorie du transport optimal a été constamment développée depuis les travaux pionniers de Y. Brenier dans les années 90 relayé par exemple par Gangbo et McCann, Ambrosio, Otto ou bien Villani. Les motivations de Y. Brenier était au départ liées aux problèmes de la mécanique des fluides mais depuis, outre la théorie du transport optimal en elle-même, son application à l'obtention d'inégalités fonctionnelles ou à l'étude de comportements asymptotiques ont été au coeur d'un grand nombre de travaux. Dans un travail en collaboration avec R. McCann [McCP], nous nous intéressons au problème de Cauchy pour des équations de transport de densité à vitesse finie. Nous considérons par exemple une version relativiste de l'équation de la chaleur étudiée via un schéma classique d'Euler par Andreu, Caselles et Mazon [128]

$$\partial_t \rho - \nabla(\rho \frac{\nabla \rho}{\sqrt{\rho^2 + |\nabla \rho|^2}}) = 0, \tag{6}$$

et mentionnée dans ce cadre par Y. Brenier dans [133] comme un flot gradient de l'entropie de Boltzmann selon les géodésiques correspondant à une fonction de coût relativiste. Nous en construisons une solution par le schéma de Jordan, Kinderlehrer, Otto [77] adapté aux flots de gradients dans lequel on doit résoudre à chaque étape un problème de transport optimal

$$\begin{aligned} \rho^h(t) &= \rho_{k+1} \\ &= \arg \inf_{\rho \text{ t.q. } \int \rho = C} \left( \int \rho \log \rho + h \inf_{\pi \text{ t.q. } \rho_k \to \rho} \int c(|\frac{x-y}{h}|) d\pi(x,y) \right), \text{ si } t \in [kh; (k+1)h], \end{aligned} \tag{7}$$

où la fonction coût est donnée par

$$\begin{aligned} c(x) &= 1 - \sqrt{1 - |x|^2} \quad \text{si } |x| < 1 \\ &= \infty, \quad \text{sinon.} \end{aligned} \tag{8}$$

Chaque étape est en fait une double minimisation, la minimisation de l'entropie de Boltzmann parmi toutes les densités atteintes après un pas de temps $h$ suivant les géodésiques associées à la fonction de coût $c$. Ce procédé suit les idées présentes dans le travail d'Otto [163] qui étend la notion de flot gradient à des fonctions de coût plus générales que le coût quadratique pour lequel la méthode est particulièrement simple à expliquer. En effet, minimiser $E(\rho) + h|\frac{\rho_0 - \rho}{h}|^2$ donne formellement $\frac{\rho_0 - \rho}{h} = E'(\rho)$ qui est une version discrète de $\partial_t \rho = E'(\rho)$, ce qui veut dire que $\rho$ est un flot gradient de $E$.

On retrouve d'ailleurs dans la deuxième partie de la minimisation le problème de Monge-Kantorovich pour le coût relativiste

$$\inf_{\pi \text{ t.q. } \rho_k \to \rho} \int_{X \times Y} c(\frac{x-y}{h}) d\pi(x,y).$$

La densité cible $\rho$ étant fixée, on cherche le plan de transport optimal $\pi(x,y)$ (densité de probabilité sur l'espace produit $X \times Y$) qui minimise le coût mesuré par la fonction $c$. Pour

construire une solution approchée de (6), nous devons montrer que ce plan de transport est supporté par le graphe d'une application que l'on appelle application optimale. Par exemple, il est connu que dans le cas d'une fonction de coût strictement convexe continue ou dans le cas de $c(z) = |z|$, le coût optimal est effectivement atteint par un plan de transport supporté par le graphe d'une application et on a donc

$$\inf_{\pi \text{ s.t. } \rho_k \to \rho} \int_{X\times Y} c(\frac{x-y}{h}) d\pi(x,y) = \inf_{T \text{ s.t. } \rho_k \to \rho} \int_Y c(\frac{T(y)-y}{h}) d\rho(x).$$

Si l'on admet l'existence de cette application optimale $S$, en dérivant formellement les équations d'optimalité du problème de minimisation (8), on obtient

$$hc(\frac{S(y)-y}{h}) = -\log \rho \quad \text{soit en prenant le gradient} \quad \nabla c(\frac{S(y)-y}{h}) = \frac{\nabla \rho}{\rho}.$$

Lorsque la fonction de coût est strictement convexe, cette relation nous permet d'expliciter l'expression de $S$ car $\nabla c^* \nabla c(x) = x$. On obtient finalement

$$\frac{S(y)-y}{h} = \nabla c^*(\frac{\nabla \rho}{\rho})$$

ce qui implique, lorsque la fonction de coût est donnée par (8)

$$\frac{S(y)-y}{h} = \frac{\nabla \rho}{\sqrt{\rho^2 + |\nabla \rho|^2}}.$$

Ceci signifie, au moins au niveau discret, que la vitesse de propagation de la densité est bien la version relativiste de la vitesse de transport de la densité dans l'équation de la chaleur $\frac{\nabla \rho}{\rho}$.

La particularité de la fonction coût relativiste a engendré de nombreuses difficultés lors de la justification rigoureuse de ce raisonnement. Notamment, cette fonction peut être infinie et n'est pas continue. Dans ce cadre, la théorie connue du transport optimal ne s'applique pas. En effet, la preuve de l'existence d'une application optimale dans le cas de fonctions de coût régulières par la dualité de Kantorovich (cf Gangbo McCann [151], Villani [172]) est basée sur la borne Lipschitz uniforme des fonctions potentiel, donnée par le module de continuité de $c$.

Plus précisément, soit $\pi_{\text{opt}}$ le plan de transport optimal, la dualité de Kantorovich s'écrit

$$\int_{\Omega\times\Omega} hc(\frac{x-y}{h}) d\pi_{\text{opt}}(x,y) = \int_\Omega \phi(x) d\rho_{i-1}(x) + \int_\Omega \psi(y) d\rho_i(y),$$

où les fonctions potentiel $\psi$ et $\phi$ sont liées par la relation

$$hc(\frac{x-y}{h}) - \phi(x) - \psi(y) \geq 0, \quad \forall (x,y) \in \Omega \times \Omega, \tag{9}$$

qui devient

$$hc(\frac{x-y}{h}) - \phi(x) - \psi(y) = 0, \quad \forall (x,y) \in \operatorname{supp} \pi_{\text{opt}}. \tag{10}$$

Quand la fonction coût $c$ est continue, les relations (9)–(10) impliquent que la fonction potentiel $\psi$ est Lipschitz — et donc différentiable — ce qui implique sur les gradients

$$\nabla \psi(y) = \nabla c(\frac{x-y}{h}). \tag{11}$$

Cette équation donne directement la forme de l'application optimale $x = S(y) = y + h\nabla c^*(\nabla \psi(y))$ vu que $\nabla c^* = (\nabla c)^{-1}$. Dans [McCP], les fonctions potentiel ne seront plus

Lipschitz et nous devons trouver un autre argument pour obtenir la différentiablilité. Pour cela, on introduit un problème régularisé et nous passons à la limite. La densité $\rho$ obtenue ne sera pas Lipschitz mais $BV(\Omega)$ ce qui donne la propriété de différentiabilité approximative qui est en fait suffisante (cf Ambrosio, Gigli et Savare [117]). Cette stratégie de régularisation a été utilisée par exemple dans [119] pour traiter le cas de fonctions non strictement convexes (voir aussi [149]).

Dans la deuxième partie de [McCP], on s'intéresse au passage à la limite lorsque le pas de temps $h$ tend vers zéro. Nous utilisons un argument de type Minty-Browder basé sur la monotonie de $\nabla c^*$ pour identifier la limite du terme non-linéaire. Dans cette étape, la difficulté vient du manque de régularité de la densité $\rho$ qui rend difficile l'écriture des fonctionnelles impliquées dans ce raisonnement.

Dans [McCP], pour pouvoir conclure à l'existence d'une application optimale, nous utilisons fondamentalement la double minimisation présente dans le schéma JKO. Mais la question de l'existence d'une application optimale pour le problème de Monge-Kantorovich pour le coût relativiste restait ouverte. Depuis, dans un travail en collaboration avec J. Bertrand [BP], nous montrons que, dans le cas de mesures à supports compacts, lorsque le coût est fini, il existe une application optimale, solution du problème de Monge pour cette fonction coût particulière.

Pour cela, on fixe deux densités de probabilité et on considère le problème de Monge Kantorovich avec un paramètre représentant le temps (ou la vitesse maximale) autorisé. Bien sûr, lorsque ce paramètre tend vers l'infini, à partir d'une certaine valeur, le transport d'une densité à l'autre ne "voit" que la partie régulière du coût et l'existence d'une application optimale découle de la théorie classique du transport de masse. De même, lorsque ce paramètre tend vers zéro, le coût devient infini. Nous montrons l'existence d'un temps critique à partir duquel le coût relativiste entre les deux densités est fini et nous montrons que pour tout temps supérieur au sens large, le transport optimal est engendré par le graphe d'une application. On utilise pour cela un argument géométrique basé sur la notion de $c$-cyclique monotonie. La question de l'existence d'une application optimale pour des fonctions de coût du même type a déjà été mentionnée dans [139]. Dans ce papier, Carlier, De Pascale et Santambrogio proposent une stratégie différente basée sur l'existence d'un maximiseur dans le problème dual. De plus, Jimenez et Santambrogio [153] ont traité la restriction du coût quadratique à un ensemble convexe autorisant les bords plats.

# Liste de publications

**Thème 1 : Limites asymptotiques d'équations de la physique des plasmas**

**Limites quasi-neutres non relativistes de l'équation de Vlasov-Maxwell**

[BMP] Y. Brenier, N. Mauser, M. Puel. *Incompressible Euler and e-MHD as scaling limits of the Vlasov-Maxwell system,*
Commun. Math. Sci. Vol 1 (2003) no 3, p.437–447.
[PSR] M. Puel, L. Saint-Raymond. *Quasineutral limit of the relativistic Vlasov-Maxwell system,*
Asymptot. Anal. Vol 40 (2004) no 3-4 p.303–352.

**Une perturbation de Navier-Stokes**

[BNP] Y. Brenier, R. Natalini, M. Puel. *On a relaxation approximation of the incompressible Navier-Stokes equations,*
Proc. Amer. Math. Soc. Vol 132 (2004) no 4, p.1021–1028.

**Approximation diffusion et homogénéisation de l'équation de Boltzmann**

[NBAPV] N. Ben Abdallah, M. Puel, M. Vogelius, *Diffusion and homogenization limits with separate scales.*
Article soumis.
[NBABP] G. Bal, M. Puel *A corrector result for diffusion and homogenization limits for the Boltzmann equation.* Article soumis.

**Diffusion anormale pour l'équation de Boltzman**

[NBAMP1] N. Ben Abdallah, A. Mellet, M. Puel *Anomalous diffusion limit for kinetic equations with degenerate collision frequency.*
Accepté dans M3AS.
[NBAMP2] N. Ben Abdallah, A. Mellet, M. Puel *Fractional diffusion limit for collisional kinetic equations : a Hilbert expansion approach.*
Accepté dans Kinet Relat. Models.

**Une étude numérique du ralentissement de particules chargées**

[CGPS] J.-F. Clouet, F. Golse, M. Puel, R. Sentis. *On the slowing down of charged particles in a heterogeneous medium.*
Kinet. Relat. Models 1 (2008), no. 3, 387–404.

**Homogénéisation d'un opérateur elliptique avec conditions de Neumann**

[ACP] G. Allaire, Y. Capdeboscq, M. Puel, *Homogeneization of a one-dimensional spectral problem for a singularly perturbed elliptic operator with Neumann boundary conditions.* Accepté dans DCDS-B.

**Thème 2 : Transport de masse relativiste**

**Le problème de Monge-Kantorovich pour une fonction de coût relativiste**

[BP] J. Bertrand, M. Puel *The optimal mass transport problem for relativistic costs.*
Preprint.

**Problème de Cauchy pour une équation de la chaleur relativiste**

[McCP] R. McCann, M. Puel. *Constructing relativistic heat flows by transport time steps,*
Annales de l'Institut Henri Poincare / Analyse non lineaire 26 (2009), pp. 2539-2580

# Chapitre 1

# Analyses asymptotiques d'équations de la physique des plasmas

## 1.1 Limite quasi-neutre non relativiste de l'équation de Vlasov-Maxwell

Nous considérons la modélisation des gaz d'ions raréfiés par le système de Vlasov-Maxwell suivant

$$\begin{aligned}
&\partial_t g + \frac{c\xi}{\sqrt{\xi^2+c^2}}\cdot\nabla_x g + \frac{Ze}{m_i}(E + \frac{c\xi}{\sqrt{\xi^2+c^2}}\wedge B)\cdot\nabla_\xi g = 0,\\
&\partial_t f + \frac{c\xi}{\sqrt{\xi^2+c^2}}\cdot\nabla_x f - \frac{e}{m_e}(E + \frac{c\xi}{\sqrt{\xi^2+c^2}}\wedge B)\cdot\nabla_\xi f = 0,\\
&\mathrm{div}_x B = 0, \quad \mathrm{div}_x E = \frac{e}{\varepsilon_0}\int (Zg - f)d\xi,\\
&\partial_t B + \ \mathrm{curl}_x E = 0, \quad -\frac{1}{c^2}\partial_t E + \ \mathrm{curl}_x B = e\mu_0 \int \frac{c\xi}{\sqrt{\xi^2+c^2}}(Zg - f)d\xi,
\end{aligned} \tag{1.1}$$

où $c$ désigne la vitesse de la lumière, $-e$ et $Ze$ les charges respectives des électrons et des ions, $\mu_0$ et $\varepsilon_0 = 1/(c^2\mu_0)$ la perméabilité et la permittivité du vide. Dans ces équations, $f$ (resp. $g$) est une fonction positive qui représente la probabilité pour un électron (resp. pour un ion) de se trouver au temps $t$, au point $x \in \Omega \subset \mathbf{R}^3$, avec un moment $\xi \in \mathbf{R}^3$.

Les deux premières équations traduisent le fait que les particules évoluent sous l'influence d'une force électromagnétique, mouvement représenté par les équations caractéristiques

$$\begin{aligned}
\frac{dX}{dt} &= V,\\
\frac{dP}{dt} &= q(E + V\wedge B),
\end{aligned}$$

où $X$ et $P$ sont la position et le moment de la particule au temps $t$, et $V = cP/\sqrt{(cm)^2+P^2}$ sa vitesse relativiste.

### 1.1.1 Mise à l'échelle correspondant au régime quasi-neutre

Dans ce travail, nous considérons tout d'abord que les ions sont immobiles car très lourds et que nous sommes dans un régime quasi-neutre, c'est à dire que l'on considère le plasma à

une échelle d'espace très grande par rapport à la longueur de Debye. Plus précisément, voici les différentes échelles intervenant dans la modélisation :

- $T$ l'échelle de temps d'observation ;
- $T_o = L/c$ une échelle de temps caractéristique correspondant à la propagation de la lumière, $L$ étant l'échelle d'espace de référence ;
- $T_p = \sqrt{m\varepsilon_0/ne^2}$ la période de plasma réciproque ($m = m_e$) ;
- $T_c = v^*\sqrt{ne^2\mu_0/m}$ la période réciproque gyrocinétique, $v^* = L/T$ étant la vitesse caractéristique dans le plasma.

L'hypothèse de quasi-neutralité revient à supposer

$$\frac{T_p}{T} = \varepsilon << 1.$$

De plus, nous considérons que le mouvement dans le plasma est non-relativiste, i.e.

$$\frac{T_o}{T} = \gamma << 1.$$

Enfin, nous introduisons un dernier paramètre

$$\alpha = \frac{T_c}{T} = \frac{\gamma}{\varepsilon},$$

qui mesure l'aspect non-relativiste ou relativiste de l'effet du champ magnétique.

Ceci nous amène à considérer le système adimensionné suivant

$$\begin{aligned}
&\partial_t f + \frac{\xi}{\sqrt{1+\gamma^2\xi^2}}\cdot\nabla_x f - (E + \alpha\frac{\xi}{\sqrt{1+\gamma^2\xi^2}} \wedge B)\cdot\nabla_\xi f = 0,\\
&\mathrm{div}_x B = 0, \quad \varepsilon^2 \mathrm{div}_x E = n - \int f d\xi,\\
&\alpha\partial_t B + \ \mathrm{curl}_x E = 0, \quad -\alpha\varepsilon^2\partial_t E + \ \mathrm{curl}_x B = -\alpha\int \frac{\xi}{\sqrt{1+\gamma^2\xi^2}} f d\xi,
\end{aligned} \tag{1.2}$$

pour $\varepsilon$ et $\gamma$ très petits.

De plus, on se place sur le tore pour éviter les problèmes au bord. Nous partons des conditions initiales suivantes

$$f_{|t=0} = f_{in}, \quad E_{|t=0} = E_{in}, \quad B_{|t=0} = B_{in}, \tag{1.3}$$

satisfaisant les équations de compatibilité

$$\varepsilon^2 \mathrm{div}_x E_{in} = 1 - \int f_{in}(x,\xi)d\xi, \quad \mathrm{div}_x B_{in} = 0, \tag{1.4}$$

et qui sont d'énergie bornée

$$\mathcal{E}_{in} = \frac{1}{\gamma^2}\iint \sqrt{1+\gamma^2\xi^2} f_{in} d\xi dx + \frac{1}{2}\int(\varepsilon^2|E_{in}|^2 + |B_{in}|^2)dx < +\infty. \tag{1.5}$$

Afin de procéder à la limite asymptotique, nous devons travailler avec des solutions globales et pour ce faire, nous utilisons la notion de solution faible du système de Vlasov Maxwell dont l'énergie décroît (cf [9]). Pour ce type de solution, nous ne pouvons assurer la conservation des quantités physiques comme pour les solutions fortes. Ainsi, nous devons réécrire les équations de conservation prenant en compte les défauts de compacité via des mesures de défaut de la façon suivante.

*Théorème* 1.1.1. Pour $\varepsilon, \alpha > 0$ fixés, soit $f_{in}^{\varepsilon,\alpha} \in L^1 \cap L^2(\Omega \times \mathbf{R}^3)$ une fonction positive et $E_{in}^{\varepsilon,\alpha}, B_{in}^{\varepsilon,\alpha} \in L^2(\Omega)$ deux champs de vecteurs satisfaisant la condition de compatibilité (1.4) et la condition d'énergie bornée (1.5). Alors, il existe une solution faible de (1.2,1.3) $(f^{\varepsilon,\alpha}, E^{\varepsilon,\alpha}, B^{\varepsilon,\alpha})$ qui satisfait

– la conservation de la masse locale

$$\partial_t \int f^{\varepsilon,\alpha} d\xi + \mathrm{div}_x \int \frac{\xi}{\sqrt{1+\gamma^2\xi^2}} f^{\varepsilon,\alpha} d\xi = 0\,, \tag{1.6}$$

au sens des distributions sur $\mathbf{R}_+^* \times \Omega$,

– la conservation locale de moment

$$\begin{aligned} &\partial_t \left( \int \xi f^{\varepsilon,\alpha}(t,x,\xi) d\xi + \gamma \int_{\sigma\in S^2} \sigma dm^{\varepsilon,\alpha}(t,x,\sigma) \right) \\ &+\mathrm{div}_x \left( \int \frac{\xi\otimes\xi}{\sqrt{1+\gamma^2\xi^2}} f^{\varepsilon,\alpha}(t,x,\xi) d\xi + \int_{\sigma\in S^2} \sigma\otimes\sigma dm^{\varepsilon,\alpha}(t,x,\sigma) \right) \\ &+nE^{\varepsilon,\alpha} - \mathrm{div}_x(\varepsilon E^{\varepsilon,\alpha}\otimes\varepsilon E^{\varepsilon,\alpha} - \frac{1}{2}|\varepsilon E^{\varepsilon,\alpha}|^2 Id) - \mathrm{div}_x(\mu_E^{\varepsilon,\alpha} - \frac{1}{2} tr(\mu_E^{\varepsilon,\alpha}) Id) \\ &-\mathrm{div}_x(B^{\varepsilon,\alpha}\otimes B^{\varepsilon,\alpha} - \frac{1}{2}|B^{\varepsilon,\alpha}|^2 Id) - \mathrm{div}_x(\mu_B^{\varepsilon,\alpha} - tr(\mu_B^{\varepsilon,\alpha}) Id) \\ &+\varepsilon\alpha\partial_t(\varepsilon E^{\varepsilon,\alpha}\wedge B^{\varepsilon,\alpha}) + \varepsilon\alpha\partial_t \mu_{EB}^{\varepsilon,\alpha} = 0, \end{aligned} \tag{1.7}$$

au sens des distributions sur $\mathbf{R}^+ \times \Omega$, avec $\mu_E, \mu_B \in L^\infty(\mathbf{R}^+, \mathcal{M}(\Omega))$ des mesures de défaut à valeurs matricielles symétriques positives, $\mu_{EB} \in L^\infty(\mathbf{R}^+, \mathcal{M}(\Omega))$ une mesure de défaut à valeurs vectorielles et $m^{\varepsilon,\alpha} \in L^\infty(\mathbf{R}^+, \mathcal{M}(\Omega \times S^2))$ une mesure de défaut scalaire. Ces mesures de défaut viennent du schéma d'approximation des équations de Vlasov-Maxwell.

– la conservation globale de l'énergie pour tout $t > 0$

$$\begin{aligned} &\frac{1}{\gamma^2} \iint \sqrt{1+\gamma^2\xi^2} f^{\varepsilon,\alpha}(t,x,\xi) d\xi dx + \iint_{\sigma\in S^2} dm^{\varepsilon,\alpha}(t,x,\sigma) \\ &+\frac{1}{2}\int (\varepsilon^2 |E^{\varepsilon,\alpha}(t,x)|^2 + |B^{\varepsilon,\alpha}(t,x)|^2) dx + \frac{1}{2}\int (tr(d\mu_E^{\varepsilon,\alpha}) + tr(d\mu_B^{\varepsilon,\alpha})) = \mathcal{E}_{in}^{\varepsilon,\alpha}, \end{aligned} \tag{1.8}$$

ainsi que l'inégalité

$$\int d\mu_{EB}^{\varepsilon,\alpha} g \le \frac{1}{2}\int g(tr(d\mu_E^{\varepsilon,\alpha}) + tr(d\mu_B^{\varepsilon,\alpha})), \tag{1.9}$$

pour toute fonction positive $g(t,x)$ avec $\mu_E^{\varepsilon,\alpha}$, $\mu_B^{\varepsilon,\alpha}$ et $m^{\varepsilon,\alpha}$ des mesures de défaut.

Nous avons étudié la limite quasi-neutre, c'est à dire $\varepsilon \to 0$, du système (1.2). Pour cela nous supposons que la densité des électrons est égale à la densité des ions à des oscillations d'ordre $\varepsilon$ près. Ces oscillations induisent des oscillations du champ électromagnétique et nous avons montré que les interactions non-linéaires de ces oscillations n'entraînent pas de termes non oscillants.

Dans cette étude, nous faisons les hypothèses techniques suivantes

*(H1) la densité de charge des ions ne dépend pas de la variable d'espace. Cette hypothèse est fondamentale pour l'étude de la perturbation singulière car sinon son noyau est beaucoup plus restreint (cf [12] pour une étude dans le cas des fluides tournants).*

*(H2) on suppose que les données initiales sont presque mono-cinétiques*

$$\iint |\xi - u^0(x)|^2 f^0(x,\xi) d\xi dx \to 0.$$

Formellement, la limite asymptotique est obtenue de la manière suivante.

En supposant que la densité reste une dirac en vitesse, on obtient les équations sur les moments

$$\begin{aligned}
&\rho = 1 - \varepsilon \mathrm{div}_x(\varepsilon E), \quad \mathrm{div}_x B = 0,\\
&\partial_t j + \mathrm{div}_x \frac{1}{\rho}(j \otimes j) + \frac{1}{\varepsilon}(1 - \varepsilon \mathrm{div}_x(\varepsilon E))\varepsilon E + \alpha j \wedge B = 0\\
&\partial_t B + \frac{1}{\alpha\varepsilon}\ \mathrm{curl}_x(\varepsilon E) = 0,\\
&\partial_t(\varepsilon E) - \frac{1}{\alpha\varepsilon}\ \mathrm{curl}_x B = \frac{1}{\varepsilon} j,
\end{aligned} \tag{1.10}$$

car la correction relativiste disparaît à la limite.

Nous décomposons (1.10) en la somme d'une équation déterminant le champ moyen et d'une perturbation singulière L définie par

$$\mathrm{L} : (u, b, e) \mapsto \left( e, \frac{1}{\alpha}\ \mathrm{curl}_x e, -u - \frac{1}{\alpha}\ \mathrm{curl}_x b \right).$$

Nous cherchons à la limite un élément $(u, e, b)$ du noyau de L c'est à dire satisfaisant

$$u = -\frac{1}{\alpha}\ \mathrm{curl}_x b, \quad e = 0 \text{ ou plutôt } e = 0 + \varepsilon \underline{e},$$

dont nous établissons l'équation limite en projetant les lois de conservation locales sur le noyau de L. Comme nous montrons qu'il n'y a pas de résonances, nous obtenons

$$\left\{ \begin{aligned}
&\partial_t v + v \cdot \nabla v + \underline{e} + \alpha v \wedge b = 0,\\
&\partial_t b + \frac{1}{\alpha} \mathrm{curl}\ \underline{e} = 0,\\
&n v + \frac{1}{\alpha}\ \mathrm{curl} b = 0.
\end{aligned} \right. \tag{1.11}$$

Dans l'équation précédente, $\underline{e}$ joue le rôle de multiplicateur de Lagrange pour la contrainte $nv + \frac{1}{\alpha}\ \mathrm{curl} b = 0$.

### 1.1.2 Cas du système de Vlasov Maxwell non relativiste

Dans un premier temps, dans [BMP] écrit en collaboration avec Y. Brenier et N. Mauser, nous avions traité le cas de solutions fortes du système de Vlasov-Maxwell non relativiste

$$\partial_t f + \xi \cdot \nabla_x f - (E - \alpha(\xi \wedge B)) \cdot \nabla_\xi f = 0, \tag{1.12}$$

$$-\alpha \partial_t B + \nabla \wedge E = 0, \qquad \varepsilon \nabla \cdot E = 1 - \rho, \tag{1.13}$$

$$\varepsilon\alpha \partial_t E + \nabla \wedge B = \alpha J, \qquad \nabla \cdot B = 0\,. \tag{1.14}$$

dont les données initiales ne dépendent que de coordonnées bi-dimensionnelles avec un champ magnétique perpendiculaire :

*Théorème* 1.1.2 ([BMP]). Soit $(f(t,x,\xi), (B,E)(t,x))$ une solution globale régulière du système de Vlasov-Maxwell non relativiste pour $x \in \mathbf{R}^3$, $\xi \in \mathbf{R}^3$, $t \geq 0$, avec $f$ suffisamment décroissante lorsque $\xi \to \infty$. Soit $(v, b, e)(t, x)$ une solution globale régulière du système de la magnéto-hydrodynamique de l'électron pour $x \in \mathbf{R}^3$, $t \geq 0$. Fixons $T > 0$ et $\alpha > 0$ et supposons que les données initiales du système de Vlasov Maxwell satisfont

$$\int f(0,x,\xi) dx d\xi = 1, \quad \int f(0,x,\xi)|\xi|^2 dx d\xi \leq C,$$

$$\varepsilon \int |E(0,x)|^2 dx \leq C\sqrt{\varepsilon}, \quad \int |B(0,x) - b(0,x)|^2 dx \leq C\sqrt{\varepsilon},$$

$$\int |\xi - v(0,x)|^2 f(0,x,\xi) dx d\xi \leq C\sqrt{\varepsilon},$$

où $C$ ne dépend pas de $\varepsilon$. Alors

$$\sup_{0\leq t\leq T} \Big( \int |\xi - v(t,x)|^2 f(t,x,\xi) dx d\xi + \int |B(t,x) - b(t,x)|^2 dx \Big) \leq C'\sqrt{\varepsilon},$$

où $C'$ ne dépend que de $T$, $C$ et $(v,b,e)$.

### 1.1.3 Convergence forte dans le cas de données bien préparées

Dans le cas relativiste traité dans [PSR] écrit en collaboration avec L. Saint Raymond, pour des données bien préparées, c'est à dire lorsque l'on suppose que les champs initiaux convergent fort vers des champs appartenant au noyau de la perturbation singulière, on obtient le résultat de convergence forte suivant.

*Théorème* 1.1.3 ([PSR]). – Soit $v_0, b_0$ deux champs à divergence nulle appartenant à $H^s(\Omega)$ $(s > 2 + \frac{3}{2})$ satisfaisant

$$nv_0 + \frac{1}{\alpha} \operatorname{curl}_x b_0 = 0,$$

avec $n$ une fonction homogène.
Alors, il existe $T^* \in ]0, +\infty]$ et $(v,b) \in L^\infty_{loc}([0,T^*[, H^s(\Omega))$ unique solution de

$$\left\{ \begin{array}{r} \partial_t v + v \cdot \nabla v + \underline{e} + \alpha v \wedge b = 0, \\ \partial_t b + \dfrac{1}{\alpha} \operatorname{curl}_x \underline{e} = 0, \\ nv + \dfrac{1}{\alpha} \operatorname{curl}_x b = 0. \end{array} \right. \tag{1.15}$$

– Soit $(f^{\varepsilon,\alpha}_{in})_{\varepsilon,\alpha}$ une famille de fonctions positives appartenant à $L^1 \cap L^2(\Omega \times \mathbf{R}^3)$, et $(E^{\varepsilon,\alpha}_{in})_{\varepsilon,\alpha}, (B^{\varepsilon,\alpha}_{in})_{\varepsilon,\alpha}$ deux familles de champs de vecteurs de $L^2(\Omega)$ satisfaisant les conditions de compatibilité (1.4) pour $n$ donné, ainsi que la convergence lorsque $\varepsilon$ tend vers 0,

$$\iint \frac{(\xi - v_0)^2}{1 + \sqrt{1 + \gamma^2 \xi^2}} f^{\varepsilon,\alpha}_{in}(x,\xi) d\xi dx + \frac{1}{2} \int \varepsilon^2 |E^{\varepsilon,\alpha}_{in}(x)|^2 dx + \frac{1}{2} \int |B^{\varepsilon,\alpha}_{in}(x) - b_0|^2 dx \to 0.$$

Soit $(f^{\varepsilon,\alpha}, E^{\varepsilon,\alpha}, B^{\varepsilon,\alpha})$, pour $\epsilon > 0, \alpha > 0$, une solution du système de Vlasov-Maxwell (1.2) avec (1.3) comme condition initiale. Alors, pour tout $T < T^*$, la densité de courant converge faiblement

$$\hat{j}^{\varepsilon,\alpha} = \int f^{\varepsilon,\alpha} \xi / \sqrt{1 + \gamma^2 \xi^2} d\xi \rightharpoonup nv \text{ dans } L^\infty([0,T], L^1(\Omega)).$$

Le champ électrique mis à l'échelle et le champ magnétique convergent fortement

$$\varepsilon E^{\varepsilon,\alpha} \to 0 \text{ et } B^{\varepsilon,\alpha} \to b \text{ dans } L^\infty([0,T], (L^2(\Omega))^2),$$

lorsque $\varepsilon \to 0, \alpha \to \overline{\alpha} < +\infty$, où $(v,b) \in L^\infty([0,T], H^s(\Omega))$ est la solution forte locale de (1.15) avec comme condition initiale $(v_0, b_0)$.

### 1.1.4 Convergence faible pour des données initiales plus générales

Dans le cas où l'on ne suppose pas que les champs initiaux limites appartiennent au noyau de la singularité L, on obtient un résultat de convergence faible.

*Théorème* 1.1.4 ([PSR]). Soit $(f^{\varepsilon,\alpha}, E^{\varepsilon,\alpha}, B^{\varepsilon,\alpha})$ pour $\epsilon > 0, \alpha > 0$ fixés, une solution du système de Vlasov-Maxwell remis à l'échelle (1.2) avec (1.3) comme conditions initiales satisfaisant

$$\iint \frac{(\xi - v_0)^2}{1+\sqrt{1+\gamma^2\xi^2}} f_{in}^{\varepsilon,\alpha}(x,\xi)d\xi dx + \frac{1}{2}\int \varepsilon^2 |E_{dans}^{\varepsilon,\alpha}(x)|^2 dx + \frac{1}{2}\int |B_{in}^{\varepsilon,\alpha}(x) - b_0|^2 dx \to 0 \text{ qd } \varepsilon \to 0,$$

où $(v_0, e_0, b_0)$ sont des champs de vecteurs réguliers choisis tels que $\mathrm{div}_x b_o = 0$.

Alors, pour tout $T < T^*$, la densité de courant

$$\hat{j}^{\varepsilon,\alpha} = \int f^{\varepsilon,\alpha}\xi/\sqrt{1+\gamma^2\xi^2}d\xi \rightharpoonup nv \text{ au sens des mesures,}$$

le champ électrique mis à l'échelle et le champ magnétique convergent faiblement

$$\varepsilon E^{\varepsilon,\alpha} \rightharpoonup 0 \text{ et } B^{\varepsilon,\alpha} \rightharpoonup b \text{ dans } L^\infty_{Loc}([0,T[,(L^2(\Omega))^2)$$

lorsque $\varepsilon \to 0, \alpha \to \overline{\alpha} < +\infty$, où $(v,b) \in L^\infty([0,T], H^s(\Omega))$ est la solution forte locale de (1.15) avec comme conditions initiales

$$\begin{cases} v_{in} = -\Delta_x(\alpha^2 \mathrm{Id} - \Delta_x)^{-1} v_0 + \alpha \ \mathrm{curl}_x(\alpha^2 \mathrm{Id} - \Delta_x)^{-1} b_0, \\ b_{in} = -\alpha(\alpha^2 \mathrm{Id} - \Delta_x)^{-1} \ \mathrm{curl}_x v_0 + \alpha^2(\alpha^2 \mathrm{Id} - \Delta_x)^{-1} b_0, \end{cases}$$

qui est la projection de $(v_0, e_0, b_0)$ sur le noyau de l'opérateur $L$.

### 1.1.5 La méthode d'énergie modulée

Le principe de la méthode de l'énergie modulée utilisée dans cette série de papiers est la suivante. Nous modulons l'énergie du système par des fonctions test $U = (v,e,b) \in C_c^\infty(\mathbf{R}^+ \times \Omega, (\mathbf{R}^3)^3)$ en écrivant

$$H_U^{\varepsilon,\alpha}(t) = \iint \frac{|\xi - v|^2}{1+\sqrt{1+\gamma^2\xi^2}} f^{\varepsilon,\alpha}(t,x,\xi)d\xi dx + \frac{1}{2}\int |\varepsilon E^{\varepsilon,\alpha} - e|^2 dx + \frac{1}{2}\int |B^{\varepsilon,\alpha} - b|^2 dx, \quad (1.16)$$

et nous obtenons des inégalités de stabilité pour cette énergie modulée lorsque $(v,e,b)$ est solution de

$$\begin{aligned} \partial_t v + v\cdot\nabla v + \frac{e}{\varepsilon} + \alpha v \wedge b = 0, \\ \partial_t b + \frac{1}{\alpha\varepsilon}\mathrm{curl}\ e = 0, \\ \partial_t e - \frac{1}{\alpha\varepsilon}\mathrm{curl}\ b - n\frac{v}{\varepsilon} + v\mathrm{div}\ e = 0, \\ \mathrm{div}\ b = 0. \end{aligned} \quad (1.17)$$

Ceci donne une notion de convergence lorsque les données sont bien préparées car les solutions de (1.17) sont alors proches du système limite.

En effet, l'énergie modulée totale satisfait

$$\begin{aligned}\mathbb{H}_U^\varepsilon(t) &= H_U^\varepsilon(t) + \iint dm^\varepsilon(t,x,\sigma) + \frac{1}{2}\int tr(d\mu_E^\varepsilon) + tr(d\mu_B^\varepsilon) \\ &= H_U^\varepsilon(0) + \int_0^t \int D_x v : (h_U^\varepsilon(s,x)dx + d\nu^\varepsilon(s,x))ds \\ &\quad + \int (A^\varepsilon(U) + O(\varepsilon)) \cdot \begin{pmatrix} \rho^\varepsilon v - \hat{j}^\varepsilon \\ b - B^\varepsilon \\ e - \varepsilon E^\varepsilon \end{pmatrix} dxds + R^\varepsilon(t),\end{aligned}$$

où $R^\varepsilon(t)$ désigne les correcteurs relativistes.

L'accélération peut être décomposée de la manière suivante

$$\begin{aligned}A^\varepsilon(U) &= \begin{pmatrix} \partial_t v + v\cdot\nabla_x v + \alpha v \wedge b \\ \partial_t b \\ \partial_t e + v\mathrm{div}_x e \end{pmatrix} + \tfrac{1}{\varepsilon}\begin{pmatrix} e \\ \frac{1}{\alpha}\ \mathrm{curl}_x e \\ -nv + \frac{1}{\alpha}\ \mathrm{curl}_x b \end{pmatrix} \\ &= \quad \partial_t U \qquad + R(U,U) \qquad + \tfrac{1}{\varepsilon} \qquad SU.\end{aligned}$$

Concernant les termes quadratiques, on a

$$h^\varepsilon + d\nu^\varepsilon \leq 6\mathbb{H}_U^\varepsilon(t).$$

Enfin, grâce au lemme de Gronwall, nous contrôlons l'énergie modulée en écrivant

$$\mathbb{H}_U^\varepsilon(t) \leq C(||U||_{H^s})(\mathbb{H}_U^\varepsilon(0) + O(A^\varepsilon(U))).$$

Pour des données plus générales, il reste à étudier les oscillations dans (1.17).

### 1.1.6 Etude des oscillations

Pour des données initiales plus générales, c'est à dire lorsque $nv_0 + \frac{1}{\alpha}\ \mathrm{curl}_x b_0 \neq 0$, nous construisons $U^{app,\varepsilon} = \overline{U} + U^{osc,\varepsilon}$ de sorte que $A^\varepsilon(U^{app,\varepsilon}) = O(\varepsilon)$ avec $\overline{U}$ champ moyen satisfaisant les équations de la (e-MHD) et $U^{osc,\varepsilon}$ partie oscillante satisfaisant une équation linéaire dont les coefficients incluent $\overline{U}$. Puis nous montrons $U^{osc,\varepsilon} \rightharpoonup 0$, ce qui implique une convergence faible.

Plus précisément, nous n'avons pas $A^\varepsilon(U^{app,\varepsilon}) = O(\varepsilon)$, nous devons alors introduire une correction $\varepsilon y^\varepsilon \rightharpoonup 0$ de sorte que $A^\varepsilon(U^{app,\varepsilon} + \varepsilon y^\varepsilon) = O(\varepsilon)$. Pour séparer $U^{app,\varepsilon} = \overline{U} + U^{osc,\varepsilon}$, nous suivons la méthode introduite dans [13]. Tout d'abord, nous composons avec le semi-groupe $W^\varepsilon = \mathcal{S}(\frac{t}{\varepsilon})U^\varepsilon$ et nous obtenons

$$\partial_t U^\varepsilon + \frac{1}{\varepsilon}SU^\varepsilon + R(U^\varepsilon, U^\varepsilon) = 0 <=> \partial_t W^\varepsilon + Q(\frac{t}{\varepsilon}, W^\varepsilon, W^\varepsilon) = 0.$$

Puis, après projection de $Q$ sur sa partie non-oscillante, on définit $\overline{W}$ appartenant au noyau de la singularité tel que

$$\partial_t \overline{W} + \overline{Q}(\overline{W}, \overline{W}) = 0.$$

L'étude de $\overline{Q}$ montre qu'il n'y a pas de couplage non linéaire constructif, c'est à dire pas de résonance. La solution peut donc s'écrire $W^\varepsilon = \overline{W} + W^{osc}$ ce qui donne finalement en recomposant avec le semi-groupe,

$$U^{app,\varepsilon} = \mathcal{S}(-\frac{t}{\varepsilon})W^\varepsilon = \overline{W} + \mathcal{S}(-\frac{t}{\varepsilon})W^{osc} = \overline{U} + U^{osc,\varepsilon}.$$

## 1.2 Une perturbation de Navier Stokes

Dans cette section, nous présentons un travail effectué en collaboration avec Y. Brenier et R. Natalini [BNP] qui s'apparente aux travaux sur le système de Vlasov-Maxwell présentés précédemment de part la nature du théorème et de part la méthode employée dans la preuve.

Nous construisons une approximation de la solution des équations de Navier Stokes. Pour cela, nous partons des équations d'Euler incompressible qui décrivent un fluide parfait incompressible

$$\begin{cases} \partial_t u + \nabla \cdot (u \otimes u) = \nabla \phi, \\ \nabla \cdot u = 0, \\ u(0,x) = u_0(x), \end{cases} \tag{1.18}$$

avec $(t,x) \in [0,T] \times \mathbf{T}^2$, où $\mathbf{T}^2$ est le tore $\mathbf{R}^2/\mathbf{Z}^2$. Les inconnues sont la vitesse du fluide $u$ à valeur dans $\mathbf{R}^2$ ainsi que la pression $\phi$.

Dans l'esprit de [20], nous introduisons sa version relaxée obtenue par une perturbation singulière du terme non linéaire $(u \otimes u)$. Ainsi, nous écrivons le système

$$\begin{cases} \partial_t u + \nabla \cdot (V) = \nabla \phi, \\ \partial_t V + a \nabla u = -\dfrac{1}{\eta}(V - u \otimes u), \\ \nabla \cdot u = 0, \\ u(0,x) = u_0(x), V(0,x) = V_0(x). \end{cases} \tag{1.19}$$

La limite asymptotique connue est celle correspondant à $\eta \to 0$ qui aboutit au moins formellement au système d'Euler (1.18). Dans ce travail, nous posons $\eta = 1$ et nous introduisons la remise à l'échelle diffusive

$$\begin{cases} u^\varepsilon(t,x) & := \dfrac{1}{\sqrt{\varepsilon}} u\left(\dfrac{x}{\sqrt{\varepsilon}}, \dfrac{t}{\varepsilon}\right), \\ V^\varepsilon(x,t) & := \dfrac{1}{\varepsilon} V\left(\dfrac{x}{\sqrt{\varepsilon}}, \dfrac{t}{\varepsilon}\right), \\ \phi^\varepsilon(x,t) & := \dfrac{1}{\varepsilon} \phi\left(\dfrac{x}{\sqrt{\varepsilon}}, \dfrac{t}{\varepsilon}\right) \end{cases} \tag{1.20}$$

pour $\varepsilon > 0$. Cette remise à l'échelle est identique à celle impliquée dans l'approximation diffusion classique que nous allons décrire pour l'équation de Boltzmann dans la section suivante. Ainsi, finalement, nous considérons le système

$$\begin{cases} \partial_t u^\varepsilon + \nabla \cdot (V^\varepsilon) = \nabla \phi^\varepsilon, \\ \sqrt{\varepsilon} \partial_t V^\varepsilon + \dfrac{a}{\sqrt{\varepsilon}} \nabla u^\varepsilon = -\dfrac{1}{\sqrt{\varepsilon}}(V^\varepsilon - u^\varepsilon \otimes u^\varepsilon), \\ \nabla \cdot u^\varepsilon = 0, \\ u^\varepsilon(0,x) = u_0^\varepsilon(x), V^\varepsilon(0,x) = V_0^\varepsilon(x). \end{cases} \tag{1.21}$$

Nous montrons que (1.21) converge, lorsque $\varepsilon$ tend vers 0, vers les solutions régulières de

l'équation de Navier Stokes incompressible

$$\begin{cases} \partial_t U + \nabla \cdot (U \otimes U) - a\Delta U = \nabla \phi, \\ \nabla \cdot U = 0, \\ U(0,x) = U_0(x). \end{cases} \tag{1.22}$$

D'ailleurs, ce scaling peut aussi être considéré comme une perturbation hyperbolique des équations de Navier Stokes, car, en éliminant $V$ dans les équations (1.21), on obtient

$$\begin{cases} \partial_t u^\varepsilon + P(\nabla \cdot (u^\varepsilon \otimes u^\varepsilon)) - a\Delta u^\varepsilon + \varepsilon \partial_{tt} u^\varepsilon = 0, \\ \nabla \cdot u^\varepsilon = 0, \end{cases} \tag{1.23}$$

où $P$ est le projecteur sur les vecteurs à divergence nulle. C'est à partir de ce système que nous prouvons le résultat de convergence énoncé ci-dessous grâce à la méthode de l'énergie modulée.

### 1.2.1 Existence de solutions et convergence asymptotique

Tout d'abord, l'existence de solutions locales régulières pour le système (1.21) est donnée par le théorème suivant.

*Théorème* 1.2.1. Supposons que les données initiales $(u_0^\varepsilon(x), V_0^\varepsilon(x))$ soient des fonctions régulières appartenant à $H^s$ pour $s \geq 2$. Alors, il existe un temps positif $T^\varepsilon$, dépendant uniquement des données initiales et une solution $(u^\varepsilon, V^\varepsilon, \phi^\varepsilon) \in C([0,T]; (H^s)^3)$ du système (1.21). De plus, si $T^\varepsilon < \infty$, alors

$$\lim_{t \to T^\varepsilon_-} ||(u^\varepsilon, V^\varepsilon)||_{H^2} \to \infty. \tag{1.24}$$

Nous énonçons alors le théorème suivant

*Théorème* 1.2.2 ([BNP]). Soit $T \geq 0$ et $U^0$ un champ de vecteur régulier à divergence nulle sur $\mathbf{T}^2$. Soit $(u_0^\varepsilon, V_0^\varepsilon)$ une donnée initiale sur $\mathbf{T}^2$ au problème (1.21). Supposons de plus qu'il existe une constante $C$ indépendante de $\varepsilon$ telle que

$$||u_0^\varepsilon||_{H^1(\mathbf{T}^2)} \leq C \tag{1.25}$$

$$||\partial_t u^\varepsilon(0,\cdot)||_{L^2(\mathbf{T}^2)} \leq C \tag{1.26}$$

$$|u_0^\varepsilon|_{H^2(\mathbf{T}^2)} < \frac{C_0}{K_s \sqrt{\varepsilon}} \tag{1.27}$$

$$\int_{\mathbf{T}^2} |u_0^\varepsilon(x) - U^0(x)|^2 dx \leq C\sqrt{\varepsilon}. \tag{1.28}$$

Alors, $u^\varepsilon$ est une solution globale du système relaxé (1.21) et converge dans $L^\infty([0,T], L^2(\mathbf{T}^2))$ lorsque $\varepsilon \to 0$, vers la solution unique $U$ des équations de Navier Stokes incompressible (1.22) ayant pour conditions initiales $U^0$. De plus

$$\sup_{t \in [0,T]} \int_{\mathbf{T}^2} |u^\varepsilon - U|^2 dx \leq C_T \sqrt{\varepsilon},$$

où $C_T$ dépend uniquement de $T$, $U$, $C$ et $C_0$.

1. où $0 < C_0 < \sqrt{a}$ et $K_s$ est la constante de Sobolev en dimension 2 pour la norme $|\cdot|_{H^2(\mathbf{T}^2)}$ et

$$|u|_{H^2(\mathbf{T}^2)} = ||u||_{L^2(\mathbf{T}^2)} + ||\text{curl } u||_{L^2(\mathbf{T}^2)} + ||\nabla(\text{curl } u)||_{L^2(\mathbf{T}^2)}$$

qui est équivalente à la norme $H^2$ pour les vecteurs à divergence nulle $\nabla \cdot u = 0$.

## 1.3 Approximation diffusion et homogénéisation de l'équation de Boltzmann

Notre point de départ, ici, est l'équation de Boltzmann, une équation cinétique comme le système de Vlasov Maxwell qui décrit le fait que l'accélération est déterminée, cette fois, par l'interaction entre les particules via leurs collisions. Dans ce cas la densité de présence est donnée par l'équation

$$\partial_t f + \mathbf{v} \cdot \nabla_{\mathbf{x}} f = Q(f) \ ,$$

à laquelle on pourrait tout à fait ajouter l'effet d'un champ électromagnétique.

Lorsque l'échelle de temps correspondant aux collisions est très petite par rapport au temps d'observation, les effets macroscopiques sont prépondérants et on peut écrire l'équation mise à l'échelle

$$\partial_t f + \frac{1}{\varepsilon}\mathbf{v} \cdot \nabla_{\mathbf{x}} f = \frac{Q(f)}{\varepsilon^2} \ ,$$

où $\varepsilon$ désigne le libre parcours moyen, temps caractéristique entre deux collisions. Lorsque $\varepsilon \to 0$, les particules épousent un profil en vitesse correspondant à un équilibre local satisfaisant $Q(f) = 0$.

Ici, nous considérons un opérateur de collision linéaire de type BGK

$$Q(f)(\mathbf{v}) = \int (\sigma(\mathbf{v}' \to \mathbf{v}) f(\mathbf{v}') - \sigma(\mathbf{v} \to \mathbf{v}') f(\mathbf{v}))\, d\nu(\mathbf{v}') \ ,$$

dont le noyau est engendré par une fonction positive que nous appellerons Maxwellienne. L'équilibre local vers lequel tend la solution de Boltzmann est alors le produit de cette Maxwellienne et d'une densité solution d'une équation de diffusion.

Nous nous plaçons dans le cas hétérogène où la section efficace, qui représente la probabilité qu'une particule passe après collision d'une vitesse $\mathbf{v}'$ à une vitesse $\mathbf{v}$, dépend de la variable d'espace. Nous allons de plus supposer que cette dépendance est périodique de période $\alpha$ très petite. Nous considérons donc l'équation

$$\partial_t f + \frac{1}{\varepsilon}\mathbf{v} \cdot \nabla_{\mathbf{x}} f = \frac{Q_\alpha(f)}{\varepsilon^2} \ , \tag{1.29}$$

où

$$Q_\alpha(f)(\mathbf{v}) = \int \left(\sigma(\frac{\mathbf{x}}{\alpha}, \mathbf{v}, \mathbf{v}') f(\mathbf{v}') - \sigma(\frac{\mathbf{x}}{\alpha}, \mathbf{v}', \mathbf{v}) f(\mathbf{v})\right) d\nu(\mathbf{v}') \ , \tag{1.30}$$

et on s'attend à ce qu'un processus d'homogénéisation apparaisse.

L'étude simultanée de l'approximation de diffusion et de l'homogénéisation de Boltzmann a déjà été étudiée dans plusieurs travaux, notamment dans Bal [46], Allaire - Bal [39], Goudon-Poupaud [74], [75] , Goudon-Mellet [72] et Ben Abdallah-Tayeb [55] mais toujours dans le cas où le libre parcours moyen et la taille de la période sont du même ordre. Dans ce cas, l'advection correspondant à la variable d'espace rapide impliquée dans le phénomène d'homogénéisation et l'opérateur de collision se contrebalancent et l'équilibre local est solution de $\mathbf{v} \cdot \nabla_{\mathbf{y}} f = Q(f)$ et non de $Q(f) = 0$, $\mathbf{y}$ étant la variable rapide $y = \frac{x}{\alpha}$.

Ici, nous supposons que le libre parcours moyen $\varepsilon$ et la taille de la cellule $\alpha$ tendent vers zéro mais nous imposons de plus $\varepsilon << \alpha$. Nous nous attendons donc à ce que l'approximation diffusion ait lieu "avant" l'homogénéisation et nous montrons d'ailleurs rigoureusement qu'à la limite, on obtient un équilibre local dont la densité satisfait une équation de diffusion homogénéisée. Rappelons que dans [102], Sentis traitait le même problème mais pour des Maxwelliennes indépendantes de la variable d'espace et pour deux paramètres $\alpha$ et $\varepsilon$ liés par

une loi du type $\alpha = \varepsilon^p$. Ici, nous introduisons un nouveau petit paramètre $\eta = \frac{\varepsilon}{\alpha}$ et nous considérerons $\alpha$ et $\varepsilon$ tels que $\alpha$, $\varepsilon$ et $\eta$ tendent vers zéro.

Nous présentons un premier résultat de convergence à deux échelles, obtenu par la méthode des moments, suivi d'un résultat plus fort requérant des données initiales plus régulières, obtenu par la méthode de Hilbert, qui permet de plus de calculer les premiers termes correctifs de l'approximation.

### 1.3.1 Convergence faible par la méthode des moments

**Notations et hypothèses**

La variable d'espace $x$ appartient à $\mathbf{R}^d$, tandis que la variable de vitesse $\mathbf{v}$ appartient à un ensemble compact symétrique $V$ de $\mathbf{R}^d$ muni d'une mesure de probabilité symétrique $\nu$. On suppose de plus que la mesure $\nu$ satisfait $\nu(\{\mathbf{v} \in V,\ |\mathbf{v}\cdot\xi| \leq h\}) \leq Ch^\gamma$, pour tout $\xi \in S^{d-1}$, $h > 0$ pour $C$ et $\gamma > 0$. La variable de temps $t$ appartient à un intervalle de temps quelconque $[0, T]$. La fonction de distribution $f^{\varepsilon,\eta}(t, \mathbf{x}, \mathbf{v})$ est l'unique solution faible de l'équation

$$\partial_t f^{\varepsilon,\eta} + \frac{1}{\varepsilon}\mathbf{v}\cdot\nabla_{\mathbf{x}} f^{\varepsilon,\eta} = \frac{1}{\varepsilon^2} Q_{\varepsilon/\eta}(f^{\varepsilon,\eta}) \quad , \quad f^{\varepsilon,\eta}(0,\mathbf{x},\mathbf{v}) = f_{ini}(\mathbf{x},\mathbf{v}) \ , \tag{1.31}$$

où

$$Q_\alpha(f)(\mathbf{v}) = Q[\frac{\mathbf{x}}{\alpha}](f) \ , \quad \text{avec} \quad Q[\mathbf{y}](f) = \int \big(\sigma(\mathbf{y},\mathbf{v},\mathbf{v}')f(\mathbf{v}') - \sigma(\mathbf{y},\mathbf{v}',\mathbf{v})f(\mathbf{v})\big)\, d\nu(\mathbf{v}') \ . \tag{1.32}$$

Concernant la section efficace $\sigma$, on suppose qu'elle satisfait

(*i*) **(H0)** $\sigma(\mathbf{y},\mathbf{v},\mathbf{v}')$ appartient à $L^\infty(V_{\mathbf{v}} \times V_{\mathbf{v}'}; C^\infty_{per}(\mathbf{R}^d_{\mathbf{y}}))$, est bornée par en dessous et au dessus par des constantes positives et est périodique de période 1 dans la variable $\mathbf{y} = \frac{x}{\alpha}$.

(*ii*) **(H1)** $\sigma$ satisfait les symétries suivantes

$$\sigma(-\mathbf{y},\mathbf{v},\mathbf{v}') = \sigma(\mathbf{y},\mathbf{v},\mathbf{v}'), \text{ et } \sigma(\mathbf{y},-\mathbf{v},-\mathbf{v}') = \sigma(\mathbf{y},\mathbf{v},\mathbf{v}').$$

(*iii*) **(H2)** $\sigma$ satisfait de plus l'équilibre détaillé, c'est à dire

$$\sigma(\mathbf{y},\mathbf{v},\mathbf{v}') = \tilde\sigma(\mathbf{y},\mathbf{v},\mathbf{v}')M(\mathbf{y},\mathbf{v}) \quad \text{avec} \quad M > 0 \quad \text{et} \quad \tilde\sigma(\mathbf{y},\mathbf{v},\mathbf{v}') = \tilde\sigma(\mathbf{y},\mathbf{v}',\mathbf{v}) \ .$$

On appelle Maxwellienne l'unique solution du problème

$$Q[\mathbf{y}](M) = 0, \quad \text{avec} \quad \int_V M(\mathbf{y},\mathbf{v})d\nu(\mathbf{v}) = 1 \ .$$

Les fonctions $M$ et $\tilde\sigma$ appartiennent à $L^\infty(V_{\mathbf{v}}; C^\infty(\mathbf{R}^d_{\mathbf{y}}))$ et $L^\infty(V_{\mathbf{v}} \times V_{\mathbf{v}'}; C^\infty(\mathbf{R}^d_{\mathbf{y}}))$, respectivement, et sont périodiques de période 1 dans la variable $\mathbf{y}$.

Ici encore, pour travailler avec des solutions globales, nous utilisons la notion de solutions faibles suivante

*Théorème* 1.3.1. [98][63]. Soit $f_{ini}(\mathbf{x},\mathbf{v})$ une fonction positive appartenant à $L^1(\mathbf{R}^d \times V)$. L'équation (1.31) admet une unique solution faible dans $C^0(\mathbf{R}^+; L^1(\mathbf{R}^d \times V))$ satisfaisant :

$$\begin{aligned} &\int_{\mathbf{R}^d\times V} f_{ini}(\mathbf{x},\mathbf{v})\phi(0,\mathbf{x},\mathbf{v})\, d\mathbf{x}\, d\nu(\mathbf{v}) \\ &\quad + \int_{\mathbf{R}^+}\int_{\mathbf{R}^d\times V} f^{\varepsilon,\eta}(\partial_t\phi + \frac{\mathbf{v}\cdot\nabla_{\mathbf{x}}\phi}{\varepsilon} + \frac{Q^*_{\varepsilon/\eta}(\phi)}{\varepsilon^2})\, d\mathbf{x}d\nu(\mathbf{v})\, dt = 0, \qquad \forall\phi \in \mathcal{T} \ . \end{aligned} \tag{1.33}$$

Les fonctions tests (les éléments de $\mathcal{T}$) sont dans $C^1$ dans les variables $(t,\mathbf{x})$, continues en $\mathbf{v}$, et sont à support compact. La solution $f^{\varepsilon,\eta}$ est positive et $\|f^{\varepsilon,\eta}(t,\cdot,\cdot)\|_{L^1} = \|f_{ini}\|_{L^1}$.

Par la méthode des moments, on montre dans [NBAPV], écrit en collaboration avec N. Ben Abdallah et M. Vogelius, le résultat de convergence double échelle suivant

*Théorème* 1.3.2 ([NBAPV]). Supposons que $\sigma(\mathbf{y},\mathbf{v},\mathbf{v}')$ satisfait **(H0)**,**(H1)** et **(H2)**. Soit $f^{\varepsilon,\eta}$ la solution de (1.31) ayant pour donnée initiale positive $f_{ini} \in L^1(\mathbf{R}^d, V)$. Soit $\mathbf{X}^* = \mathbf{X}^*(\mathbf{y},\mathbf{v})$, l'unique solution du problème de cellule

$$Q^*[\mathbf{y}](\mathbf{X}^*) = -\mathbf{v} \ , \quad \int_V \mathbf{X}^* d\nu(\mathbf{v}) = 0 \ ,$$

où $Q^*[\mathbf{y}]$ est l'adjoint de l'opérateur $Q[\mathbf{y}]$ pour le produit scalaire $L^2(V)$. Soit $M$, la Maxwellienne locale, et soit $\lambda = \lambda(\mathbf{y},\mathbf{v})$ l'unique solution de

$$Q[\mathbf{y}](\lambda) = \mathbf{v}\cdot\nabla_{\mathbf{y}} M \ , \quad \int_V \lambda(\mathbf{y},\mathbf{v}) d\nu(\mathbf{v}) = 0 \ .$$

Pour toutes suites $\varepsilon_k$, $\eta_k$, telles que $\varepsilon_k \to 0$, $\eta_k \to 0$, et $\varepsilon_k/\eta_k \to 0$, lorsque $k \to \infty$, il existe une mesure de Borel régulière $\mathcal{F}(dt, d\mathbf{x}, d\mathbf{y}, d\mathbf{v})$ telle que $f^{\varepsilon_k,\eta_k}$ converge à deux échelles (à l'échelle 1 et $\frac{\varepsilon_k}{\eta_k}$) vers $\mathcal{F}(dt, d\mathbf{x}, d\mathbf{y}, d\mathbf{v})$. La mesure limite se décompose ainsi $\mathcal{F}(dt, d\mathbf{x}, d\mathbf{y}, d\mathbf{v}) = N(dt, d\mathbf{x}, d\mathbf{y}) M(\mathbf{y},\mathbf{v}) d\nu(\mathbf{v})$, tandis que la densité $N(t,\mathbf{x},\mathbf{y})$ se décompose elle-même $N(t,\mathbf{x},\mathbf{y}) = \rho(\mathbf{y}) d\mathbf{y}\, n(dt, d\mathbf{x})$. La densité microscopique $\rho(\mathbf{y})$ est périodique de période 1 et est la solution de

$$L(\rho(\mathbf{y})) = 0 \ , \quad \int_Y \rho(\mathbf{y}) = 1 \ ,$$

avec

$$L(\rho(\mathbf{y})) = -\mathrm{div}_{\mathbf{y}}(D(\mathbf{y})\nabla_{\mathbf{y}}\rho(\mathbf{y})) + \mathrm{div}_{\mathbf{y}}(U(\mathbf{y})\rho(\mathbf{y})) \ ,$$

et

$$D(\mathbf{y}) = \int_V \mathbf{X}^*(\mathbf{y},\mathbf{v}) \otimes \mathbf{v} M(\mathbf{y},\mathbf{v}) d\nu(\mathbf{v}) \ , \quad U(\mathbf{y}) = \int_V \mathbf{v}\lambda(\mathbf{y},\mathbf{v}) d\nu(\mathbf{v}) \ .$$

La densité macroscopique $n(dt, d\mathbf{x})$ est l'unique solution de l'équation

$$\partial_t n - \mathrm{div}_{\mathbf{x}}(\mathbf{D}\nabla_{\mathbf{x}} n) = 0 \quad , \quad n(t=0,\mathbf{x}) = n_{ini}(\mathbf{x}) = \int_V f_{ini}(\mathbf{x},\mathbf{v})\, d\nu(\mathbf{v}) \ . \tag{1.34}$$

La matrice homogénéisée est donnée par

$$\mathbf{D} = \int_Y \rho(\mathbf{y}) D(\mathbf{y})\, d\mathbf{y} - \int_Y \mathbf{\Theta}^{[-1]}(\mathbf{y}) \otimes H(\mathbf{y})\, d\mathbf{y} \ , \tag{1.35}$$

où

$$H(\mathbf{y}) = \nabla_{\mathbf{y}} \cdot (\rho D^\top(\mathbf{y})) + D(\mathbf{y})\nabla_{\mathbf{y}}\rho - \rho U(\mathbf{y}), \ \mathbf{\Theta}^{[-1]}(\mathbf{y}) = L^{*-1}\Big(\int M(\mathbf{y},\mathbf{v})(\mathbf{v}\cdot\nabla_{\mathbf{y}})\, \mathbf{X}^*(\mathbf{y},\mathbf{v}) d\nu(\mathbf{v})\Big)$$

et $L^*$ est l'opérateur adjoint de $L$ pour le produit scalaire de $L^2(Y)$.

### 1.3.2 La méthode des moments

La méthode des moments est une méthode basée sur la reformulation faible de l'équation considérée. Dans ce travail, nous la couplons avec la notion de convergence deux échelles en utilisant des fonctions test oscillantes ce qui permet d'introduire la variable d'espace rapide dans l'équation.

Rappelons tout d'abord la définition précise de la convergence deux échelles que nous utilisons, ces deux échelles étant dans notre cas 1 et $\frac{\eta}{\varepsilon} = \alpha$.

*Définition* 1.3.1. Convergence à deux échelles

*Une suite de mesures positives $\mu_{\varepsilon\eta}$ converge deux échelles vers $\mu$ si et seulement si, pour toute fonction test $\phi \in \mathcal{C}^0_{c,\#}([0,T], \mathbf{R}^d, \mathbf{R}^d, V)$, on a*

$$\int_0^\infty \int_{\mathbf{R}^d} \int_V \phi(t, \mathbf{x}, \frac{\eta \mathbf{x}}{\varepsilon}, \mathbf{v}) \mu_{\varepsilon\eta}(dt, d\mathbf{x}, d\mathbf{v})$$

$$\longrightarrow_{\varepsilon\eta \to 0} \int_0^\infty \int_{\mathbf{R}^d} \int_Y \int_V \phi(t, \mathbf{x}, \mathbf{y}, \mathbf{v}) \mu(dt, d\mathbf{x}, d\mathbf{y}, dt).$$

Nous montrons qu'à une sous suite près, $f^{\varepsilon,\eta}(t, \mathbf{x}, \mathbf{v})$ converge deux échelles vers $\mathcal{F}(t, \mathbf{x}, \mathbf{y}, \mathbf{v})$. Pour identifier cette limite, nous utilisons une série de fonctions test bien choisies. De plus, nous utilisons le développement asymptotique des fonctions de cellule dépendantes de $\eta$

$$\eta v \cdot \nabla_y F^\eta + Q(F^\eta) = 0 \quad \text{ou} \quad \eta v \cdot \nabla_y \mathbf{X}^{\eta *} + Q(\mathbf{X}^{\eta *}) = v.$$

Le point clé du processus d'homogénéisation vient du caractère singulier de $\mathbf{X}^{\eta *}$. De manière générale, une solution d'équation de cellule du type

$$\eta v \cdot \nabla_y u^\eta + Q(u^\eta) = g,$$

satisfait $||u||_{L^2(Y \times V)} \leq \frac{1}{\eta^2} ||g||_{L^2(Y \times V)}$. Mais dès lors que $\int g dv = 0$, on gagne une puissance de $\eta$. Une autre manière de retrouver ce résultat et, par exemple, de montrer que $F^\eta$ est bornée indépendamment de $\eta$, consiste à écrire le développement asymptotique de Hilbert. En effet, les parités imposées sur la fonction $\sigma$ permettent d'écrire le développement de $F^\eta$ en partant de l'ordre zéro en $\eta$. Ce n'est pas le cas pour $\mathbf{X}^{*\eta}$ et de manière générale pour $u^\eta$. Dans le cas général, il faut commencer le développement à l'ordre $-2$ en $\eta$ pour avoir suffisamment de degrés de liberté pour continuer le développement. Dans le cas de $\mathbf{X}^{*\eta}$, il suffit de le commencer à l'ordre $-1$. La première équation "fictive" ajoutée sera alors $Q^*(\mathbf{X}^{*-1}) = 0$. On obtient ainsi

$$\mathbf{X}^{*\eta}(\mathbf{y}, \mathbf{v}) = \frac{1}{\eta} \theta_{-1}(\mathbf{y}) + \mathbf{X}^*_{[0]}(\mathbf{y}, \mathbf{v}) \text{ avec } \mathbf{X}^*_{[0]}(\mathbf{y}, \mathbf{v}) = \mathbf{X}^*(\mathbf{y}, \mathbf{v}) - \mathbf{X}^*(\mathbf{y}, \mathbf{v}) \cdot \nabla_{\mathbf{y}} \theta_{-1}(\mathbf{y}). \quad (1.36)$$

La méthode des moments peut alors s'écrire de la manière suivante. Tout d'abord, pour toute fonction test $\phi(t, \mathbf{x}, \mathbf{y}, \mathbf{v})$ périodique en $\mathbf{y}$, on a

$$\frac{d}{dt} \int_V \int_{\mathbf{R}^d} f^{\varepsilon\eta} \phi d\mathbf{x} d\mathbf{v} = \int_V \int_{\mathbf{R}^d} f^{\varepsilon\eta} (\partial_t \phi + \frac{1}{\varepsilon} v \cdot \nabla_x \phi + \frac{\eta}{\varepsilon^2} v \cdot \nabla_y \phi + \frac{1}{\varepsilon^2} Q^*(\phi)) d\mathbf{x} d\mathbf{v},$$

et donc

$$\lim_{\varepsilon\eta \to 0} \int_0^T \int_{\mathbf{R}^d} \int_V f^{\varepsilon\eta}(t, \mathbf{x}, \mathbf{v}) Q^*(\phi(t, \mathbf{x}, \eta \frac{\mathbf{x}}{\varepsilon}, \mathbf{v})) d\mathbf{v} d\mathbf{x} dt = 0.$$

Ce qui implique $\forall\psi$ t.q. $\int_V \psi M d\mathbf{v} = 0$, $\int_0^\infty \int_{\mathbf{R}^d} \int_Y \int_V \psi \mathcal{F} d\mathbf{v} d\mathbf{y} d\mathbf{x} dt = 0$ et donc

$$\mathcal{F}(t,\mathbf{x},\mathbf{y},\mathbf{v}) = n(t,\mathbf{x},\mathbf{y})M(\mathbf{y},\mathbf{v}).$$

Puis, en utilisant la fonction test particulière

$$\phi(t,\mathbf{x},\eta\frac{\mathbf{x}}{\varepsilon},\mathbf{v}) = \psi(t,\mathbf{x},\eta\frac{\mathbf{x}}{\varepsilon}) + \eta z(t,\mathbf{x},\eta\frac{\mathbf{x}}{\varepsilon},\mathbf{v}) + \varepsilon w(t,\mathbf{x},\eta\frac{\mathbf{x}}{\varepsilon},\mathbf{v}),$$

avec $Q^*(z) = -\mathbf{v}\cdot\nabla_{\mathbf{y}}\psi$, et $Q^*(w) = -\mathbf{v}\cdot\nabla_{\mathbf{x}}\psi$, on obtient

$$\frac{d}{dt}\int_V\int_{\mathbf{R}^d} f^{\varepsilon\eta}\phi = \int_V\int_{\mathbf{R}^d} f^{\varepsilon\eta}(\partial_t\psi+\eta\partial_t z+\varepsilon\partial_t w) + \partial_t f^{\varepsilon\eta}(\psi+\eta z+\varepsilon w)$$

$$= \int_V\int_{\mathbf{R}^d} f^{\varepsilon\eta}(\partial_t\psi+\eta\partial_t z+\varepsilon\partial_t w+\frac{\eta}{\varepsilon}\mathbf{v}\cdot\nabla_{\mathbf{x}}z+\mathbf{v}\cdot\nabla_{\mathbf{x}}w+\frac{\eta^2}{\varepsilon^2}\mathbf{v}\cdot\nabla_{\mathbf{y}}z+\frac{\eta}{\varepsilon}\mathbf{v}\cdot\nabla_{\mathbf{y}}w).$$

En passant à la limite et en utilisant la relation $z = -\mathbf{X}^*\cdot\nabla_{\mathbf{y}}\psi$, on arrive à

$$\int_0^T \int_{\mathbf{R}^d} \int_Y \int_V n(t,\mathbf{x},\mathbf{y})M(\mathbf{y},\mathbf{v})\mathbf{v}\cdot\nabla_{\mathbf{y}}(-\mathbf{X}^*(\mathbf{y},\mathbf{v})\cdot\nabla_{\mathbf{y}}\psi(t,\mathbf{x},\mathbf{y}))dtd\mathbf{x}d\mathbf{y}d\mathbf{v} = 0,$$

ce qui donne après intégration par partie

$$\int \mathrm{div}_{\mathbf{y}}[-\mathbf{v}\cdot\nabla_{\mathbf{y}}n(t,\mathbf{x},\mathbf{y})M(\mathbf{y},\mathbf{v})\mathbf{X}^*(\mathbf{y},\mathbf{v}) - n(t,\mathbf{x},\mathbf{y})\mathbf{v}\cdot\nabla_{\mathbf{y}}M(\mathbf{y},\mathbf{v})\mathbf{X}^*(\mathbf{y},\mathbf{v})]d\mathbf{v} = 0,$$

ce qui n'est autre que $L(n) = 0$ ce qui implique $n(t,\mathbf{x},\mathbf{y}) = \rho(\mathbf{y})\tilde{n}(t,\mathbf{x})$.

Pour obtenir l'équation de diffusion pour $\tilde{n}$, il faut choisir la fonction test

$$\phi(t,\mathbf{x},\eta\frac{\mathbf{x}}{\varepsilon},\mathbf{v}) = \varphi(t,\mathbf{x}) + \varepsilon\psi^\eta(t,\mathbf{x},\eta\frac{\mathbf{x}}{\varepsilon},\mathbf{v})$$

avec $\eta\mathbf{v}\cdot\nabla_{\mathbf{y}}\psi^\eta + Q^*(\psi^\eta) = -\mathbf{v}\cdot\nabla_{\mathbf{x}}\varphi$ i.e. $\psi^\eta = -\mathbf{X}^{\eta*}\cdot\nabla_{\mathbf{x}}\phi$

et remplacer $\mathbf{X}^{*\eta}$ par son développement en $\eta$ (1.36). On obtient

$$\begin{aligned}
&-\int_{\mathbf{R}^d} n_{ini}(\mathbf{x})\varphi(0,\mathbf{x})d\mathbf{x} \\
&\quad= \int_{\mathbf{R}^+}\int_{\mathbf{R}^d}\int_Y\int_V \rho(\mathbf{y})M(\mathbf{y},\mathbf{v})[\partial_t\varphi(t,\mathbf{x})+\mathbf{v}\cdot\nabla_{\mathbf{x}}(\mathbf{X}^*_{[0]}(\mathbf{y},\mathbf{v})\cdot\nabla_{\mathbf{x}}\varphi(t,\mathbf{x}))]d\mathbf{y}d\nu(\mathbf{v})\,n(dt,d\mathbf{x}) \\
&\qquad+ \lim_{\varepsilon\eta\to\infty}\frac{1}{\eta}\int_{\mathbf{R}^+}\int_V\int_{\mathbf{R}^d} f^{\varepsilon\eta}\mathbf{v}\cdot\nabla_{\mathbf{x}}(\mathbf{\Theta}^{[-1]}(\mathbf{y})\cdot\nabla_{\mathbf{x}}\varphi)|_{\mathbf{y}=\frac{\eta\mathbf{x}}{\varepsilon}}d\mathbf{x}d\nu(\mathbf{v})dt\ .
\end{aligned} \tag{1.37}$$

Le terme qui pourrait poser problème est celui en $\frac{1}{\eta}$ mais celui-ci est traité en introduisant $\Psi(\mathbf{x},\mathbf{y},\mathbf{v})$ défini par $Q^*[\mathbf{y}](\Psi) = -\mathbf{v}\cdot\nabla_{\mathbf{x}}(\theta^{-1}(\mathbf{y})\cdot\nabla_{\mathbf{x}}\varphi)$. On a alors

$$\frac{1}{\eta}\int_{\mathbf{R}^+}\int_V\int_{\mathbf{R}^d} f^{\varepsilon\eta}\mathbf{v}\cdot\nabla_{\mathbf{x}}(\theta_{-1}(\mathbf{y})\cdot\nabla_{\mathbf{x}}\varphi)|_{\mathbf{y}=\eta\frac{\mathbf{x}}{\varepsilon}}d\mathbf{v}d\mathbf{x}dt$$

$$= -\frac{1}{\eta}\int_{\mathbf{R}^+}\int_V\int_{\mathbf{R}^d} f^{\varepsilon\eta}Q^*(\Psi)d\mathbf{v}d\mathbf{x}dt$$

$$= \frac{\varepsilon^2}{\eta}\int f_{ini}(\mathbf{x},\mathbf{v})\Psi(0,\mathbf{x},\mathbf{y},\mathbf{v})d\mathbf{x}d\mathbf{y}d\mathbf{v} + \int_{\mathbf{R}^+}\int_V\int_{\mathbf{R}^d} f^{\varepsilon\eta}(\frac{\varepsilon^2}{\eta}\partial_t\Psi + \frac{\varepsilon}{\eta}\mathbf{v}\cdot\nabla_{\mathbf{x}}\Psi+\mathbf{v}\cdot\nabla_{\mathbf{y}}\Psi)d\mathbf{v}d\mathbf{x}dt.$$

Ce qui donne le résultat en passant à la limite sans oublier que $\frac{\varepsilon}{\eta}$ tend vers zéro.

### 1.3.3 Convergence forte et calcul de correcteurs par une méthode de Hilbert

Dans [NBABP], écrit en collaboration avec G. Bal, au lieu de considérer une équation d'évolution, on regarde l'équation

$$\mathbf{v}\cdot\nabla f^{\varepsilon\eta}+\varepsilon f^{\varepsilon\eta}+\frac{1}{\varepsilon}Q_\alpha f^{\varepsilon\eta}=\varepsilon S_\alpha(\mathbf{x},\mathbf{v}):=\varepsilon S(\mathbf{x},\frac{\mathbf{x}}{\alpha},\mathbf{v}). \tag{1.38}$$

Cette équation se ramène à une équation d'évolution par le changement d'inconnue $u^{\varepsilon,\alpha}=e^t f^{\varepsilon,\alpha}$ et le terme source joue le rôle des conditions initiales. On se place dans un cadre un peu plus général que dans le paragraphe précédant. Notamment, on va supposer que le noyau de $Q^*$ est engendré par $\psi^*$ une fonction ne dépendant que de $\mathbf{v}$ alors que précédemment, on avait $\psi^*=1$ (Remarquons que nous changeons la notation désignant les équilibres pour insister sur le fait qu'on ne suppose plus la conservation de la masse). Pour cela, on remplace l'hypothèse **H2** par

- **(H2')** $\sigma$ satisfait l'équilibre détaillé

  $\sigma(\mathbf{y},\mathbf{v},\mathbf{v}')=\tilde\sigma(\mathbf{y},\mathbf{v},\mathbf{v}')\psi(\mathbf{y},\mathbf{v})\psi^*(\mathbf{y},\mathbf{v}')$ avec $\psi>0,\ \psi^*>0$ et $\tilde\sigma(\mathbf{y},\mathbf{v},\mathbf{v}')=\tilde\sigma(\mathbf{y},\mathbf{v}',\mathbf{v})$ .

- On suppose que

  $$Q(f)=\int\sigma(y,\mathbf{v},\mathbf{v}')f(\mathbf{v}')d\nu(\mathbf{v}')-\nu(\mathbf{v})f(\mathbf{v}),$$

  avec

  $$\nu(\mathbf{y},\mathbf{v})=\int_V\tilde\sigma(\mathbf{y},\mathbf{v},\mathbf{v}')\psi(\mathbf{v}')d\nu(\mathbf{v}').$$

*Remarque* 1.3.1. L'hypothèse (**H2'**) n'est pas nécessaire mais elle permet une construction simple des fonctions propres $\psi$ et $\psi^*$ dont l'existence est par ailleurs assurée par le théorème de Krein Rutman.

En utilisant la méthode des développements de Hilbert, on montre un résultat de convergence forte qui permet, de plus, de calculer le premier terme correctif. Notons qu'il diffère selon le régime asymptotique valide, déterminé par les échelles respectives des paramètres $\alpha$ et $\eta$.

Nous montrons le théorème suivant.

*Théorème* 1.3.3 ([NBABP]). Soit $f^{\varepsilon,\eta}$ une solution de (1.38) ayant pour condition initiale $f_{ini}(\mathbf{x},\mathbf{v})$, on montre le développement

$$\begin{aligned}&||f^{\varepsilon,\eta}-n^{0,0}(\mathbf{x})\rho^0(\tfrac{\eta\mathbf{x}}{\varepsilon})\psi(\tfrac{\eta\mathbf{x}}{\varepsilon},\mathbf{v})\\&-\eta[n^{0,0}(\mathbf{x})Q^{-1}(-v\cdot\nabla_{\mathbf{y}}(\rho^0(\tfrac{\eta\mathbf{x}}{\varepsilon})\psi(\tfrac{\eta\mathbf{x}}{\varepsilon},\mathbf{v})))+n^{0,1}(\mathbf{x})\rho^0(\tfrac{\eta\mathbf{x}}{\varepsilon})\psi(\tfrac{\eta\mathbf{x}}{\varepsilon},\mathbf{v})]\\&-\tfrac{\varepsilon}{\eta}(\theta^{-1}(\mathbf{y}))\psi(\tfrac{\eta\mathbf{x}}{\varepsilon},\mathbf{v})\cdot\nabla_{\mathbf{x}}n^{0,0}+n^{1,-1}(\mathbf{x})\rho^0(\tfrac{\eta\mathbf{x}}{\varepsilon})\psi(\tfrac{\eta\mathbf{x}}{\varepsilon},\mathbf{v})||=o(\eta+\tfrac{\varepsilon}{\eta})\end{aligned} \tag{1.39}$$

où le profil $\psi$ est la maxwellienne satisfaisant $Q(\psi)=0,\ \int\psi d\nu(\mathbf{v})=1$ pour tout $\mathbf{y}\in Y$. Dans ce développement :

- la densité microscopique $\rho^0(\mathbf{y})$ est donnée par l'équation elliptique

  $$L(\rho^0)=0\quad\text{normalisée par}\quad\int_Y\rho^0(\mathbf{y})d\mathbf{y}=1$$

  avec

  $$L(\rho)=-\int_V\psi^*(\mathbf{v})\mathbf{v}\cdot\nabla_{\mathbf{y}}(Q^{-1}(\mathbf{v}\cdot\nabla_{\mathbf{y}}(\psi(\mathbf{y},\mathbf{v})\rho(\mathbf{y}))))d\nu(\mathbf{v}),$$

– la fonction $\theta^{-1}$ est donnée par

$$\theta^{-1} = L^{-1}\left(\int_V (\mathbf{v}Q^{-1}(-\mathbf{v}\cdot\nabla_{\mathbf{y}}\rho^0(\mathbf{y})\psi(\mathbf{y},\mathbf{v})) - \mathbf{v}\cdot\nabla_{\mathbf{y}}(\mathbf{X}(\mathbf{y},\mathbf{v})\rho^0(\mathbf{y})))\psi^*(\mathbf{v})d\nu(\mathbf{v})\right),$$

– et la densité macroscopique est déterminée par l'équation de diffusion

$$n^{0,0}(\mathbf{x}) - \nabla_{\mathbf{x}}\cdot(\mathbf{D}\cdot\nabla_{\mathbf{x}}n^{0,0}(\mathbf{x})) = \int_Y\int_V S(\mathbf{x},\mathbf{y},\mathbf{v})\psi^*(\mathbf{v})d\mathbf{y}d\nu(\mathbf{v}),$$

où le coefficient de diffusion homogénéisé est donné par

$$\mathbf{D} = \int_V\int_Y (\overline{\mathbf{X}}^{*0}(\mathbf{y},\mathbf{v})\otimes\mathbf{v}\rho^0(\mathbf{y})\psi(\mathbf{y},\mathbf{v}) + \theta^{*-1}(\mathbf{y})\psi^*(\mathbf{v})\otimes\mathbf{v}Q^{-1}(-v\cdot\nabla_{\mathbf{y}}(\rho^0(\mathbf{y})\psi(\mathbf{y},\mathbf{v})))d\nu(\mathbf{v})d\mathbf{y},$$

où

$$\overline{\mathbf{X}}^{*0} = Q^{*-1}(\mathbf{v}\psi^* + \mathbf{v}\cdot\nabla_{\mathbf{y}}\mathbf{X}^{*-1})$$

et

$$L^*(\theta^{*-1}) = -\int_V \psi(\mathbf{y},\mathbf{v})\mathbf{v}\cdot\nabla_{\mathbf{y}}Q^{*-1}(\mathbf{v}\psi^*(\mathbf{v}))d\nu(\mathbf{v}).$$

Les correcteurs $n^{0,1}$ et $n^{1,-1}$ sont définis par

$$n^{0,1} - \nabla_{\mathbf{x}}\cdot(\mathbf{D}\cdot\nabla_{\mathbf{x}}n^{0,1}) = \nabla_{\mathbf{x}}\cdot(\mathbf{D^1}\cdot\nabla_{\mathbf{x}}n),$$

avec

$$\begin{aligned}
\mathbf{D} &= \int_V\int_Y (\chi^{*0}(y,v)\otimes v\psi^0(y,v) + \chi^{*-1}(y,v)\otimes v\psi^1(y,v))d\nu(v)dy \\
&= \int_V\int_Y Q^{*-1}(v\psi^* + v\cdot\nabla_y\theta^{*-1}\psi^*)(y,v)\otimes v\rho^0(y)\psi(y,v) \\
&\qquad + \theta^{*-1}(y)\psi^*(v)\otimes vQ^{-1}(-v\cdot\nabla_y(\rho^0(y)\psi(y,v)))d\nu(v)dy
\end{aligned},$$

$$\begin{aligned}
\mathbf{D^1} &= \int_V\int_Y (\chi^{*-1}(y,v)\otimes v\psi^2(y,v) + \chi^{*0}(y,v)\otimes v\psi^1(y,v) + \chi^{*1}(y,v)\otimes v\psi^0(y,v))d\nu(v)dy \\
&= \int_V\int_Y (\theta^{*-1}(y)\psi^*(v)\otimes vQ^{-1}(-v\cdot\nabla_y(Q^{-1}(-v\cdot\nabla_y(\rho^0(y)\psi(y,v)))))d\nu(v)dy \\
&\quad \int_V\int_Y [Q^{*-1}(v\psi^* + v\cdot\nabla_y\theta^{*-1}\psi^*) + \theta^{*0}\psi^*]\otimes vQ^{-1}(-v\cdot\nabla_y(\rho^0(y)\psi(y,v))d\nu(v)dy \\
&\quad \int_V\int_Y Q^{*-1}(v\cdot\nabla_y Q^{*-1}(v\psi^* + v\cdot\nabla_y\theta^{*-1}\psi^*) + \theta^{*1}(y)\psi^*(y,v))\otimes v\rho^0(y)\psi(y,v)d\nu(v)dy.
\end{aligned}$$

et

$$n^{1,-1} - \nabla_{\mathbf{x}}\cdot(\mathbf{D}\cdot\nabla_{\mathbf{x}}n^{1,-1}) = S^{1,-1}(\mathbf{x}),$$

avec

$$\begin{aligned}
S^{1,-1}(x) &= -\nabla_x\cdot\int_Y\int_V [-n^{0,0}(x)\psi(y,v) + S(x,y,v) - v\cdot\nabla_x(-\chi^0(y,v)\cdot\nabla_x n^{0,0}(x))] \\
&\qquad \chi^{*-1}(y,v)d\nu(v)dy \\
&\quad -\nabla_x\cdot\int_Y\int_V [-v\cdot\nabla_x(-\chi^{-1}(y,v)\cdot\nabla_x n^{0,0}(x))]\chi^{*0}(y,v)d\nu(v)dy.
\end{aligned}$$

**Commentaires**

• Ce résultat implique que si $\eta << \frac{\varepsilon}{\eta}$ (collisions relativement faibles), le premier terme correcteur est un correcteur de type homogénéisation, donné par

$$\theta^{-1}\psi(\frac{\eta\mathbf{x}}{\varepsilon},\mathbf{v})\cdot\nabla_{\mathbf{x}}n(\mathbf{x})+n^{1,-1}(\mathbf{x})\rho(\frac{\eta\mathbf{x}}{\varepsilon})\psi(\frac{\eta\mathbf{x}}{\varepsilon},\mathbf{v}),$$

alors que si $\frac{\varepsilon}{\eta} << \eta$ (collisions relativement fortes), il est donné par un terme collisionel

$$n(\mathbf{x})Q^{-1}(\nabla_{\mathbf{y}}(\rho(\frac{\eta\mathbf{x}}{\varepsilon})\psi(\frac{\eta\mathbf{x}}{\varepsilon},\mathbf{v})))+n^{0,1}(\mathbf{x})\rho(\frac{\eta\mathbf{x}}{\varepsilon})\psi(\frac{\eta\mathbf{x}}{\varepsilon},\mathbf{v}).$$

• De plus, ce résultat est une première étape dans l'étude de l'équation non linéaire de Fermi-Dirac dans l'esprit des articles de N. Ben Abdallah, H. Chaker et C. Schmeiser [51, 52].

### 1.3.4 La méthode de Hilbert

Le principe de ce travail repose sur les développements de Hilbert. En effet, on écrit

$$f^{\varepsilon,\eta}=n^{0,\eta}(\mathbf{x})\psi^{\eta}(\frac{\eta\mathbf{x}}{\varepsilon},v)+\varepsilon f^{1,\eta}(\mathbf{x},\frac{\eta\mathbf{x}}{\varepsilon},v)+\varepsilon^2 f^{2,\eta}(\mathbf{x},\frac{\eta\mathbf{x}}{\varepsilon},v)+\varepsilon^3 f^{3,\eta}(\mathbf{x},\frac{\eta\mathbf{x}}{\varepsilon},v)+r^{\varepsilon,\eta}(\mathbf{x},v), \tag{1.40}$$

avec

$$\begin{aligned}
\eta\mathbf{v}\cdot\nabla_{\mathbf{y}}f^{0,\eta}+Q(f^{0,\eta}) &= 0,\\
\eta\mathbf{v}\cdot\nabla_{\mathbf{y}}f^{1,\eta}+Q(f^{1,\eta}) &= -\mathbf{v}\cdot\nabla_{\mathbf{x}}f^{0,\eta}(\mathbf{x},\mathbf{y},\mathbf{v})=-\psi^{\eta}(\mathbf{y},\mathbf{v})v\cdot\nabla_{\mathbf{x}}n^{0,\eta}(\mathbf{x}),\\
\eta\mathbf{v}\cdot\nabla_{\mathbf{y}}f^{2,\eta}+Q(f^{2,\eta}) &= -f^{0,\eta}(\mathbf{x},\mathbf{y},\mathbf{v})+S(\mathbf{x},\mathbf{y},\mathbf{v})-\mathbf{v}\cdot\nabla_{\mathbf{x}}f^{1,\eta}(\mathbf{x},\mathbf{y},\mathbf{v}),\\
\eta\mathbf{v}\cdot\nabla_{\mathbf{y}}f^{3,\eta}+Q(f^{3,\eta}) &= -\mathbf{v}\cdot\nabla_{\mathbf{x}}f^{2,\eta}(\mathbf{x},\mathbf{y},\mathbf{v})-f^{1,\eta}(\mathbf{x},\mathbf{y},\mathbf{v}),
\end{aligned}$$

où $\psi^{\eta}$ est solution de $\eta\mathbf{v}\cdot\nabla_{\mathbf{y}}\psi^{\eta}+Q(\psi^{\eta})=0$.

A priori, le terme de reste $r^{\varepsilon,\eta}$ satisfait

$$||r^{\varepsilon,\eta}||_{L^2_{\mathbf{xv}}}\leq\varepsilon^2||f^{3,\eta}||_{H^2_{\mathbf{x}}L^{\infty}_yL^2_v}+\varepsilon^2||f^{2,\eta}||_{L^2_{\mathbf{x}}L^{\infty}_yL^2_v}. \tag{1.41}$$

On estime les différents termes en calculant leurs développements en $\eta$ qui nous montrent que

$$f^{1,\eta}\sim\frac{1}{\eta}\quad f^{2,\eta}\sim\frac{1}{\eta^2}\quad f^{3,\eta}\sim\frac{1}{\eta^3},$$

ce qui a priori ne permet pas de conclure. On pourrait pousser le développement pour augmenter la puissance mais quoi qu'il en soit, on aurait besoin de l'existence d'un entier $n$ tel que $\dfrac{\varepsilon^n}{\eta^{n+1}}\to 0$.

Pour éviter cette hypothèse, on montre l'estimation fondamentale sur $f^{\varepsilon\eta}$ solution de (1.38)

$$\|f^{\varepsilon\eta}\|+\frac{1}{\varepsilon}\|f^{\varepsilon\eta}-\psi^{\eta}\int_V\frac{f^{\varepsilon\eta}}{\psi^{\eta}}\|\leq C\Big(\|\varepsilon S_{\alpha}\|+\|\int_V(S_{\alpha}\psi^*)\|\Big). \tag{1.42}$$

Ceci nous permet d'estimer le terme de reste dans le développement de Hilbert car on obtient

$$\begin{aligned}
||r^{\varepsilon,\eta}||_{L^2_{\mathbf{xv}}} \leq\ & \varepsilon^2||\textstyle\int_V v\cdot\nabla_{\mathbf{x}}f^{3,\eta}\psi^*dv||_{L^2_{\mathbf{x}}L^{\infty}_yL^2_v}+\varepsilon^3||v\cdot\nabla_{\mathbf{x}}f^{3,\eta}||_{L^2_{\mathbf{x}}L^{\infty}_yL^2_v}\\
&+\varepsilon^2||f^{2,\eta}||_{L^2_{\mathbf{x}}L^{\infty}_yL^2_v}+\varepsilon^3||f^{3,\eta}||_{L^2_{\mathbf{x}}L^{\infty}_yL^2_{\mathbf{v}}}.
\end{aligned} \tag{1.43}$$

Le terme d'ordre supérieur de $f^{3,\eta}$ vient d'une équation "fictive" $Q(f^{3,-3}) = 0$ (équation rajoutée pour gagner un degré de liberté comme dans le développement de $\mathbf{X}^{*\eta}$) et donc, par parité, ce terme est d'intégrale nulle contre $v\psi^*$ et ainsi, on gagne la puissance de $\varepsilon$ qui permet de conclure que le terme de reste tend vers zéro.

L'équation de compatibilité correspondant à l'équation déterminant $f^{2,\eta}$ nous donne l'équation de diffusion dépendant de $\eta$ satisfaite par la densité $n^{0,\eta}$. Il reste alors à utiliser les développements en $\eta$ pour passer à la limite dans cette dernière équation.

## 1.4 Diffusion anormale pour l'équation de Boltzmann

### 1.4.1 Mise à l'échelle d'un problème de diffusion anormale

Dans le cas de l'approximation de diffusion que nous appellerons "classique", on approche la solution de l'équation de Boltzmann par un équilibre local associé à une densité satisfaisant une équation de diffusion. Le coefficient de diffusion impliqué dans cette équation est de la forme

$$D = \int_V \frac{v \otimes v}{\nu(v)} M(v) dv \quad \text{avec } \nu(x,v) = \int_{\mathbf{R}^d} \sigma(x,v,v') M(v') dv'.$$

Ainsi, dans certains cas, ce coefficient peut être infini.

En effet, lorsque l'ensemble des vitesses n'est plus borné, si l'équilibre $M$ n'est plus du type Gaussien mais est une fonction à décroissance de type puissance, comme

$$M(v) \sim |v|^{-d-\alpha} \quad \text{lorsque } |v| \to \infty, \qquad \text{pour } \alpha \in (0,2), \tag{1.44}$$

le second moment de $M$ n'est plus borné et on ne peut plus définir le coefficient de diffusion correspondant au scaling classique de l'approximation diffusion. De même, $D$ n'est plus défini lorsque la section efficace est dégénérée lorsque les vitesses tendent vers zéro, par exemple si

$$\nu(v) \sim \nu_0 |v|^{d+2+\beta}, \qquad \text{lorsque } |v| \to 0 \quad \text{pour } \beta > 0. \tag{1.45}$$

Nous interprétons ceci comme le fait que nous considérons une échelle de temps trop grande. Ainsi, nous procédons à une nouvelle mise à l'échelle en temps de l'équation de Boltzmann $t \sim \varepsilon^{-\delta}$ avec $\delta$ choisi en fonction de $\alpha$ ou $\beta$. Nous obtenons alors une équation de type équation de diffusion impliquant un opérateur elliptique non local de type opérateur de diffusion fractionnaire.

### 1.4.2 Cas des équilibres à décroissance de type puissance

Dans un premier article en collaboration avec N. Ben Abdallah et A. Mellet [NBAMP1], nous considérons le cas où les profils d'équilibre sont des fonctions dont le second moment en vitesse est infini pour lesquelles on ne peut définir le coefficient de diffusion classique. Ce travail vient après [92] dans lequel le cas d'une section efficace indépendante de la variable d'espace est traité par une méthode basée sur la réécriture du problème en variable de Fourier et après [91] dans lequel le cas de sections efficaces plus générales est traité par la méthode des moments. Dans le travail présenté ici, nous cherchons à obtenir un résultat de convergence forte pour la norme à poids $L^2_{F^{-1}}$ par une méthode de type Hilbert. Cette méthode requière bien sûr une hypothèse de plus grande régularité sur les conditions initiales. Nous allons voir qu'une réorganisation du développement est nécessaire et pour cela, nous nous inspirons des résultats de [91] pour ordonner les différents termes.

Plus précisément, nous considérons l'équation de Boltzmann mise à l'échelle

$$\varepsilon^\alpha \partial_t f^\varepsilon + \varepsilon v \cdot \nabla_x f^\varepsilon = Q(f^\varepsilon), \tag{1.46}$$

où $\alpha$ correspond au paramètre introduit dans (1.44) et où l'opérateur de collision peut être décomposé en un terme de gain et un terme de perte

$$Q(f) = Q^+(f) - Q^-(f),$$

où

$$Q^+(f) = \int_{\mathbf{R}^d} \sigma(x, v, v') f(v') M(v) \text{ et } Q^-(f) = \nu(x, v) f(v).$$

Le premier résultat de diffusion anormale dans le cas de sections efficaces inhomogènes [91] a été obtenu par la méthode des moments en choisissant une fonction test particulière. En effet, la formulation faible de Boltzmann s'écrit

$$-\int_0^\infty \int_{X\times V} f^\varepsilon \partial_t \chi^\varepsilon dxdvdt - \int_{X\times V} f_0 \chi^\varepsilon(0, x, v) dxdv$$

$$= \varepsilon^{-\alpha} \int_0^\infty \int_{X\times V} (Q^+(f^\varepsilon)\chi^\varepsilon - \nu f^\varepsilon \chi^\varepsilon + f^\varepsilon v \cdot \nabla_x \chi^\varepsilon) dxdvdt.$$

Comme dans [NBAPV], le choix de la fonction test est fondamental. Ici, le choix est inspiré par les calculs de [92] dans le cas où la section efficace ne dépend pas de la variable d'espace. On définit

$$\chi^\varepsilon = (\nu - \varepsilon v \cdot \nabla_x)^{-1}(\nu\varphi),$$

que l'on peut réécrire

$$\chi^\varepsilon = \int_0^\infty e^{-\int_0^z \nu(x+\varepsilon v s, v) ds} \nu(x + \varepsilon v z, v) \varphi(x + \varepsilon v z) dz.$$

En écrivant $f^\varepsilon = n^\varepsilon M + g^\varepsilon$, on obtient

$$-\int_0^\infty \int_{X\times V} f^\varepsilon \partial_t \chi^\varepsilon dxdvdt - \int_{X\times V} f_0 \chi^\varepsilon(0, x, v) dxdv$$

$$= \varepsilon^{-\alpha} \int_0^\infty \int_{X\times V} n^\varepsilon \nu M[\chi^\varepsilon - \varphi] dxdvdt$$

$$+\varepsilon^{-\alpha} \int_0^\infty \int_{X\times V} Q^+(g^\varepsilon)[\chi^\varepsilon - \varphi] dxdvdt.$$

On montre que le dernier terme tend vers zéro et comme $\chi^\varepsilon \to \varphi$, on a, en passant à la limite,

$$-\int_0^\infty \int_X \rho \partial_t \varphi dxdt - \int_X n_0 \varphi(0, x) dxdv$$

$$= \lim_{\varepsilon\to 0} \varepsilon^{-\alpha} \int_0^\infty \int_{X\times V} n^\varepsilon \nu M[\chi^\varepsilon - \varphi] dxdvdt$$

$$= \lim_{\varepsilon\to 0} \varepsilon^{-\alpha} \int_{\mathbf{R}\times\mathbf{R}^d\times\mathbf{R}^d} n^\varepsilon \nu(x, v) M \int_0^\infty e^{-\int_0^z \nu(x+\varepsilon v s, v) ds} \nu(x + \varepsilon v z, v)[\varphi(x + \varepsilon v z) - \varphi(x, v)] dxdzdtdv$$

On obtient en utilisant $M \sim |v|^{-d-\alpha}$ après un changement de variables,

$$-\int_0^\infty \int_{X\times V} f^\varepsilon \partial_t \chi^\varepsilon dxdvdt - \int_{X\times V} f_0 \chi^\varepsilon(0,x,v)dxdv$$

$$= \mathrm{VP} \int_{\mathbf{R}\times\mathbf{R}^d\times\mathbf{R}^d} n(t,x)\nu_0(x)\nu_0(x+w) z^\alpha e^{-z\int_0^1 \nu_0(x+sw)ds} \frac{\varphi(t,x)-\varphi(t,y)}{|x-y|^{d+\alpha}} dydxdt, \tag{1.47}$$

qui est une équation elliptique non locale pour $n = \lim_{\varepsilon\to 0} n^\varepsilon$ (VP désigne la valeur principale). Dans le cas le plus simple, où $\sigma$ ne dépend pas de $x$, c'est une équation de diffusion fractionnaire que l'on peut écrire dans les variables de Fourier (cf [92])

$$\partial_t \hat{n} + \kappa|\xi|^{\frac{\alpha}{2}}\hat{n} = 0.$$

Concernant la méthode de Hilbert, a priori, le développement classique s'écrirait

$$f^\varepsilon = f^0 + \varepsilon f^1 + \varepsilon^2 f^2 + r^\varepsilon,$$

dont les différents termes satisfont la série d'équations

$$\begin{aligned} Q(f^0) &= 0 \quad \Rightarrow f^0 = n(t,x)M(v), \\ Q(f^1) &= v\cdot\nabla_x f^0, \\ Q(f^2) &= v\cdot\nabla_x f^1 + \partial_t f^0. \end{aligned}$$

La parité de $M$ assure l'existence d'une solution pour $f^1$ et on a alors

$$f^1 = -\chi\cdot\nabla_x n \quad \text{avec} \quad Q(\chi) = -vM.$$

C'est l'équation de compatibilité de l'équation qui détermine $f^2$ et qui donne l'équation de diffusion satisfaite par $n$

$$\partial_t n - \nabla_x(D\nabla_x n) = 0 \quad \text{avec} \quad D = \int v\otimes\chi dv = \int \chi^*\otimes vMdv. \tag{1.48}$$

Mais nous avons vu que ce coefficient n'est pas défini si $M \sim |v|^{-d-\alpha}$. Or, dans la méthode des moments, il apparaît que $Q^+$ est d'ordre inférieur à $Q^-$, et donc, nous proposons une réorganisation du développement de la manière suivante

$$f^\varepsilon = f^0 + \varepsilon f^1 + \varepsilon^2 f^2 + \varepsilon^3 f^3 + r^\varepsilon,$$

avec

$$\begin{aligned} Q(f^0) &= 0 \quad \Rightarrow f^0 = n(t,x)M(v), \\ (\nu+\varepsilon v\cdot\nabla_x)(f^1) &= -\varepsilon v\cdot\nabla_x f^0, \\ Q(f^2) &= -Q^+(f^1) + \varepsilon^\alpha \partial_t f^0, \\ (\nu+\varepsilon v\cdot\nabla_x)(f^3) &= -\varepsilon v\cdot\nabla_x f^2. \end{aligned}$$

Ainsi, l'équation de compatibilité de l'équation déterminant $f^2$ donne

$$\partial_t n - \varepsilon^{-\alpha}\int_V Q^+((\nu+\varepsilon v\cdot\nabla_x)^{-1}(\varepsilon vM\cdot\nabla_x(n)) = 0.$$

Notons que l'on retrouve le résultat de [91] car, par analogie avec l'approximation diffusion classique, on pourrait décomposer la fonction test utilisée par A. Mellet de la manière suivante

$$\chi^\varepsilon = \varphi + \psi^\varepsilon,$$

avec

$$(\nu - \varepsilon v \cdot \nabla_x)\psi^\varepsilon = -\varepsilon v \cdot \nabla\varphi.$$

Ainsi, l'opérateur obtenu par A. Mellet peut s'écrire

$$\int_0^\infty\!\!\int_{X\times V} n^\varepsilon \nu M[\chi^\varepsilon - \varphi]dydxdv = \int_0^\infty\!\!\int_{X\times V} n^\varepsilon \nu M[\psi^\varepsilon]dydxdv$$

$$= \int_0^\infty\!\!\int_{X\times V} Mn^\varepsilon Q^{+*}[(\nu - \varepsilon v\cdot\nabla_x)^{-1}(-\varepsilon v\cdot\nabla_x\varphi)],$$

et donc (1.47) est bien la formulation faible de (1.48).

Avant de préciser le théorème obtenu, nous rappelons les hypothèses concernant ce travail.

On suppose qu'il existe des constantes $\sigma_0$ et $\sigma_1$ telles que

$$0 < \sigma_0 \le \sigma(x,v,v') = \sigma(x,v',v) \le \sigma_1 < \infty \quad \text{pour tout } (x,\, v,\, v') \in \mathbf{R}^{3n}. \tag{1.49}$$

En particulier, ceci implique que la fréquence de collision

$$\nu(x,v) = \int_{\mathbf{R}^n} \sigma(x,v,v')F(v')dv'$$

satisfait

$$\sigma_0 \le \nu(x,v) \le \sigma_1 \quad \text{pour tout } (x,v) \in \mathbf{R}^{2n}.$$

On a aussi besoin de supposer que la fréquence de collision est suffisamment régulière et qu'elle a un bon comportement asymptotique pour les grandes vitesses. Pour cela, on suppose :

$$\begin{aligned} &\partial_x^k \nu \in L^\infty(\mathbf{R}^n \times \mathbf{R}^n) \text{ pour tout multi-indice } k \text{ tel que } |k| \le 4 \\ &\nu(x,v) = \nu_0(x) \quad \forall |v| > 1. \end{aligned} \tag{1.50}$$

L'hypothèse la plus importante concerne le comportement pour les grandes vitesses de la fonction d'équilibre $F(v)$. On suppose :

$$F(v) = \frac{\kappa}{|v|^{n+\alpha}} \qquad \text{pour } |v| \ge 1, \tag{1.51}$$

pour $\alpha \in (0,2)$ et

$$F(v) \le \kappa \qquad \text{pour tout } v \in \mathbf{R}^n. \tag{1.52}$$

Enfin, on suppose que l'on a les symétries suivantes :

$$F(-v) = F(v), \qquad \nu(x,-v) = \nu(x,v) \quad \text{pour tout } (x\, v) \in \mathbf{R}^{2n}. \tag{1.53}$$

Ceci implique en particulier que le flux $\int_{\mathbf{R}^n} vF(v)\,dv = 0$ lorsque $\alpha > 1$, et nous savons que cette condition est nécessaire de manière générale dans l'étude des limites de diffusion.

On peut alors énoncer notre résultat principal :

*Théorème* 1.4.1 ([NBAMP1]). Supposons (1.49), (1.50), (1.51), (1.52), (1.53), soit $f^\varepsilon$ la solution de (1.46) avec pour condition initiale $f_{in}(x,v) = \rho_{in}(x)F(v)$ telle que $\rho_{in} \in H^4(\mathbf{R}^n)$, alors

$$||f^\varepsilon - \rho_0 F||_{L^\infty(0,\infty;L^2_{F^{-1}}(\mathbf{R}^n\times\mathbf{R}^n))} \longrightarrow 0, \qquad \text{lorsque } \varepsilon \to 0, \tag{1.54}$$

où $\rho_0$ satisfait

$$\begin{cases} \partial_t \rho_0 + L(\rho_0) = 0, & \text{dans } [0,\infty)\times\mathbf{R}^n, \\ \rho_0(0,x) = \rho_{in}(x), & \text{dans } \mathbf{R}^n \end{cases} \tag{1.55}$$

et $L$ est un opérateur elliptique défini par l'intégrale singulière :

$$L(\rho) = \text{P.V.} \int_{\mathbf{R}^n} \gamma(x,y)\frac{\rho(x)-\rho(y)}{|y-x|^{n+\alpha}}\,dy. \tag{1.56}$$

avec

$$\gamma(x,y) = \nu_0(x)\,\nu_0(y)\int_0^\infty z^\alpha e^{-z\int_0^1 \nu_0((1-s)x+sy)\,ds}\,dz.$$

### 1.4.3 Cas de sections efficaces dégénérées

Dans [NBAMP2], nous considérons typiquement que les Maxwelliennes peuvent s'écrire $M(v) = \frac{1}{(2\pi T)^{N/2}}e^{-\frac{v^2}{2T}}$, (pour une température donnée $T > 0$). La dégénérescence que nous étudions vient cette fois de la forme de la section efficace. En effet, nous considérons des sections efficaces $\sigma(v,v')$ satisfaisant

$$\sigma(v,v') = \sigma(v',v) \geq 0 \tag{1.57}$$

et nous nous limitons même au cas où

$$\sigma(v,v') = a(v)a(v'), \tag{1.58}$$

avec $a(v) \geq 0$.

L'opérateur de collision $Q$ s'écrit alors

$$Q(f) = a(v)\int a(v')f(v')\,dv'M(v) - \nu(v)f(v),$$

où

$$\nu(v) := \int b(v,v')M(v')\,dv' = a(v)\int a(v')M(v')\,dv'$$

représente la fréquence de collision. L'opérateur $Q$ peut aussi s'écrire

$$Q(f) = \nu(v)[n_\nu M(v) - f(v)] \qquad \text{avec} \quad n_\nu := \frac{\displaystyle\int \nu(v)f(v)\,dv}{\displaystyle\int \nu(v)M(v)\,dv.} \tag{1.59}$$

Nous considérons une dégénérescence de $\nu$ de la forme suivante. Il existe $\delta > 0$, $\beta > 0$ et $\nu_0 > 0$ tels que

$$\nu(v) = \nu_0|v|^{d+2+\beta}, \qquad \text{si } |v| \leq \delta \tag{1.60}$$

et nous supposons que la Maxwellienne satisfait

$$M(v) = M_0 > 0\,, \qquad \text{si } |v| \leq \delta,$$

$$\int_{\mathbf{R}^d} vM(v)\,dv = 0, \tag{1.61}$$

qui est souvent une conséquence de la parité de $M$.

Enfin, nous supposons une dernière hypothèse technique satisfaite pour un large spectre de fonction $\nu(v)$ si $M(v)$ est une Gaussienne et qui traduit le fait que le problème de définition du coefficient de diffusion est uniquement dû aux petites vitesses et plus particulièrement à la dégénérescence de $\nu$.

$$\int_{|v|\geq\delta} \frac{|v|^2}{\nu(v)} M(v)\,dv < \infty, \qquad \int_{\mathbf{R}^d} \nu(v)^2 M(v)\,dv < \infty. \tag{1.62}$$

Nous obtenons alors le résultat suivant

*Théorème* 1.4.2 ([NBAMP2]). Supposons (1.60), (1.61) et (1.62). Soit

$$\alpha = \frac{\beta+2d+2}{\beta+d+1} = 2 - \frac{\beta}{\beta+d+1}. \tag{1.63}$$

Alors la solution $f^\varepsilon$ de (2) converge faiblement dans $L^2_{\nu M^{-1}}(\mathbf{R}^d\times\mathbf{R}^d\times(0,T))$ vers $n(x,t)M(v)$ où $n$ est une solution de

$$\begin{cases} \partial_t n + \kappa(-\Delta)^{\alpha/2} n = 0, \\ n(x,0) = n_0, \end{cases} \tag{1.64}$$

avec

$$\kappa = \frac{1}{c_{d,\alpha}} \frac{1}{\beta+d+1} \frac{M_0}{\nu_0^{\alpha-1}} \int_0^\infty z^\alpha e^{-z}\,dz.$$

La constante $c_{d,\alpha}$ est celle qui apparaît dans la définition du Laplacien Fractionnaire comme une intégrale singulière

$$(-\Delta)^{\alpha/2} u = c_{d,\alpha} \text{P.V.} \int_{\mathbf{R}^d} \frac{u(x)-u(x+y)}{|y|^{d+\alpha}}\,dy.$$

Remarquons que dans ce cadre, on ne peut obtenir d'opérateur de diffusion fractionnaire d'ordre inférieur à 1 ce qui n'était pas le cas dans le problème adressé précédemment.

La preuve de ce théorème repose sur la méthode des moments inspirée de [91].

## 1.5 Une étude numérique du ralentissement de particules chargées

Dans le cadre des expériences sur la fusion par confinement inertiel, on considère la réaction

$$D + D \Longrightarrow T + \text{proton}$$

où $D$ représente les ions Deuterieum et $T$ représente les ions Tritium. Les ions Tritium émis peuvent réagir avec le Deuterium autour. Mais le milieu est en fait pollué et pour le bon fonctionnement de la fusion, il faut réussir à estimer le taux de Tritium perdu dans la zone polluée. Ceci est l'objet de l'article [CGPS], écrit en collaboration avec J.F. Clouët, F. Golse et R. Sentis. Le mouvement des particules est modélisé par un phénomène de ralentissement. On suppose que les particules se déplacent en ligne droite, c'est à dire que l'on néglige le scattering. Le milieu pollué est peu connu et est modélisé de façon statistique. Ce problème a déjà été traité mais à chaque fois pour des milieux markoviens. Ici, nous considérons qu'il est formé de boules de rayon fixé dont les centres sont tirés de manière aléatoire. Nous différentions deux

cas extrêmes, celui correspondant au cas où le rayon des boules est très petit et à celui où le rayon des boules est très grand. Dans ces deux cas, il est possible d'estimer la proportion de particules qui stoppe dans la partie polluée du domaine de façon analytique. Puis, on procède à des calculs numériques pour déterminer une formule qui permet de donner une bonne estimation de cette proportion. Le but final du CEA était d'intégrer ce calcul dans un code de manière à éviter de faire cette estimation coûteuse à chaque étape.

### 1.5.1 Modèle mathématique

Nous utilisons les notations suivantes : $\overrightarrow{v} = \overrightarrow{\Omega} w$ désigne la vitesse des particules, $w \in [0, w_{\max}]$ et $\overrightarrow{\Omega} \in \mathbf{S}^2$, la sphère unité. Enfin, $f = f(t, x, w, \overrightarrow{\Omega})$ désigne la densité de charge des particules.

Elle satisfait l'équation de transport suivante

$$\frac{\partial f}{\partial t} + w\overrightarrow{\Omega}.\frac{\partial f}{\partial x} = k(x)S(w)\frac{\partial}{\partial w}(f), \tag{1.65}$$

avec comme conditions initiales

$$f^{ini}(w, \overrightarrow{\Omega})dw = \delta_{w_{\max}}(dw) \qquad \text{la masse de Dirac en } w_{\max}$$

et

$$f(0, x, w, \overrightarrow{\Omega}) = f^0(x, w, \overrightarrow{\Omega}) \equiv f^{ini}(w)\mathbf{1}_{\mathcal{G}^c}(x).$$

Nous voulons estimer la probabilité pour une particule de stopper dans le milieu non pollué, i.e.

$$P = \lim_{t\to\infty} \frac{\int\int\int \mathbf{1}_{\mathcal{G}^c}(\mathbf{x})u(t, x, \Omega, w)\frac{d\Omega}{4\pi}d\mathbf{x}dw}{\int \mathbf{1}_{\mathcal{G}^c}(\mathbf{x})d\mathbf{x}}, \tag{1.66}$$

où $\mathcal{G}$ désigne le domaine pollué.

Nous considérons ici un modèle simplifié où la fonction de ralentissement peut être factorisée. Ainsi, on écrit

$$\mathcal{S} = S(w) * k(x),$$

où $S(w) = \dfrac{\gamma}{w} + w^2$ est une fonction universelle dans laquelle l'interaction avec les ions est modélisée par le terme $\dfrac{\gamma}{w}$ alors que celle avec les électrons est modélisée par le terme $w^2$. Le coefficient de ralentissement $k(x)$ est hétérogène, il vaut $k(x) = k_g$ dans la partie polluée du milieu et $k(x) = k_l$ dans la partie non polluée.

On peut alors calculer la distance d'arrêt. En effet, les équations caractéristiques correspondant à l'équation (1.65) sont, pour un angle $\overrightarrow{\Omega}$ fixé, données par

$$dY_t = W_t dt, \quad \text{ et } dW_t = -k(x)S(W_t)dt.$$

En définissant la fonction potentiel

$$H_{wmax} = \int_0^{w_{max}} \frac{\lambda}{S(\lambda)}d\lambda,$$

si la particule stoppe dans le milieu non pollué, on peut écrire

$$H_{wmax} = k_l(Y^1 - Y^0) + k_g(Y^{1,s} - Y^1) + k_l(Y^2 - Y^{1,s})... + k_l(Y^n - Y^{n-1})$$

alors que si elle stoppe dans le milieu pollué, on a

$$H_{wmax} = k_l(Y^1 - Y^0) + k_g(Y^{1,s} - Y^1) + k_l(Y^2 - Y^{1,s})... + k_g(Y^{n,s} - Y^n).$$

D'un point de vue géométrique, le milieu pollué est décrit par une cellule périodique de taille $L = H_{wmax}/k_l$, où le polluant est concentré dans des boules de centres $q$ distribués selon une loi de Poisson

$$\mathbf{P}(\sharp(\text{points } q \text{ dans } A) = n) = (\xi|A|L^{-3})^n \exp(-\xi|A|L^{-3})/n!,$$

où $\xi$ est le nombre moyen de boules dans le tore $[0, L[^3$. Notons que nous supposons aussi que les boules peuvent se recouvrir.

Nous avons vu que nous considérons deux cas extrêmes. Pour cela, nous introduisons des longueurs caractéristiques

$$f_g \leq f_g^{\max} = 0.6 \text{ la fraction volumique du milieu pollué,}$$

$$k_g^{-1} \leq k_l^{-1} \leq L, \text{ les coefficients de ralentissement et } \quad r = l_g \text{ le rayon des boules}$$

Tout d'abord, lorsque le rayon des boules de polluant est très petit, on suppose qu'une particule va rencontrer un grand nombre de particules avant de s'arrêter et on s'attend alors au résultat d'homogénéisation suivant

$$P = \frac{k_l f_l}{k_l f_l + k_g l_g} = 1 - f_g \text{ si } k_g = k_l.$$

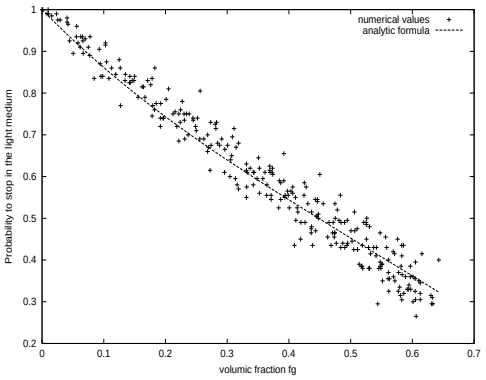
Lorsque le rayon des boules est très grand, nous pouvons supposer que les particules stoppent dans la première boule rencontrée. On peut donc faire un calcul simple permettant de déterminer la proportion de boules qui rencontrera au moins une boule.

On obtient alors

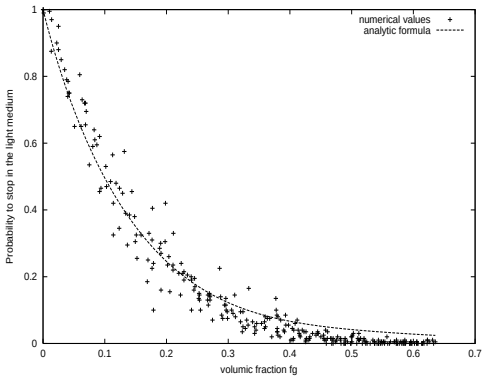
$$P = \frac{2}{\pi} \text{ en dimension 2 et } P = \frac{3}{4r} \text{ en dimension 3}$$

### 1.5.2 Résultats numériques et formule analytique

Dans le cas d'un rayon très petit, nous obtenons les résultats numériques suivants

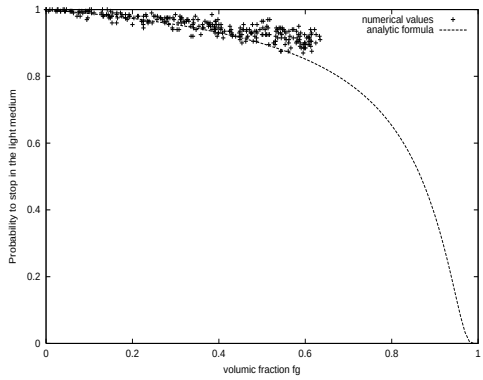



$k_g = k_l$, **Dimension 2**

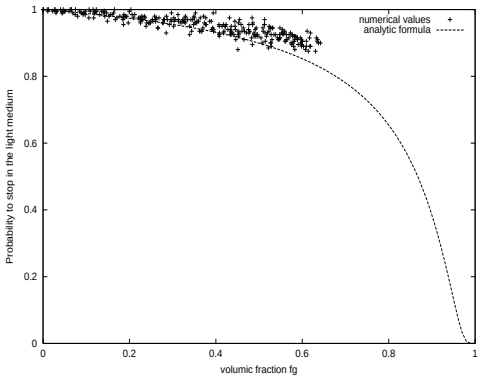



$k_g = 9k_l$, **Dimension 2**

alors que pour de grands rayons, nous obtenons




$k_g = k_l$, **Dimension 2**




$k_g = 9k_l$, **Dimension 2**

Nous proposons alors la formule analytique interpolant ces résultats

$$P = exp(\frac{-A * f_g}{1 - f_g})(1 - exp(-k_g r/L)) + exp(-k_g r/L)\frac{k_l f_l}{k_l f_l + k_g l_g},$$

$$\text{avec } A = \frac{2}{\pi r} \text{ si dim=2 et } A = \frac{3}{4r} \text{ si dim=3.}$$

Elle représente correctement la probabilité pour une particule de stopper dans le milieu non polluant car elle satisfait notre critère : être précise dans les cas extrêmes tout en donnant une valeur raisonnable dans les cas intermédiaires non significatifs.

## 1.6 Homogénéisation d'un opérateur elliptique avec conditions de Neumann

Dans [ACP], écrit en collaboration avec G. Allaire et Y. Capdeboscq, nous étudions l'homogénéisation d'un opérateur elliptique. L'originalité de ce travail vient du fait que nous considérons des conditions limites de Neumann et non de Dirichlet. On se place en dimension 1. On considère un domaine borné que nous fixons égal à $\Omega = ]0,1[$ et nous considérons l'opérateur

$$\begin{cases} -\varepsilon^2 \text{div}\left(a\left(\frac{x}{\varepsilon}\right)\nabla u^\varepsilon\right) + \varepsilon b\left(\frac{x}{\varepsilon}\right)\nabla u^\varepsilon + c\left(\frac{x}{\varepsilon}\right)u^\varepsilon = \lambda^\varepsilon \rho\left(\frac{x}{\varepsilon}\right)u^\varepsilon, \text{ dans } \Omega, \\ \\ a\left(\frac{x}{\varepsilon}\right)\nabla u^\varepsilon = 0, \text{ sur } \partial\Omega, \end{cases} \tag{1.67}$$

dont les coefficients $a, b, c$ et $\rho$ sont supposés continus et périodiques de période 1. La variable d'espace $x$ représente la variable macroscopique dans $\Omega$, alors que $y$ représente la variable microscopique $Y$, liée à $x$ par la relation $y = x/\varepsilon$. On suppose de plus qu'il existe une constante $C > 0$ telle que

$$\forall y \in Y, \quad 0 < C < a(y) < C^{-1}, \quad 0 < C < \rho(y) < C^{-1}.$$

Nous nous intéressons au comportement asymptotique du premier couple propre, c'est à dire du couple composé de la première valeur propre et d'un vecteur propre associé $(\lambda^\varepsilon, u^\varepsilon)$, lorsque $\varepsilon$ tend vers zéro. Contrairement au cas où les conditions de bord sont du type Dirichlet, le comportement ici dépend de la taille de la fraction de cellule située à droite du domaine (par exemple).

Nous choisissons d'ailleurs $\varepsilon$ de la forme

$$\varepsilon = \frac{1}{n + \delta}, \tag{1.68}$$

où $n$ est un entier et $0 \leq \delta < 1$ est la portion restante de cellule à droite du domaine.

Bien sûr, le cas particulier où $\delta = 0$ se traite simplement.

*Proposition* 1.6.1. Si $\delta = 0$ dans (1.68), soit $(\lambda_N, u_N)$ le premier couple propre du problème de cellule avec conditions de Neumann

$$\begin{cases} -\mathrm{div}_y\left(a(y)\nabla_y u_N\right) + b(y)\nabla_y u_N + c(y)u_N = \lambda_N \rho(y) u_N, \text{ dans } Y, \\ \qquad\qquad a(0)\nabla_y u_N(0) = a(1)\nabla_y u_N(1) = 0, \\ \qquad\qquad u_N(0) = 1. \end{cases} \tag{1.69}$$

Soit $\theta_N = \log\left(u_N(1)\right)$. Alors la fonction $w_N(y) = e^{-\theta_N y} u_N(y)$ est périodique de période 1 et le premier couple propre de (1.67) est donné par

$$\lambda^\varepsilon = \lambda_N, \qquad u_\varepsilon(x) = e^{\frac{\theta_N x}{\varepsilon}} w_N\left(\frac{x}{\varepsilon}\right).$$

Dans le cas où $0 < \delta < 1$, la particularité du problème avec conditions de Neumann vient du fait que l'on s'attend à ce que deux problèmes de cellule interviennent,

$$\begin{cases} -\mathrm{div}_y\left(a(y)\nabla_y u_N^t\right) + b(y)\nabla_y u_N^t + c(y)u_N^t = \lambda_N^t \rho(y) u_N^t \text{ in } (t-1, t), \\ \qquad\qquad a(t)\nabla_y u_N^t(t) = a(t-1)\nabla_y u_N^t(t-1) = 0, \end{cases} \tag{1.70}$$

$u_N^t(t-1) = 1$, pour $t = 0$ ou $t = \delta$.

Plus précisément, des résultats numériques établis dans le cas auto-adjoint nous montrent différents comportements asymptotiques pouvant se produire selon les paramètres en jeu. Ainsi, en prenant des coefficients arbitraires, nous obtenons soit une concentration de la solution à une des deux extrémités du domaine comme dans la figure 1.1, soit une concentration aux deux extrémités du domaine comme dans la figure 1.2 ou bien un phénomène d'homogénéisation où l'on voit que la solution tend vers zéro aux extrémités comme dans la figure 1.3 .

### 1.6.1 Problèmes de cellules exponentiels périodiques

Vu les phénomènes dévoilés par les résultats numériques, inspirés par les résultats de [42, 56, 60, 97] nous introduisons des fonctions exponentielles périodiques, solutions des problèmes de cellules tout en étant le produit d'une exponentielle et d'une fonction périodique. Ces fonctions sont en effet caractéristiques des phénomènes de concentration dans le cas de conditions aux limites de Dirichlet avec dérive. Pour $\theta \in \mathbb{R}$, on écrit

$$\begin{cases} -\mathrm{div}_y(a(y)\nabla_y \psi_\theta) + b(y)\nabla_y \psi_\theta + c(y)\psi_\theta = \lambda_\theta \rho(y)\psi_\theta, \quad \text{dans } Y, \\ y \to e^{-\theta y}\psi_\theta(y), \quad Y\text{-périodique}, \end{cases} \tag{1.71}$$

et son problème adjoint par rapport au produit scalaire $L^2(Y)$ devient

$$\begin{cases} -\mathrm{div}_y(a(y)\nabla_y \psi_\theta^*) - b(y)\nabla_y \psi_\theta^* + (c(y) - \mathrm{div}_y b(y))\psi_\theta^* = \lambda_\theta \rho(y)\psi_\theta^*, \quad \text{dans } Y, \\ y \to e^{\theta y}\psi_\theta^*(y), \quad Y\text{-périodique}. \end{cases} \tag{1.72}$$

Dans les équations (1.71) et (1.72), $\lambda_\theta$ est la première valeur propre et $\psi_\theta > 0, \psi_\theta^* > 0$ sont les premières fonctions propres associées données par le théorème de Krein Rutman. Il s'avère qu'une de ces fonctions exponentielles périodiques joue un rôle particulier car elle permet grâce à un procédé de factorisation de se ramener à un problème auto-adjoint. En effet, cela est dû aux propriétés de cette fonction particulière résumées dans la proposition suivante.

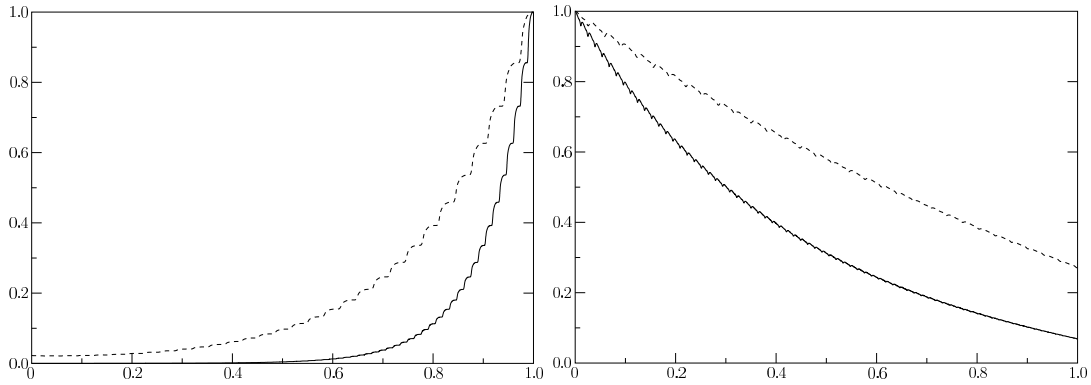



FIGURE 1.1 – A gauche : concentration en $x = 1$. A droite : concentration en $x = 0$.

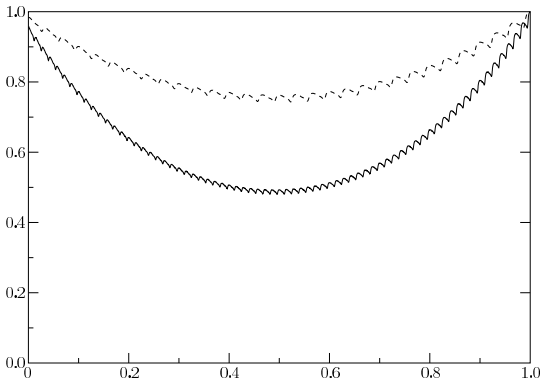



FIGURE 1.2 – Concentration aux deux extrémintés.

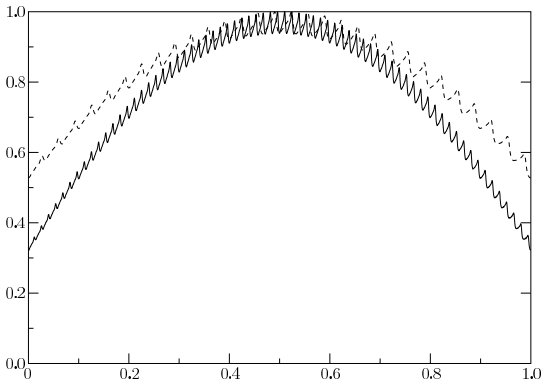



FIGURE 1.3 – Phénomène d'homogénéisation .

*Proposition* 1.6.2. Les propriétés suivantes sont satisfaites.

(*i*) L'application $\theta \to \lambda_\theta$ est strictement concave et $\lim_{\theta\to\pm\infty} \lambda_\theta = -\infty$.

(*ii*) Pour $\theta_\infty$ défini de telle sorte que $\lambda_\theta$ est maximal, les fonctions propres $\psi_\infty \equiv \psi_{\theta_\infty}$ et $\psi^*_\infty \equiv \psi^*_{\theta_\infty}$ satisfont

$$a(y)\left(\psi_\infty \nabla_y \psi^*_\infty(y) - \psi^*_\infty \nabla_y \psi_\infty(y)\right) + b(y)\psi^*_\infty(y)\psi_\infty(y) \equiv 0 \text{ dans } Y. \tag{1.73}$$

(*iii*) Pour tout $y \in Y$, l'application $\theta \to \frac{1}{\psi_\theta(y)}\nabla_y \psi_\theta(y)$ est strictement croissante et bijective de $\mathbb{R}$ vers $\mathbb{R}$.

En fait, les solutions de (1.70) pour $t = 0$ ou $t = \delta$, $u^t_N$, sont elles aussi des fonctions exponentielles périodiques. Ainsi, il existe $\theta^t_N \in \mathbf{R}$ tel que $u^t_N(y) = e^{\theta^t_N y} w^t_N(y)$ où $w^t_N$ est une fonction 1 périodique.

Nous allons voir dans le théorème ci-dessous que selon la position de $\theta^0_N$ et $\theta^\delta_N$ par rapport à $\theta_\infty$, nous allons obtenir différents comportements asymptotiques de $u_\varepsilon$ lorsque $\varepsilon$ tend vers zéro.

### 1.6.2 Dérivation des différents comportements asymptotiques possibles

Tout d'abord, si $u^0_N(y) = \frac{u^\delta_N(y)}{u^\delta_N(-1)}$, le résultat est très simple comme dans le cas $\delta = 0$.

*Proposition* 1.6.3 ([ACP]). Si les solutions $u^0_N$ et $u^\delta_N$ de (1.70), pour $t = 0$ et $t = \delta$ respectivement, sont égales à un facteur multiplicatif près, le premier couple propre de (1.67) est donné par

$$\lambda^\varepsilon = \lambda^0_N, \qquad u_\varepsilon(x) = e^{\frac{\theta^0_N x}{\varepsilon}} \frac{w^0_N\left(\frac{x}{\varepsilon}\right)}{w^0_N(0)},$$

où la fonction $w^0_N(y) = e^{-\theta^0_N y} u^0_N(y)$, avec $\theta^0_N = \log\left(u^0_N(0)\right)$, est la fonction périodique de période 1 définie plus haut.

Lorsque $u^0_N(y) \neq \frac{u^\delta_N(y)}{u^\delta_N(-1)}$, le résultat est donné par les théorèmes suivants.

*Théorème* 1.6.1 ([ACP]). Le premier vecteur propre de (1.67) se concentre sur une des extrémités de $\Omega$ dans les deux cas suivants.

– Si $\left(\theta^0_N < \theta_\infty \text{ et } \theta^\delta_N \leq \theta_\infty\right)$ ou bien si $\left(\theta^0_N < \theta_\infty < \theta^\delta_N \text{ et } \lambda^0_N < \lambda^\delta_N\right)$, alors

$$\left|\lambda^\varepsilon - \lambda^0_N\right| = \gamma_0 e^{2(\theta^0_N - \theta_\infty)/\varepsilon}(1 + o(1)),$$

et

$$\left\| u^\varepsilon(x) - u^\varepsilon(0) e^{\frac{\theta^0_N x}{\varepsilon}} \frac{w^0_N\left(\frac{x}{\varepsilon}\right)}{w^0_N(0)} \right\|_{L^\infty(\Omega)} \leq \frac{C}{\varepsilon} e^{(\theta^0_N - \theta_\infty)/\varepsilon} \|u^\varepsilon\|_{L^1(\Omega)},$$

où $\gamma_0$ est une constante positive indépendante de $\varepsilon$.

– Si $\left(\theta^\delta_N > \theta_\infty \text{ et } \theta^0_N \geq \theta_\infty\right)$ ou bien si $\left(\theta^0_N < \theta_\infty < \theta^\delta_N \text{ et } \lambda^0_N > \lambda^\delta_N\right)$, alors

$$\left|\lambda^\varepsilon - \lambda^\delta_N\right| = \gamma_1 e^{2(\theta_\infty - \theta^\delta_N)/\varepsilon}(1 + o(1)),$$

et

$$\left\| u^\varepsilon(x) - u^\varepsilon(1) \frac{e^{\frac{\theta^\delta_N x}{\varepsilon}} w^\delta_N\left(\frac{x}{\varepsilon}\right)}{e^{\frac{\theta^\delta_N}{\varepsilon}} w^\delta_N(\delta)} \right\|_{L^\infty(\Omega)} \leq \frac{C}{\varepsilon} e^{(\theta_\infty - \theta^\delta_N)/\varepsilon} \|u^\varepsilon\|_{L^1(\Omega)},$$

où $\gamma_1$ est une constante positive indépendante de $\varepsilon$.

Le premier vecteur propre de (1.67) se concentre sur une ou deux extrémités de $\Omega$ dans le troisième cas suivant.

– Si $\left(\theta_N^0 < \theta_\infty < \theta_N^\delta \text{ et } \lambda_N^\delta = \lambda_N^0\right)$, c'est à dire $\theta_N^\delta - \theta_\infty = \theta_\infty - \theta_N^0 > 0$, alors

$$\lambda^\varepsilon - \lambda_N^0 = -\gamma_\delta e^{(\theta_N^0 - \theta_\infty)/\varepsilon}(1 + o(1)),$$

et

$$\left\| u^\varepsilon(x) - u^\varepsilon(0) e^{\frac{\theta_N^0 x}{\varepsilon}} \frac{w_N^0\left(\frac{x}{\varepsilon}\right)}{w_N^0(0)} - u^\varepsilon(0) c_\delta e^{\frac{\theta_N^0 - \theta_\infty}{\varepsilon}} \frac{\psi_\infty(\delta-1) e^{\frac{\theta_N^\delta x}{\varepsilon}} w_N^\delta\left(\frac{x}{\varepsilon}\right)}{\psi_\infty(-1) w_N^0(0)} \right\|_{L^\infty(\Omega)} \leq \frac{C}{\varepsilon} e^{(\theta_N^0 - \theta_\infty)/\varepsilon} \|u^\varepsilon\|_{L^1(\Omega)},$$

où $\gamma_\delta > 0$ et $c_\delta$ sont des constantes positives indépendantes de $\varepsilon$.

*Remarque* 1.6.1. Dans les deux premiers cas, la fonction propre $u_\varepsilon$ est approchée par le produit d'une fonction périodique et d'une exponentielle. Selon le signe de $\theta_N^0$ et de $\theta_N^\delta$, la concentration a lieu à l'une ou l'autre des extrémités. Dans le troisième cas, la fonction propre $u_\varepsilon$ se concentre à une extrémité de $\Omega$ si $\theta_\infty \neq 0$ et aux deux extrémités dans le cas particulier où $\theta_\infty = 0$. En effet, l'approximation aboutit en $x = 0$ à

$$u^\varepsilon(x) \approx u^\varepsilon(0) e^{\frac{\theta_N^0 x}{\varepsilon}} \frac{w_N^0\left(\frac{x}{\varepsilon}\right)}{w_N^0(0)},$$

alors qu'en $x = 1$, nous obtenons

$$\begin{aligned} & u^\varepsilon(x) - u^\varepsilon(0) e^{\frac{\theta_N^0 x}{\varepsilon}} \frac{w_N^0\left(\frac{x}{\varepsilon}\right)}{w_N^0(0)} - u^\varepsilon(0) c_\delta e^{\frac{\theta_N^0 - \theta_\infty}{\varepsilon}} \frac{\psi_\infty(\delta-1) e^{\frac{\theta_N^\delta x}{\varepsilon}} w_N^\delta\left(\frac{x}{\varepsilon}\right)}{\psi_\infty(-1) w_N^0(0)} \\ = \quad & u^\varepsilon(x) - u^\varepsilon(0) e^{\frac{\theta_N^0 x}{\varepsilon}} \frac{w_N^0\left(\frac{x}{\varepsilon}\right)}{w_N^0(0)} - u^\varepsilon(0) c_\delta e^{\frac{\theta_\infty}{\varepsilon}} e^{\frac{\theta_N^\delta (x-1)}{\varepsilon}} \frac{\psi_\infty(\delta-1) w_N^\delta\left(\frac{x}{\varepsilon}\right)}{\psi_\infty(-1) w_N^0(0)}, \end{aligned}$$

ce qui implique

$$u^\varepsilon(x) \approx u^\varepsilon(0) c_\delta e^{\frac{\theta_\infty}{\varepsilon}} e^{\frac{\theta_N^\delta (x-1)}{\varepsilon}} \frac{\psi_\infty(\delta-1) w_N^\delta\left(\frac{x}{\varepsilon}\right)}{\psi_\infty(-1) w_N^0(0)}.$$

Dans ce dernier cas, la localisation est déterminée par le signe de $\theta_\infty$. Si $\theta_\infty < 0$, la localisation a lieu en $x = 0$, et si $\theta_\infty > 0$ la localisation a lieu en $x = 1$. Dans le cas particulier où $\theta_\infty = 0$ (qui inclut le cas auto-adjoint), une double localisation aux deux extrémités a lieu.

Le dernier cas où $\theta_N^0 \geq \theta_\infty$ et $\theta_N^\delta \leq \theta_\infty$, correspond au régime d'homogénéisation. Dans ce cas, les fonctions propres ne se concentrent pas aux extrémités.

*Théorème* 1.6.2 ([ACP]). Si $\theta_N^0 \geq \theta_\infty$ et $\theta_N^\delta \leq \theta_\infty$, le premier couple propre de (1.67) est de la forme

$$u^\varepsilon(x) \approx \psi_\infty\left(\frac{x}{\varepsilon}\right) u(x) \text{ et } \lambda_\varepsilon = \lambda_\infty + \varepsilon^2(\lambda_0^* + o(1)),$$

où $\psi_\infty$ est une fonction périodique et $(u, \lambda_0^*)$ est le premier couple propre du problème homogénéisé suivant

$$\begin{cases} -d^* \Delta u & = & \lambda_0^* s^* u \text{ dans } \Omega, \\ u \in H^1(\Omega) & \text{avec soit} & u(0) = 0 \text{ soit } u(1) = 0, \end{cases}$$

avec $d^*$ et $s^*$ deux constantes positives.

Notons que le caractère périodique des coefficients est très important et dans le cas où ces coefficients ne seraient pas périodiques, on pourrait obtenir des comportements tout à fait différents [44], [45].

La preuve de ces théorèmes repose sur trois étapes. Tout d'abord, on procède à la factorisation par $\psi_\infty$ qui permet de se ramener à un problème auto-adjoint. Puis on écrit le quotient de Rayleigh qui nous permet d'encadrer les valeurs propres en construisant des fonctions tests bien adaptées à partir des différents problèmes de Neumann associés à chaque partie de la cellule. On obtient par là, à la fois la convergence des valeurs propres et la convergence de la dérive de la fonction exponentielle. Enfin, une fois que l'on a la convergence de la dérive, on décompose la solution de (1.67) sur une base de solutions de problèmes de cellule dont nous savons exprimer un développement par le théorème de Kato.

# Chapitre 2

# Transport de masse relativiste

## 2.1 Le problème de Monge-Kantorovich pour une fonction de coût relativiste

Dans un article en collaboration avec J. Bertrand [BP], nous nous intéressons au problème de Monge-Kantorovich dans le cadre du transport relativiste. Nous distinguons deux manières de lier deux mesures de probabilité.

*Définition* 2.1.1. Soit $\rho_1 \in P(\mathbf{R}^d)$ et $S$ une application borélienne $S : \mathbf{R}^d \to \mathbf{R}^d$. On dit que $\rho_0$ est la *mesure image* de $\rho_1$ par $S$, ou $S_\# \rho_0 = \rho_1$, si pour toute fonction borélienne $\varphi$

$$\int_{\mathbf{R}^d} \varphi(S(x)) d\rho_1(x) = \int_{\mathbf{R}^d} \varphi(x) d\rho_0(x).$$

*Définition* 2.1.2. Soient deux mesures de probabilité $\rho_0$ et $\rho_1$, l'ensemble des *plans de transport* entre $\rho_0$ et $\rho_1$ désigne les mesures de probabilité sur $\mathbf{R}^d \times \mathbf{R}^d$ ayant $\rho_0$ et $\rho_1$ comme marginales :

$$\Pi(\rho_0, \rho_1) = \{\pi(x,y) \in P(\mathbf{R}^d \times \mathbf{R}^d) \quad \text{t.q.} \int_{\mathbf{R}^d \times \mathbf{R}^d} \varphi(x) d\pi(x,y) = \int_{\mathbf{R}}^d \varphi(x) d\rho_0(x)$$

$$\text{et} \int_{\mathbf{R}^d \times \mathbf{R}^d} \varphi(y) d\pi(x,y) = \int_{\mathbf{R}}^d \varphi(y) d\rho_1(y)\},$$

pour toute fonction borélienne $\varphi : \mathbf{R}^d \longrightarrow \mathbf{R}$.

Le problème de Monge est le suivant. On se demande si

$$\inf_{S_\# \rho_0 = \rho_1} \int_{\mathbf{R}^d} c(x, S(x)) d\rho_0(x)$$

est atteint parmi toutes les applications $S$ telles que $S_\# \rho_0 = \rho_1$. La fonction $c$ ci dessus est appelée fonction de coût. Mais ce problème peut ne pas avoir de solution.

Ceci a amené Kantorovich à considérer le problème suivant. Soient deux mesures $\rho_0$ et $\rho_1$ dans $\mathcal{P}(\mathbf{R}^d)$, on cherche les plans optimaux, c'est à dire les plans qui atteignent le minimum défini par

$$\min_{\pi \in \Pi(\rho_0, \rho_1)} \int_{\mathbf{R}^d \times \mathbf{R}^d} c(x,y) d\pi(x,y).$$

Le problème de Kantorovich est une relaxation du problème de Monge et, dès lors que la fonction de coût est positive semi-continue inférieurement, le problème de Kantorovich admet une solution.

La question posée ici est l'existence d'une application optimale lorsque la fonction de coût appartient à ce que nous appelons l'ensemble des coûts relativistes.

### 2.1.1 Le coût relativiste

Nous sommes partis de la fonction de coût relativiste pour la chaleur $c(x,y) = \mathbf{h}(x-y)$ avec

$$\mathbf{h}(z) = \begin{cases} 1 - \sqrt{1-|z|^2}, & |z| \leq 1, \\ \infty, & |z| > 1. \end{cases} \tag{2.1}$$

La particularité de ce coût est qu'il est strictement convexe mais n'est pas continu, ce qui implique qu'il peut être infini. De plus, son gradient tend vers l'infini lorsque $|z| \to 1$.

Notons que la discontinuité de $c$, impliquant sa partie infinie, est intimement liée au caractère borné de $\nabla c^*(z)$ et donc au phénomène relativiste. En effet, comme $c$ est défini à partir de $c^*$ par

$$c(z) = \sup_{w \in \mathbf{R}^d} (z \cdot w) - c^*(w),$$

le fait que $c$ n'est pas fini quand $|z| > 1$ vient du fait que $z = \nabla c^*(w)$ ne peut être atteint quel que soit $w \in \mathbf{R}^d$. Ce qui signifie que $|\nabla c^*| \leq 1$.

Il se trouve que nous pouvons généraliser notre résultat d'existence d'une application optimale à des fonctions de coût plus générales incluant

$$c(x,y) = f(x-y) + \delta(x-y|\mathbf{C}), \tag{2.2}$$

où $\delta(\cdot|\mathbf{C})$ est l'indicatrice d'un ensemble strictement convexe $\mathbf{C}$ contenant $0$, définie par $\delta(x|\mathbf{C}) = 0$ si $x \in \mathbf{C}$ et $+\infty$ sinon alors que $f : \mathbf{R}^d \to \mathbf{R}^+$ est une fonction strictement convexe et différentiable satisfaisant $f(0) = 0$.

Plus précisément, nous traitons les coûts définis ci-dessous

*Définition* 2.1.3 (fonctions de coût relativistes). On dit que $c(x,y) = \mathbf{h}(x-y)$ est une fonction de coût relativiste si $\mathbf{h} : \mathbf{R}^d \to [0,+\infty]$ est une fonction strictement convexe telle que $\mathbf{h}^{-1}([0,+\infty)) = \mathbf{C}$ est la fermeture d'un ouvert borné strictement convexe. On suppose de plus que $\mathbf{h}(0) = 0$ et que $0 \in \overset{\circ}{\mathbf{C}}$. Enfin, on suppose que $\mathbf{h}$ (respectivement $\mathbf{h}'$) est continue sur $\mathbf{C}$ (respectivement sur $(\mathbf{h}')^{-1}(\mathbf{R}) \subset IDom$).

Dans cette définition, $\mathbf{h}'$ désigne la dérivée directionnelle définie par

*Définition* 2.1.4 (Dérivées directionnelles). Soit $\mathbf{h} : \mathbf{R}^d \to [0,+\infty]$ une fonction convexe telle que $\mathbf{h}^{-1}([0,+\infty)) = \mathbf{C}$ est la fermeture d'un ouvert borné convexe. Posons $\mathbf{h}'$ la dérivée directionnelle

$$\begin{aligned} \mathbf{h}' : IDom &\longrightarrow \mathbf{R} \cup \{-\infty\} \\ (x,w) &\longmapsto \mathbf{h}'(x;w) \end{aligned}$$

où $IDom = \{(x,w) \in \mathbf{C} \times \mathbf{R}^d; x + w \in \overset{\circ}{\mathbf{C}}\}$ et

$$\mathbf{h}'(x;w) = \lim_{\lambda \downarrow 0} \frac{\mathbf{h}(x+\lambda w) - \mathbf{h}(x)}{\lambda}.$$

Remarquons que ces conditions autorisent la fonction de coût relativiste pour la chaleur.

### 2.1.2 Existence d'une application optimale pour le coût relativiste

Le principe de [BP] est le suivant. Nous fixons deux mesures de probabilité et nous voulons montrer que dès lors que le coût relativiste optimal liant ces deux probabilités est fini, le plan de transport optimal est engendré par une application. Pour ce faire, nous introduisons un problème paramétré pour un temps –ou une vitesse– $t$

$$c_t(x,y) = \mathbf{h}(\frac{x-y}{t})$$

et la fonction de coût total

$$C(t) = \min_{\pi\in\Pi(\rho_0,\rho_1)} \int_{\mathbf{R}^{2d}} c(\frac{x-y}{t})\, d\pi(x,y).$$

L'idée est que si la vitesse limite autorisée est très petite, le coût sera infini quel que soit le plan de transport, alors que si la vitesse maximale est très grande, le coût optimal sera obligatoirement fini et la discontinuité de la fonction coût ne devrait pas intervenir.

Nous montrons dans un premier temps l'existence d'une vitesse limite critique suivant la définition suivante

*Définition* 2.1.5 (vitesse critique).

$$\text{Il existe une vitesse } T \text{ telle que pour tout } t \geq T,\ C(t) < +\infty \text{ et } C(t) = +\infty \text{ sinon.} \quad (2.3)$$

Puis nous montrons l'existence d'une application de transport optimale pour les temps sur-critiques par le théorème suivant.

*Théorème* 2.1.1. Soient $\rho_0$ et $\rho_1$ deux mesures de probabilité à support compact sur $\mathbf{R}^d$ et $c(x,y) = \mathbf{h}(x-y)$ une fonction de coût relativiste. On suppose que $\rho_0 = \rho_0 dx$ est absolument continue par rapport à la mesure de Lebesgue. Alors, pour toute vitesse critique ou sur-critique $t \geq T$, il existe un unique plan de transport optimal $\pi_0^t$ pour le coût $c_t = \mathbf{h}(\frac{\cdot}{t})$ engendré par une application, i.e.

$$\pi_0^t = (Id, S)_\sharp \rho_0.$$

De plus, si nous supposons que $\rho_0$ est régulière et $t > T$, il existe une application $c_t$-concave $\phi$ telle que $S$ est définie $\pi_0^t$ presque partout sur $p_x(\{x-y \in t\overset{\circ}{\mathbf{C}}\} \cap \operatorname{supp}\pi_0^t)$ par la formule

$$S(x) = x - t\nabla\mathbf{h}^*(|\tilde{\nabla}\phi(x)|), \quad (2.4)$$

où $\mathbf{h}^*$ désigne la transformée de Legendre de $\mathbf{h}$ et $\tilde{\nabla}\phi$ est le gradient approximatif de $\phi$.

*Définition* 2.1.6. Une mesure de probabilité $\rho = \rho dx$ sur $\mathbf{R}^d$ est dite régulière s'il existe un domaine $\Omega$ (*i.e.* un ouvert borné connexe) sur lequel $\rho$ est concentrée et tel que

$$\exists c > 0 \text{ t.q.} \quad \rho(x) \geq c$$

pour presque tout $x \in \Omega$.

L'argument majeur de la preuve de la première partie de ce théorème repose sur la notion de $c$-cyclique monotonie, propriété qui traduit que le support d'un plan de transport optimal est constitué de couples dont tout réarrangement produirait un coût supérieur.

Pour cela, on raisonne par contradiction. On suppose qu'un point du domaine $x_0$ est associé par le plan de transport optimal à deux points différents du domaine image $y_1$ et $y_2$.

On utilise alors un lemme très important de [142] qui permet de construire un autre couple $(x', y')$ au voisinage par exemple de $(x_0, y_1)$ de telle sorte que $|x' - y_2| < |x_0 - y_2|$

et $|x_0 - y'| < |x' - y'|$ ce qui contredit la $c$-cyclique monotonie dans le cas de fonctions de coût croissantes de la distance. Ce raisonnement se généralise ensuite au cas de fonctions strictement convexes en utilisant les dérivées directionnelles et notamment les dérivées rentrantes.

La deuxième partie de ce théorème concerne la construction d'une fonction potentiel associée à l'application de transport optimale $S$. Une stratégie alternative au raisonnement cité ci-dessus serait la construction de l'application de transport optimale à partir d'une fonction potentiel $c$-concave. Cette fonction $c$-concave peut être fournie soit par la dualité de Kantorovich soit par la $c$-cyclique monotonie. En effet, le support d'un plan de transport optimal est $c$-cyclique monotone et par là, est inclus dans le $c$-surdifférentiel d'une fonction $c$-concave que nous appelons potentiel (cf Ruschendorf [167]). Toute la difficulté dans le cas de fonctions coût relativistes réside dans l'obtention d'une borne $L^\infty$ pour ce potentiel et d'une propriété de différentiabilité. Pour résoudre le premier problème, on utilise un argument géométrique d'estimation de la croissance du volume d'un domaine élargi des voisinages de ses points. Ainsi, on montre qu'un couple appartenant au support du plan de transport optimal peut être relié à tout autre couple du support de ce transport optimal après une série finie de liaisons sachant qu'on dit que les deux couples $(x_0, y_0)$ et $(x_1, y_1)$ sont liés si $|x_1 - y_0| \leq t$, $t$ étant la vitesse limite fixée. Ceci implique que la fonction potentiel $c$-concave construite par Ruschendorf est bornée. La propriété de différentiabilité vient du fait que la fonction potentiel peut être vue comme une extension d'une suite de fonctions Lipschitz obtenues en se restreignant à des couples du support satisfaisant $|x_i - y_i| < t - \delta$ avec $\delta > 0$. C'est pourquoi cette propriété n'est obtenue qu'en dehors de l'ensemble des couples extrémaux du support du transport optimal, caractérisés par $|x_i - y_i| = t$.

## 2.2 Problème de Cauchy pour une équation de la chaleur relativiste

Nous présentons enfin [McP], écrit en collaboration avec R. McCann, dont le but est de montrer l'existence de solution pour l'équation

$$\partial_t \rho = \operatorname{div}(\rho \frac{\nabla \rho}{\sqrt{\rho^2 + |\nabla \rho|^2}}) = \operatorname{div}(\rho \frac{\nabla \log \rho}{\sqrt{1 + |\nabla \log \rho|^2}}), \tag{2.5}$$

que nous appelons équation de la chaleur relativiste introduite par Mihalas–Mihalas [162] ou Rosenau [166] et qui a été mentionnée par Brenier dans [133]. Cette équation représente un phénomène relativiste dans le sens où la vitesse de propagation $\frac{\nabla \rho}{\sqrt{\rho^2 + |\nabla \rho|^2}}$ -qui n'est autre que la version relativiste de la vitesse de propagation impliquée dans l'équation de la chaleur- est bornée. Dans [133], Brenier propose d'appliquer à (2.5) la méthode introduite par Jordan, Kinderlehrer, Otto [154] ou Otto [163], basée sur le transport optimal qui s'applique formellement à toute équation du type

$$\partial_t \rho = \operatorname{div}(\rho \nabla c^*(\nabla(F'(\rho)))), \tag{2.6}$$

où $c^*$ est appelée fonction de mobilité et $F$ est une fonction convexe représentant l'entropie. Le principe est que cette équation peut être vue comme le flot gradient de l'entropie relativement à une fonction de coût qui est la transformée de Legendre de la fonction de mobilité $c^*$ ( voir sur ce sujet [116], [117], [121], [172]).

Concernant le schéma JKO, nous citons par exemple [154], où Jordan, Kinderlehrer et Otto traitent le cas de la distance de Wasserstein liée au coût $c(z) = \frac{|z|^2}{2}$ et l'entropie $F(\rho) =$

$\rho \log \rho - V\rho$. Ainsi, ils obtiennent un résultat d'existence pour l'équation de Fokker-Plank

$$\partial_t \rho = \mathrm{div}(\rho \nabla V) + \beta^{-1}\Delta\rho, \text{ où } V \text{ est une fonction régulière donnée.}$$

D'autre part, dans [163], Otto considère le cas où $c(z) = \frac{|z|^q}{q}$ avec $F(\rho) = \frac{n\rho^b}{b(b-1)}$ et $b = n + (p-2)/(p-1)$ aboutissant à une équation doublement dégénérée

$$\partial_t \rho = \mathrm{div}(|\nabla \rho^n|^{p-2}\nabla\rho^n), \quad \text{avec } \frac{1}{p} + \frac{1}{q} = 1, \quad p \geq 2.$$

Enfin, dans [114], Agueh regarde les fonctions données par

$$\beta|z|^q \leq c(z) \leq \alpha(|z|^q + 1) \text{ pour tout } z \in \mathbf{R}^d, \text{ avec } \alpha, \beta > 0 \text{ et } q > 1,$$

qui incluent de nombreuses équations comme l'équation de Fokker Planck ou celle des milieux poreux.

Mais pour l'équation de la chaleur relativiste, ce n'est plus le coût quadratique qui est en jeu mais le coût relativiste pour la chaleur, introduit plus tôt pour lequel, avant le travail récent en collaboration avec J. Bertrand, la théorie de Monge Kantorovich n'était pas complète. Les seuls résultats disponibles étaient ceux sur la dualité de Kantorovich de [119] mais l'existence d'une application optimale n'était pas connue. Dans ce travail antérieur à [BP], nous avions contourné ce problème en régularisant la fonction de coût par le procédé de Yosida. Le passage à la limite dans la régularisation est alors rendu possible par la compacité des densités obtenue grâce à une inégalité reliant l'information de Fisher et l'entropie et est donc particulier à la double minimisation caractéristique au schéma JKO.

### 2.2.1 Description de la stratégie basée sur le transport optimal

Notons tout d'abord que le cadre étudié ici est un peu plus général et concerne les fonctions de coût satisfaisant

$$c(z) = \begin{cases} \tilde{c}(|z|) \geq 0, & \text{if } |z| \leq 1, \\ +\infty, & \text{if } |z| > 1, \end{cases}$$

où $\tilde{c}$ est une fonction strictement continue sur $[0,1]$ appartenant à $C^2([0,1[)$ satisfaisant $|\nabla c(z)| \to \infty$ lorsque $|z| \to 1$ et où les fonctions d'entropie sont du type $F \in C^2(\mathbf{R})$ telles que $\frac{F(\lambda)}{\lambda} \to \infty$ lorsque $\lambda \to \infty$ et telles que $\lambda^d F(\lambda^{-d})$ est convexe.

Nous décrivons ici le schéma discret en temps grâce auquel nous construisons, à partir de $\rho_0 \in P(\Omega)$ donnée, la fonction constante par morceaux $\rho^h(t,y) \in P([0,T] \times \Omega)$ déterminée par

$$\begin{cases} \rho^h(0,y) = \rho_0(y), \\ \rho^h(t,y) = \rho_i^h(y), \quad \text{pour } t \in ]ih;(i+1)h], \end{cases} \tag{2.7}$$

où $P(\Omega)$ désigne l'ensemble des mesures de probabilité sur $\Omega$ et où $h$ est le pas de temps. Dans (2.7), $\rho_i^h(y)$ est une solution du problème de minimisation $\mathcal{P}_i^h$ défini par

$$I(\rho_{i-1}^h, \rho_i^h) = \inf_{\rho \in P(\Omega)} I(\rho_{i-1}^h, \rho),$$

avec

$$I(\rho_{i-1}^h, \rho) = \int_\Omega F(\rho(y))dy + h \inf_{\pi \in \Pi_i^h(\rho_{i-1}^h, \rho)} \int_{\Omega\times\Omega} c(\frac{x-y}{h})d\pi(x,y),$$

$\Pi_i^h$ désignant l'ensemble des mesures de probabilité qui ont pour marginales $\rho_{i-1}^h$ et $\rho$ et où $c$ est la fonction de coût convexe.

Les différentes étapes de la construction de solution de (2.6) sont les suivantes :

**Etape 1** On montre que le problème de minimisation admet une unique solution, le plan de transport optimal $\pi$.

**Etape 2** On introduit l'application optimale associée à ce plan de transport et on dérive les équations d'Euler -Lagrange du problème de minimisation, ce qui donne formellement

$$\nabla c\left(\frac{S_i^h(y)-y}{h}\right) = \nabla(F'(\rho_i^h)),$$

où $S_i^h$ désigne l'application optimale pour laquelle la mesure image de $\rho_i^h$ est $\rho_{i-1}^h$.

Comme $\nabla c^*(\nabla c(y)) = y$, ceci implique

$$S_i^h(y) - y = h\nabla c^*(\nabla(F'(\rho_i^h))). \tag{2.8}$$

**Etape 3** On construit une équation discrète en temps approchée et on passe à la limite lorsque le pas de temps tend vers zéro. En effet, en multipliant l'équation (2.8) par $\rho\nabla\xi$ où $\xi$ est une fonction test régulière, on obtient alors au sens des distributions

$$\rho_i^h - \rho_{i-1}^h = h\ \mathrm{div}(\rho_i^h\nabla c^*(\nabla(F'(\rho_i^h))) + \text{termes correctifs.} \tag{2.9}$$

Pour passer à la limite lorsque $h$ tend vers zéro pour obtenir (2.6), il reste à identifier la limite $\overline{A}$ du terme non linéaire $\rho_i^h\nabla c^*(\nabla(F'(\rho_i^h)))$. Pour cela, nous utilisons un argument du type Minty Browder [148] basé sur les propriétés de monotonie du gradient de la fonction de mobilité $c^*$.

En effet, nous allons voir que l'on a pour toute fonction test $\xi(t,x) \geq 0$ et $\zeta(t,x) \in C^\infty(\mathbf{R}^d, \mathbf{R}^d)$

$$\int \xi(\overline{A} - \overline{\rho}\nabla c^*(\zeta))(D^{ac}(F'(\overline{\rho})) - \zeta) \geq 0, \tag{2.10}$$

ce qui implique $\overline{A} = \overline{\rho}\nabla c^*(D^{ac}(F'(\overline{\rho})))$ en prenant $\zeta = D^{ac}(F'(\overline{\rho})) + \delta w$, et en passant successivement à la limite en supposant $\delta \searrow 0$ puis $\delta \nearrow 0$.

Formellement, pour obtenir (2.10), on veut passer à la limite $\varepsilon < h \to 0$ dans la relation de monotonie suivante

$$\int_0^T \int_\Omega \xi(t,y)\rho^{\varepsilon h}(t,y)(\nabla c^*(\nabla(F'(\rho^{\varepsilon h}(t,y)))) - \nabla c^*(\zeta(t,y)))\cdot(\nabla(F'(\rho^{\varepsilon h}(t,y))) - \zeta(t,y))dydt \geq 0.$$

Pour cela, on utilise

1- la convexité de déplacement ("displacement convexity") [160] de la fonctionnelle d'entropie (impliquée par la monotonie de $F$ et la convexité de $\lambda^d F(\lambda^{-d})$ et sa conséquence, l'inégalité impliquant l'information de Fisher et l'entropie satisfaite par $\rho^h$ solution du problème $\mathcal{P}_i^h$

$$\int_0^T \int_\Omega \rho^h(t,y)\nabla c^*(\nabla(F'(\rho^h(t,y)))) \cdot \nabla(F'(\rho^h(t,y)))dydt \leq \int_\Omega (F(\rho^h(0,y)) - F(\rho^h(T,y)))dy, \tag{2.11}$$

ou plus précisément, sa version locale

$$\begin{aligned}\int_0^T \int_\Omega \xi(t,y)\rho^h(t,y)\nabla c^*(\nabla(F'(\rho^h(t,y)))) \cdot \nabla(F'(\rho^h(t,y)))dydt \\ \leq -\int_0^T \int_\Omega \nabla\xi(t,y)\rho^h(t,y)\cdot\nabla c^*(\nabla(F'(\rho^h(t,y))))F'(\rho^h(t,y))dydt \\ -\int_0^T \int_\Omega \xi(t,y)\partial_t F(\rho^h(t,y))dydt,\end{aligned} \tag{2.12}$$

où $\xi$ est une fonction test régulière.

2- L'égalité correspondante satisfaite formellement par toute solution régulière $\tilde{\rho}$ de (2.5) obtenue en multipliant l'équation par $F'(\tilde{\rho}(t,y))$ et en intégrant par partie :

$$\int_\Omega \tilde{\rho}(t,y)\nabla F'(\tilde{\rho}(t,y))\cdot\nabla c^*(\nabla F'(\tilde{\rho}(t,y)))dydt = \int_\Omega [(F(\rho_0(y)) - F(\tilde{\rho}(T,y))]dy. \tag{2.13}$$

C'est ainsi que nous contournons le problème de non linéarité car la borne BV de $\rho^h$ implique une convergence forte qui permet de passer à la limite dans le terme de droite de (2.11) que nous remplaçons alors par le terme de gauche de (2.11).

### 2.2.2 Théorème d'existence de solution

*Théorème* 2.2.1. Soit $\Omega \subset \mathbf{R}^d$ un domaine borné convexe, et $0 < m < \rho_0 < M$ qui implique $\int_\Omega F(\rho_0) < \infty$. Soit $c : \mathbf{R}^d \longrightarrow [0,\infty]$ une fonction de coût sur $\mathbf{R}^d$ donnée par $c(z) = \tilde{c}(|z|)$, où $\tilde{c} \in C^0([0,1]) \cap C^2([0,1[)$ avec $|\nabla c(z)| \to \infty$ lorsque $|z| \to 1$, $\tilde{c}(0) = 0 = \tilde{c}'(0)$ et $\tilde{c}''(\lambda) > 0$ sur $[0,1[$. Soit $F \in C^2(\mathbf{R})$ une fonction convexe telle que $\frac{F(\lambda)}{\lambda} \to \infty$ quand $\lambda \to \infty$ et telle que $\lambda^d F(\lambda^{-d})$ est convexe.

(i) **Caractérisation du support de la mesure optimale : vitesse de propagation finie**
Soit $\pi_i^h$ la mesure optimale pour le problème de minimisation $(\mathcal{P}_i^h)$ ayant pour seconde marginale $\rho_i^h$, alors $\rho_i^h \in BV(\Omega)$ et

$$\operatorname{supp}\pi_i^h \subset \left\{(x,y) \quad | \quad \frac{|x-y|}{h} < 1\right\} \cup Z_i^h \text{ avec } \pi_i^h(Z_i^h) = 0.$$

(ii) **Equation d'Euler-Lagrange : un schéma discret**
Il existe une application $S_i^h \in L^\infty(\Omega;\Omega)$ définie presque partout selon la mesure de Lebesgue par

$$S_i^h(y) = y + h\nabla c^*(D^{ac}(F'(\rho_i^h(y)))), \tag{2.14}$$

telle que $\pi_i^h = (S_i^h \times id)_\# \rho_i^h$ et $D^{ac}(F'(\rho_i^h(y))) = F''(\rho_i^h(y))D^{ac}\rho_i^h$.

(iii) **Convergence de la mesure** $\rho^h$. Soit $\rho^h$ la fonction constante par morceau définie à partir de $\rho_i^h$ via (2.7). Lorsque $h \to 0$, une sous suite de $\rho^h$ converge fort dans $L^1([0,T]\times\Omega)$ vers $\overline{\rho} \in L^\infty([0,T]\times\Omega) \cap L^1_w([0,T], BV(\Omega))$ et $D\rho^h \rightharpoonup D\overline{\rho}$ au sens des mesures.

(iv) **Equation limite**
La limite $\overline{\rho}$ appartient à $L^1_w([0,T], BV(\Omega)) \cap L^\infty([0,T]\times\Omega)$ et satisfait

$$\partial_t \overline{\rho} = \operatorname{div}(\overline{\rho}\nabla c^*(D^{ac}(F'(\overline{\rho})))), \tag{2.15}$$

au sens des distributions.

Ce théorème est établi à une sous suite près car nous n'avons pas unicité de la solution de l'équation limite. Pour cela, il faudrait définir comme dans [128] une notion de solution entropique ce que nous pensons possible.

**Cas particulier : Dimension 1.**

Notons qu'en dimension 1, le problème devient beaucoup plus simple (voir [138]). Soit $\rho \in P(\Omega)$, on peut l'étendre à $\mathbf{R}$ et on peut introduire des fonctions de distribution cumulatives [172],

$$l(x) = \int_{-\infty}^{x} d\rho = d\rho[(-\infty, x]].$$

La fonction $l$ est continue à droite, croissante, et a pour limites $l(-\infty) = 0$ et $l(+\infty) = 1$. On peut alors définir une fonction inverse généralisée de $l$ sur $[0, 1]$

$$l^{-1}(t) = \inf\{x \in \mathbf{R} \text{ s. t. } l(x) > t\}.$$

De plus, en dimension 1, le plan de transport optimal et donc l'application optimale ne dépend pas de la fonction de coût à partir du moment où $c(x, y) = c(|x - y|)$ (cf. [161]). Et le plan de transport optimal est donné par

$$\pi_i = (l_{i-1}^{-1} \times l_i^{-1})_{\#} d\mathcal{H}^1,$$

où $l_{i-1}$ est la fonction de distribution cumulative pour $\rho_{i-1}$ et $l_i$, la fonction de distribution cumulative pour $\rho_i$. Lorsque $\rho_{i-1}$ ne charge pas les points, le plan optimal est donné par

$$\rho_{i-1} = (l_{i-1}^{-1} \circ l_i)_{\#} \rho_i.$$

Les équations d'Euler-Lagrange (2.8) impliquent donc

$$l_{i-1}^{-1} \circ l_i(y) - y = h \nabla c^*(\nabla(F'(l_i'(y)))),$$

ou

$$\begin{aligned} \frac{l_{i-1}^{-1}(x) - l_i^{-1}(x)}{h} &= \nabla c^*(\nabla(F'(l_i'(l_i^{-1}(x))))) \\ &= \nabla c^*(\nabla(F'(\frac{1}{(l_i^{-1})'(l_i \circ l_i^{-1}(x))}))) = \nabla c^*(\nabla(F'(\frac{1}{(l_i^{-1})'(x)}))), \end{aligned}$$

ce qui correspond à un schéma d'Euler implicite $l^{-1}$.

### 2.2.3 Rappels sur les fonctions BV

Le principal problème pour appliquer la troisième étape de la stratégie présentée plus haut, est le manque de régularité de la densité $\rho^h$. En effet, comme on suppose que la densité est minorée par une constante positive, l'inégalité reliant l'information de Fisher et le saut d'entropie implique que la suite de fonctions constantes par morceau $\rho^h(t, x)$ est bornée dans $L_w^1([0, T], BV(\Omega))$. Par exemple, la solution limite ne peut pas être prise comme fonction test pour obtenir l'égalité (2.13).

Rappelons que l'ensemble des fonctions $BV$ est l'ensemble des fonctions $\rho \in L^1(\Omega)$ dont le gradient au sens des distributions est une mesure de Radon vectorielle dont la variation totale, définie par

$$||\rho||_{BV} = ||D\rho||_{TV} = \sup\{\int_\Omega \rho \text{div}(\xi) dx \text{ où } \xi \in C_0^\infty(\Omega; \mathbf{R}^d) \text{ t.q. } |\xi(x)| \leq 1\}$$

est finie.

Le gradient $D\rho$ d'une fonction $BV$ notée $\rho$ peut être décomposé comme la somme de sa partie absolument continue par rapport à la mesure de Lebesgue $D^{ac}\rho$, qui coïncide avec la

dérivée approximative de $\rho$, et sa partie singulière $D^s\rho$, divisée en deux, une partie correspondant aux sauts $D^j\rho$ et une partie appelée partie de Cantor $D^c\rho$. Ainsi, on écrit

$$D\rho = D^{ac}\rho + D^s\rho = D^{ac}\rho + D^j\rho + D^c\rho$$

Comme $BV$ n'est pas séparable, nous devons faire la différence entre l'espace $L^1_w([0,T], BV(\Omega))$ dans lequel la solution limite vit et l'espace $L^1([0,T], BV_2(\Omega))$ où $BV_2(\Omega) = BV(\Omega) \cap L^2(\Omega)$ dont le dual est $L^\infty([0,T], BV_2(\Omega)')$. Ainsi, on utilisera donc le produit de dualité

$$\langle \rho; \tilde{\rho} \rangle = \int_0^T \langle \rho(t); \tilde{\rho}(t) \rangle dt,$$

lorsque $\rho \in L^1([0,T], BV_2(\Omega))$ et $\tilde{\rho} \in L^\infty([0,T], BV_2(\Omega)')$.

C'est ainsi que nous définissons $(z; D\rho)$ dans le terme de droite de (2.12) comme une distribution lorsque $z \in L^\infty([0,T] \times \Omega; \mathbf{R}^d)$, $\rho \in L^1([0,T], BV_2(\Omega)) \cap L^\infty([0,T] \times \Omega)$, et div$z = B$ au sens des distributions, $B$ étant la limite faible$^*$ d'une suite $B^h$ bornée dans le dual de $L^1([0,T], BV_2(\Omega))$ et $z$ étant la limite faible$^*$ d'une suite bornée $z^h$. En effet, soit $\xi$ une fonction test, on écrit

$$< \xi; (z; D\rho) >= -\int_0^T < B; \rho\xi >_{L^1([0,T],BV_2(\Omega))'; L^1([0,T],BV_2(\Omega))} dt - \int_0^T \int_\Omega z\rho\nabla\xi dt dy.$$

Enfin, si $z$ est la limite faible$^*$ d'une suite bornée $z^h$, on montre que

$$< \xi; (z; D\rho) >= \lim_{h\to 0} < \xi; (z^h; D\rho) >= \lim_{h\to 0} \left( -\int_0^T \int_\Omega \text{div} z^h \rho\xi dt dy - \int_0^T \int_\Omega z^h \rho\nabla\xi dt dy \right).$$

Pour cela, il faut que div$z^h = B^h$ dans $L^1(\Omega)$, pour $h$ et $t$ fixé.

Nous utilisons par ailleurs pour le terme de gauche dans (2.12) la notion introduite par Dal Maso [145] de fonctionnelle de fonction $BV$ et sa propriété de $L^1$ semi-continuité démontrée dans le papier de De Cicco, Fusco and Verde [146]. Pour $f(x, \lambda, \xi)$ une intégrande dépendant de $x$, de $\lambda \in \mathbf{R}$ et du champ de vecteur $\xi$, Dal Maso définit la fonctionnelle $\mathcal{F}(\rho, D\rho)$ de $\rho$, une fonction $BV$, par

$$\begin{aligned} \mathcal{F}(\rho, D\rho) &= \int_0^T \int_\Omega f(x, \rho, D^{ac}\rho) dt dx + \int_\Omega f^0(x, \rho, \frac{D\rho}{|D\rho|}) |D^c\rho| \\ &\quad + \int_{J_\rho} \left( \int_{\rho_-(x)}^{\rho_+(x)} f^0(x, s, \nu_\rho(x)) ds \right) d\mathcal{H}^{d-1}(x), \end{aligned} \tag{2.16}$$

où $f^0$ est la fonction de récession donnée par $\lim_{t\to 0} t f(x, \lambda, \frac{\xi}{t})$, où $J_\rho$ est l'ensemble des sauts de $\rho$, $\nu_\rho = D\rho/|D\rho|$ la dérivée de Radon Nikodym de $D\rho$ par rapport à sa variation totale $|D\rho|$. C'est cette extension des fonctionnelles aux fonctions $BV$ qui nous permet de définir

$$\int_0^T \int_\Omega DF'(\rho) \cdot \nabla c^*(D(F'(\rho))),$$

pour $\rho \in L^1([0,T], BV(\Omega))$ en décomposant $\xi \cdot \nabla c^*(\xi) = c(\nabla c^*(\xi)) + c^*(\xi)$ en appliquant le théorème de semi-continuité inférieure à $c^*(\xi)$.

# Chapitre 3

# Perspectives

Nous présentons dans ce chapitre des travaux en cours ou à venir en liaison avec les travaux présentés dans ce mémoire.

## 3.1 Etude de la limite semi-classique d'une équation de Schrödinger confinée

Dans un travail à venir en collboration avec F. Delebecque et F. Méhats, nous partons du système de Vlasov soumis à un potentiel de confinement

$$\left\{ \begin{array}{l} i\varepsilon\partial_t\psi = -\frac{\varepsilon^2}{2}\Delta_x\psi - \frac{1}{2}\partial_{zz}\psi + V\psi \\ -\Delta V = |\psi|^2 \end{array} \right. \tag{3.1}$$

avec une condition initiale bien préparée, c'est à dire concentrée sur la première sous-bande définie par

$$\psi_{|t=0} = \phi_1\chi_1 \tag{3.2}$$

avec

$$\left\{ \begin{array}{rcl} -\frac{1}{2}\partial_{zz}\chi_1 + V\chi_1 & = & \mathcal{E}_1[|\phi|^2]\chi_1 \\ -\Delta V & = & |\phi_1|^2|\chi_1|^2. \end{array} \right. \tag{3.3}$$

Nous voulons montrer la convergence des observables associées à ce système vers les solutions du système de Vlasov-sous bande suivant

$$\partial_t f + v\cdot\nabla_x f - \nabla_x\mathcal{E}\cdot\nabla_v f = 0 \tag{3.4}$$

couplée à

$$\left\{ \begin{array}{rcl} -\frac{1}{2}\partial_{zz}\chi + V\chi & = & \mathcal{E}\chi \\ -\Delta V & = & \displaystyle\int f dv|\chi|^2. \end{array} \right. \tag{3.5}$$

Pour montrer ce résultat, nous comptons utiliser la méthode de l'énergie modulée présentée dans ce mémoire dans le cadre des équations cinétiques mais qui a été largement développée pour le système de Schrödinger-Poisson.

La justification formelle de ce passage à la limite repose sur l'introduction d'un système intermédiaire défini par l'équation

$$i\varepsilon\partial_t\phi = -\frac{\varepsilon^2}{2}\Delta_x\phi - \mathcal{E}_1[|\phi|^2]\chi_1 \tag{3.6}$$

couplée avec

$$\left\{ \begin{array}{rcl} -\frac{1}{2}\partial_{zz}\chi_1 + V\chi_1 & = & \mathcal{E}_1[|\phi|^2]\chi_1 \\ -\Delta V & = & |\phi|^2|\chi_1|^2. \end{array} \right. \tag{3.7}$$

Ce système est associé à une énergie formellement conservée donnée par

$$\overline{E} = \frac{\varepsilon}{2}\int |\nabla\phi|^2 dx + \int \mathcal{E}_1|\phi|^2 dx.$$

En se plaçant dans le cadre de l'approximation adiabatique on considère que lorsque la fonction d'onde $\psi$ a une donnée initiale concentrée sur la première sous-bande, elle reste concentrée au cours du temps et $\psi$ est alors proche de $\phi\chi_1$. Ainsi, la limite semi-classique formelle aboutit à (3.4-3.5).

## 3.2 Mise à l'échelle alternative entre l'homogénéisation et diffusion pour l'équation de Boltzmann et approximation de diffusion pour l'équation de Boltzmann Pauli

Dans les travaux présentés plus haut, [NBAPV] et [NBABP], nous étudions l'interaction du phénomène d'homogénéisation et de l'approximation de diffusion pour l'équation de Boltzmann

$$\partial_t f^{\varepsilon,\alpha} + \frac{1}{\varepsilon}\mathbf{v}\cdot\nabla_{\mathbf{x}} f^{\varepsilon,\alpha} = \frac{1}{\varepsilon^2}Q_\alpha(f^{\varepsilon,\alpha}) \quad , \quad f^{\varepsilon,\alpha}(0,\mathbf{x},\mathbf{v}) = f_{ini}(\mathbf{x},\mathbf{v})$$

dans le cas où le paramètre $\varepsilon$ correspondant à l'approximation de diffusion, le libre parcours moyen, tend vers zéro "plus vite" que la taille des hétérogénéités $\alpha$. Nous avons vu que nous pouvons obtenir grâce aux développements de Hilbert une convergence forte vers un profil en vitesse appartenant au noyau de l'opérateur de collision associé à une densité solution d'une équation de diffusion homogénéisée. Dans un travail en cours avec G. Bal, on s'intéresse au cas où la taille des hétérogénéités est beaucoup plus petite que le libre parcours moyen. D'un point de vue fromel, on s'attend à obtenir l'approximation diffusion de l'équation de Boltzmann homogénéisée obtenue dans [66]. Toute la différence réside dans le développement de l'opérateur $\eta v\nabla_y + Q$ où $\eta$ est le rapport des ordres de grandeur de la taille des hétérogénéités et du libre parcours moyen qui dans ce nouveau cadre tend vers l'infini. Nous nous basons sur les techniques employées dans [66] nécessitant l'introduction d'un correcteur et en conséquence, nous nous attendons à n'obtenir cette fois qu'une convergence faible.

L'autre question naturelle à la suite des travaux [NBAPV] et [NBABP] concerne l'équation de Boltzmann non linéaire et dans un premier temps, l'équation de Boltzmann Pauli qui a l'avantage de préserver au cours du temps des propriétés $L^\infty$. Le premier résultat de convergence que nous avions obtenu via la méthode des moments était un résultat de convergence deux échelles, un résultat de convergence faible contre des fonctions test oscillantes. Le manque de compacité de la densité était alors un obstacle à la généralisation au cas non linéaire mais à présent que nous avons un résultat de convergence forte par les dévelopements de Hilbert, nos envisageons par des techniques de linéarisation autour d'un équilibre d'obtenir un résultat dans le cas de Boltzmann Pauli où l'équation de diffusion serait alors une équation non linéaire. Ceci fait l'objet d'une collaboration avec H. Chaker.

## 3.3 Diffusion anormale pour l'équation de Fokker Plank

Dans le cadre de la thèse d'Elissar Nasreddine, en co-encadrement avec P. Laurençot, nous étudions la limite de diffusion d'une équation de Fokker Plank centrée sur des équilibres à

décroissance du type puissance. Le cas de diffusion classique – lorsque la puissance associée à l'équilibre est telle que le coefficient de diffusion est bien défini – a fait l'objet du stage de Master recherche d'Elissar coencadré avec N. Ben Abdallah. Comme pour l'équation de Boltzmann, pour certaines puissances, le coefficient de diffusion n'est plus défini et on s'attend à un phénomène de diffusion anormale.

## 3.4 Régularité de la fonction potentiel du problème de Monge-Kantorovich pour le coût relativiste

Concernant le transport relativiste, et plus précisément, le problème de Monge Kantorovich pour le coût relativiste, nous montrons dans [JB] que nous pouvons – en se restreignant aux points non extrémaux du support du plan de transport optimal – exprimer l'application de transport optimal en fonction d'une fonction potentiel candidate comme solution du problème dual. La construction de cette fonction potentiel fait apparaître son caractère $L^\infty$ mais peut-on obtenir une meilleur caractérisation ? Au vu du résultat obtenu dans [McP] pour l'équation de la chaleur relativiste, on s'attend au mieux à une régularité $BV$ du potentiel. Mais cette propriété est obtenue dans [McP] après avoir relié le potentiel $\phi$ et la densité $\rho$ solution de l'équation de la chaleur relativiste par l'équation $\phi = \log \rho$ en écrivant à la fois l'application optimale et les équations d'Euler Lagrange du problème de double minimisation. La régularité $BV$ est alors obtenue pour $\rho$ par une inégalité fonctionnelle impliquant l'information de Fisher et l'entropie qui n'est autre qu'une inégalité log-Sobolev inverse correspondant au flot de la chaleur relativiste. A priori, cette régularité est liée au problème de double minimisation mais nous cherchons dans une collaboration à venir avec J. Bertrand à l'obtenir dans le cadre du problème de Monge Kantorovich en reliant les deux densités données par un flot construit en interpollant ces densités grâce à l'application optimale pour le coût relativiste qui formellement nous donne une solution de l'équation de la chaleur relativiste et donc la régularité $BV$.

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# Deuxième partie

# Publications

# Chapitre 4

# Incompressible Euler and e-MHD as scaling limits of theVlasov-Maxwell system

Yann Brenier [1], Norbert J. Mauser [2], Marjolaine Puel [3]

**abstract**

We consider asymptotic limits of the Vlasov-Maxwell system with two different scalings concerning the magnetic field as a relativistic effect. The combined "quasi-neutral" and non-relativistic limits thus lead to two different limit systems. In the case when we keep the magnetic field in the non-relativistic limit, we obtain the so-called electron magneto-hydrodynamics equations. Otherwise we obtain the incompressible Euler equations with no more magnetic field left. The key tool of the proof is the method of modulated energy.

## 4.1 Introduction

Our starting point is the Vlasov-Maxwell system which couples a kinetic equation for the dynamics of the charge distribution, with the Maxwell equations for the electromagnetic field. We deal with two different scalings of this system, that lead to two different limit equations. On the one hand, the magnetic field can be viewed as a merely relativistic effect that vanishes in the non-relativistic limit $c \to \infty$ simultaneously with the "quasi-neutral" limit, which yields thus the incompressible Euler equations. On the other hand, we can choose a scaling where the magnetic field occurs also in the limit equation which is then a magneto hydrodynamic model called "electron-MHD" (e-MHD) [Ho]. In the following figure (fig 1) we sketch all the formal asymptotic limits that can be considered.

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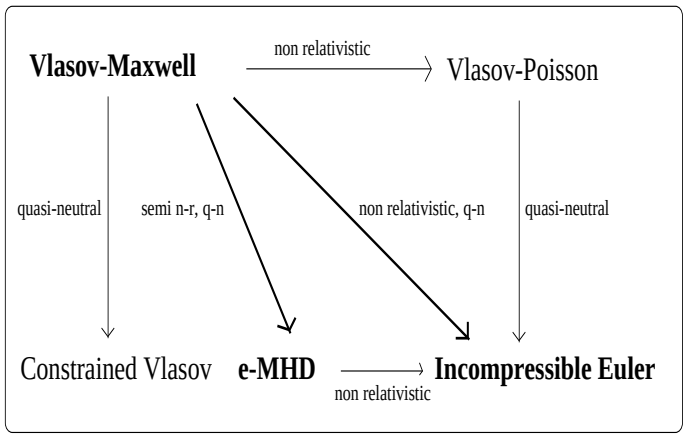


**Figure 1**

The system that we study is the following

$$\partial_t f + \xi \cdot \nabla_x f - (E - \alpha(\xi \wedge B)) \cdot \nabla_\xi f = 0, \tag{4.1}$$

$$-\alpha \partial_t B + \nabla \wedge E = 0 \quad , \quad \varepsilon \nabla \cdot E = 1 - \rho, \tag{4.2}$$

$$\varepsilon\alpha \partial_t E + \nabla \wedge B = \alpha J \quad , \quad \nabla \cdot B = 0\,. \tag{4.3}$$

*Remarque* 4.1.1. Note the convention of sign for the magnetic field that we use consistently. Often, the convention in the Maxwell equations is with the opposite sign.

The parameter $\alpha$ is chosen according to the desired scaling and the density $\rho$ and the current $J$ are given by

$$\rho = \int f(t,x,\xi) d\xi \quad , \quad J = \int \xi f(t,x,\xi) d\xi\,. \tag{4.4}$$

In (4.1)–(4.3), $f(t,x,\xi) \geq 0$ is a nonnegative distribution function, the "charge density", that represents the probability for particles to be at time $t > 0$ on a point $x \in \mathbf{R}^3$ with the velocity $\xi \in \mathbf{R}^3$. The fields $E$ and $B$ are respectively the electric field and the magnetic field. They satisfy the Maxwell equations and act on the particles via the Lorentz force in the Vlasov equation.

*Remarque* 4.1.2. In our theory, we will consider solutions that are nearly monokinetic, i.e. $f(t,x,\xi) \sim \rho(t,x)\delta(\xi - \frac{J(t,x)}{\rho(t,x)})$, which means that the electronic temperature is close to zero.

Taking the first two moments of (4.1), we obtain the following system,

$$\partial_t \rho + \nabla \cdot J = 0 \tag{4.5}$$

$$\partial_t J + \nabla : (\int \xi \otimes \xi \; f d\xi) + E\rho - \alpha J \wedge B = 0 \tag{4.6}$$

*Remarque* 4.1.3. The notation $a : (b \otimes c)$ for three vectors $a, b, c \in \mathbf{R}^3$ stands for $\sum_{j=1}^3 a_j b_i c_j$.

Starting from this system, we consider three different scalings.

**Scaling 1 :** $\alpha << 1$**,** $\varepsilon << 1$

This scaling corresponds to a fully non-relativistic limit, where the magnetic field disappears, coupled with a quasi-neutral limit (the ion density instantaneously adjusts itself to the electron density). Formally, it is easy to guess the limit of (4.5),(4.6), (4.2),(4.3) when $\varepsilon,\ \alpha \to 0$, provided that the distribution function tends to be monokinetic, i.e.

$$\nabla : \int \xi \otimes \xi f d\xi \to \nabla : (J \otimes J).$$

This limit is the Euler equation of incompressible fluids

$$\partial_t J + \nabla : (J \otimes J) - E = 0, \tag{4.7}$$

$$\nabla \cdot J = 0 \quad , \quad \rho = 1, \tag{4.8}$$

$$\nabla \wedge E = 0 \quad , \quad B = 0, \quad . \tag{4.9}$$

**Scaling 2 :** $\alpha = 1$**,** $\varepsilon << 1$ : the magnetic field does not disappear any longer in the limit equations, that we name "e-MHD" following D.D. Holm [Ho] :

$$\partial_t J + \nabla : (J \otimes J) - E + J \wedge B = 0, \tag{4.10}$$

$$J = \nabla \wedge B \quad , \quad \rho = 1, \tag{4.11}$$

$$-\partial_t B + \nabla \wedge E = 0\,, \quad \nabla \cdot B = 0. \tag{4.12}$$

As above, provided that the distribution function tends to be monokinetic, the formal limit is obvious.

**Scaling 3 :** $\alpha \to \infty$ : in this limit, we obtain trivially that $\rho = 1$ and $J = 0$. Therefore, only the case of non vanishing temperature is interesting (when $f$ is not monokinetic). It describes the fluctuations around the equilibrium $\rho = 1$ and $J = 0$. By rescaling $\tilde{B} = \alpha B$, we obtain the following system

$$\partial_t f + \xi \cdot \nabla f - (E - (\xi \wedge \tilde{B})) \cdot \nabla_\xi f = 0, \tag{4.13}$$

$$\partial_t \tilde{B} = \nabla \wedge E, \quad \nabla \cdot \tilde{B} = 0\,, \quad J = 0, \quad \rho = 1, \tag{4.14}$$

which has been addressed (from a very different point of view) in [BrPu].

In this paper, we prove the desired results for the case 1 and the case 2. We use the modulated energy method introduced in [14] used in a quantum context in [Pu1], [Pu2] and based on the concept of dissipative solutions due to P.-L. Lions [32]. This method can be seen as a variant of both the relative entropy method by H.T. Yau [Ya] and the hamiltonian energy method by E. Grenier [5].

The key idea is to estimate a modulation of the energy by the solution of the formal limit equation. Typically, we replace the kinetic energy $\int |\xi|^2 f d\xi$ by $\int |\xi - v|^2 f d\xi$ where $v$ is a smooth solution to the limit equations.

To keep the proofs as simple as possible we consider only global smooth solutions of the Vlasov-Maxwell system, with sufficient decay at infinity in $x$ and $\xi$ or periodicity in $x$ and decay at infinity in $\xi$, so that we can perform all required integration by parts without care.

## 4.2 Convergence to the electron-Magnetohydrodynamics (e-MHD)

### 4.2.1 Result

In this section, we consider the Vlasov-Maxwell system (4.1), (4.2), (4.3), (4.4) : We consider here that $\alpha > 0$ is a fixed constant of order one and $\varepsilon$ is a small parameter. (This means that the magnetic field is, in some sense, non-relativistic.)

*Remarque* 4.2.1. Using $\tilde{B} = \alpha B$ as a new variable, we can write equivalently

$$\partial_t f + \xi \cdot \nabla f - (E - \xi \wedge \tilde{B})) \cdot \nabla_\xi f = 0 \tag{4.15}$$

$$-\partial_t \tilde{B} + \nabla \wedge E = 0 \quad , \quad \varepsilon \nabla \cdot E = 1 - \rho \tag{4.16}$$

$$\varepsilon \alpha^2 \partial_t E + \nabla \wedge \tilde{B} = \alpha^2 J \quad , \quad \nabla \cdot B = 0, \tag{4.17}$$

$$\rho = \int f d\xi \quad , \quad J = \int \xi f d\xi. \tag{4.18}$$

The Vlasov-Maxwell system is known to admit global strong solutions for smooth initial data depending only on two space coordinates, with a perpendicular magnetic field (see [GlSc] for details and [DiLi] for global weak solutions). The system that we consider for the limit $\varepsilon \to 0$, is the so-called e-MHD system [Ho]

$$\begin{gathered} \partial_t v + (v \cdot \nabla)v + e - v \wedge b = 0 \\ \text{with } -\alpha \partial_t b + \nabla \wedge e = 0,\ \alpha v = \nabla \wedge b \text{ and } \nabla \cdot b = 0. \end{gathered} \tag{4.19}$$

Introducing $\omega = \nabla \wedge (v + \alpha A)$ where $A$ is the magnetic potential such that $\nabla \wedge A = b$, we can write e-MHD in a different way :

$$\partial_t \omega + v \cdot \nabla \omega = \omega \cdot \nabla v, \text{ with } -\Delta v + \alpha^2 v = \nabla \wedge \omega.$$

If the initial data depends only on the two first coordinates $x_1$, $x_2$, then $\omega$ is aligned with the third coordinates and can be seen as a scalar field satisfying

$$\partial_t \omega + v \cdot \nabla \omega = 0, \text{ with } -\Delta v + \alpha^2 v =^{\perp} \nabla \omega = (-\partial_1 \omega_2, \partial_2 \omega_1, 0),$$

Therefore, the existence results are the same as for the incompressible Euler equations (see [32], for instance). In particular, we have global strong solutions for smooth initial conditions depending only on two space coordinates. Let us now state our main result

*Theorem* 4.2.1. Let $(f(t,x,\xi),(B,E)(t,x))$ be a global smooth solution to the VM system for $x \in \mathbb{R}^3$, $\xi \in \mathbb{R}^3$, $t \geq 0$, with sufficient decay of $f$ as $\xi \to \infty$. Let $(v,b,e)(t,x)$ be a global smooth solution to the e-MHD system for $x \in \mathbb{R}^3$, $t \geq 0$, Let us fix $T > 0$ and $\alpha > 0$ and assume the initial conditions of the VM system to satisfy

$$\int f(0,x,\xi) dx d\xi = 1, \tag{4.20}$$

$$\int f(0,x,\xi)|\xi|^2 dx d\xi \leq C, \tag{4.21}$$

$$\varepsilon \int |E(0,x)|^2 dx \leq C\sqrt{\varepsilon}, \tag{4.22}$$

$$\int |B(0,x) - b(0,x)|^2 dx \leq C\sqrt{\varepsilon}, \tag{4.23}$$

$$\int |\xi - v(0,x)|^2 f(0,x,\xi) dx d\xi \leq C\sqrt{\varepsilon} \tag{4.24}$$

where $C$ does not depend on $\varepsilon$. Then

$$\sup_{0 \leq t \leq T} \Big( \int |\xi - v(t,x)|^2 f(t,x,\xi) d\xi dx + \int |B(t,x) - b(t,x)|^2 dx \Big) \leq C' \sqrt{\epsilon},$$

where $C'$ depends only on $T$, $C$ and $(v,b,e)$.

### 4.2.2 Moment equations and modulated energy

We deduce from the Vlasov system the following equations for the moments $\rho$ and $J$ :

$$\partial_t \rho + \nabla \cdot J = 0 \tag{4.25}$$

and

$$\partial_t J + \nabla : (\int \xi \otimes \xi f d\xi) + \rho E - \alpha J \wedge B = 0$$

and the conserved energy is

$$H = \frac{1}{2}\int |\xi|^2 f(t,x,\xi) d\xi dx + \int \frac{\varepsilon |E(t,x)|^2 + |B(t,x)|^2}{2} dx.$$

Following [14], we define a modulated energy $H_{v,b}(t)$ by

$$H_{v,b}(t) := \frac{1}{2}\int |\xi - v(t,x)|^2 f(t,x,\xi) d\xi dx + \int \frac{\varepsilon |E(t,x)|^2 + |B(t,x) - b(t,x)|^2}{2} dx$$

where $v$ and $b$ are arbitrarily chosen smooth divergence free vector fields depending on $t \geq 0$ and $x \in \mathbb{R}^3$.

*Proposition* 4.2.2. Let $(f(t,x,\xi),(B,E)(t,x))$ be a global smooth solution to the VM system for $x \in \mathbb{R}^3$, $\xi \in \mathbb{R}^3$, $t \geq 0$, with sufficient decay of $f$ as $\xi \to \infty$. Then, for all smooth vector fields $(v,b,e)(t,x)$ defined for $t \geq 0$ and $x \in \mathbb{R}^3$ such that $b$ and $v$ are divergence free, the modulated energy $H_{v,b}$ satisfies the following ODE

$$\frac{d}{dt}(H_{v,b} - \theta) = \int \sum_{i,j=1}^{d} \partial_j v_i [ - \int (\xi - v)_i (\xi - v)_j f d\xi + (B-b)_i (B-b)_j + \varepsilon E_i E_j ] dx + \eta + r \tag{4.26}$$

where

$$\theta = \varepsilon \int E \cdot (e + \alpha v \wedge (B - b))$$

$$\eta = -\varepsilon \int E \cdot (\partial_t e + \nabla (e \cdot v) - \alpha \partial_t (v \wedge b) + \alpha \partial_t v \wedge B),$$

and

$$r = \int (e - E + \alpha v \wedge B) \cdot (\frac{\nabla \wedge b}{\alpha} - v)$$

$$- \int (B - b) \cdot (\partial_t b - \frac{\nabla \wedge e}{\alpha}) - \int ((\partial_t + v \cdot \nabla) v + e - \alpha v \wedge b) \cdot (J - \rho v).$$

Theorem 4.2.1 follows easily from Proposition 4.2.2. Indeed, we first assume $(v,b,e)$ to be solution to the e-MHD equation. As a consequence, $r$ vanishes in the right-hand side of (4.26). Next, it follows from the assumptions of Theorem 4.2.1 that $\eta(t)$ and $\theta(t)$ are uniformly bounded by $C_1\sqrt{\varepsilon}$, for $t \in [0,T]$ where $C_1$ depends only on $T$, $C$ and $(v,b,e)$. Thus, we get from (4.26),

$$\frac{d}{dt}(H_{v,b} - \theta) \leq 2\lambda H_{v,b} + \eta \leq 2\lambda (H_{v,b} - \theta) + 2\lambda\theta + \eta \leq 2\lambda (H_{v,b} - \theta + C_2\sqrt{\varepsilon}),$$

where

$$\lambda = \sup_{t,x,\xi} \sum_{i,j=1}^{d} \partial_j v_i(t,x) \frac{\xi_i \xi_j}{|\xi|^2}$$

and $C_2$ depends only on $T$, $C$ and $(v, b, e)$. Thus

$$H_{v,b}(t) - \theta(t) + C_2\sqrt{\varepsilon} \leq \exp(2\lambda t)(H_{v,b}(0) - \theta(0) + C_2\sqrt{\varepsilon}).$$

From the assumptions of Theorem 4.2.1, we know that $H_{v,b}(0) \leq 1.5C\sqrt{\varepsilon}$ and conclude that

$$H_{v,b}(t) \leq C'\sqrt{\varepsilon}$$

where $C'$ depends only on $T$, $C$ and $(v, b, e)$. and the result of Theorem 4.2.1 follows.

### 4.2.3 Proof of Proposition 4.2.2

N.B. To perform the calculations required by the proof, we use notations with indices $i, j, k \in \{1, 2, 3\}$, implicit summation on repeated indices and abridged notations with comma for partial derivatives. For instance, $v_{,t} + v_{i,j}v_j$ stands for $\partial_t v + (v \cdot \nabla)v$. Consistently, we use the signature symbol $\epsilon_{ijk} = +1, -1$ to denote the wedge product and the curl operator. With these notations, the equations satisfied by $(\rho, J, E, B)$ are

$$\rho_{,t} = -J_{j,j}, \tag{4.27}$$

$$-J_{i,t} = (\int \xi_i \xi_j f d\xi)_{,j} - \alpha\epsilon_{ijk}J_j B_k + \rho E_i, \tag{4.28}$$

$$\alpha B_{i,t} = \epsilon_{ijk}E_{k,j}, \quad B_{i,i} = 0, \tag{4.29}$$

$$\alpha J_i = \varepsilon\alpha E_{i,t} + \epsilon_{ijk}B_{k,j}, \quad \rho = 1 - \varepsilon E_{i,i}. \tag{4.30}$$

Let us notice that we can write equation (4.28) in a different way, using equations (4.29) and (4.30), namely :

$$-J_{i,t} = (\int \xi_i \xi_j f d\xi)_{,j} - \epsilon_{ijk}\epsilon_{jpq}B_{q,p}B_k - \varepsilon\alpha\epsilon_{ijk}E_{j,t}B_k + E_i(1 - \varepsilon E_{j,j})$$

$$= (\int \xi_i \xi_j f d\xi)_{,j} + B_k B_{k,i} - B_k B_{i,k} - \varepsilon\alpha\epsilon_{ijk}(E_j B_k)_{,t} + \varepsilon\alpha\epsilon_{ijk}E_j B_{k,t} + E_i(1 - \varepsilon E_{j,j})$$

$$= (\int \xi_i \xi_j f d\xi)_{,j} + B_j(B_{j,i} - B_{i,j}) - \varepsilon\alpha\epsilon_{ijk}(E_j B_k)_{,t} + \varepsilon\epsilon_{ijk}\epsilon_{kpq}E_j E_{q,p} + E_i(1 - \varepsilon E_{j,j})$$

(using (4.29). Thus,

$$-J_{i,t} = (\int \xi_i \xi_j f d\xi)_{,j} + B_j(B_{j,i} - B_{i,j}) - \varepsilon\alpha\epsilon_{ijk}(E_j B_k)_{,t} + \varepsilon E_j(E_{j,i} - E_{i,j}) + E_i(1 - \varepsilon E_{j,j}). \tag{4.31}$$

The modulated energy is defined by

$$H_{b,v} = \frac{1}{2}(\int\int |\xi - v|^2 f d\xi + \int |B - b|^2 + \int \varepsilon |E|^2). \tag{4.32}$$

We have

$$H_{b,v} = H + \int(-J_i v_i + \rho v_i v_i/2) + \int(-B_i b_i + b_i b_i/2)$$

where the energy

$$H = H_{0,0,0} = \frac{1}{2}(\int\int |\xi|^2 f d\xi + \int |B|^2 + \int \varepsilon |E|^2)$$

is conserved and therefore does not depend on $t$. Thus

$$\frac{dH_{b,v}}{dt} = Q_1 + Q_2 + Q_3 + L_1 + L_2 + L_3, \tag{4.33}$$

where

$$Q_1 = -\int v_i J_{i,t}, \quad Q_2 = \int \rho_{,t} v_i v_i/2, \quad Q_3 = -\int B_{i,t} b_i, \tag{4.34}$$

$$L_1 = -\int J_i v_{i,t}, \quad L_2 = \int \rho v_i v_{i,t}, \quad L_3 = -\int (B-b)_i b_{i,t}. \tag{4.35}$$

From equation (4.31), we deduce

$$Q_1 = \int v_i (\int \xi_i \xi_j f d\xi)_{,j} + \int v_i B_j (B_{j,i} - B_{i,j}) - \varepsilon\alpha \int \epsilon_{ijk} v_i (E_j B_k)_{,t}$$

$$+\varepsilon \int v_i E_j (E_{j,i} - E_{i,j}) + \int v_i E_i (1 - \varepsilon E_{j,j}).$$

Thus, after integrating by part and using that $v_{j,j} = B_{j,j} = 0$,

$$Q_1 = \int v_{i,j} (-\int \xi_i \xi_j f d\xi + B_j B_i + \varepsilon E_j E_i) + \int v_i E_i + \eta_0 + \frac{d}{dt}\theta_0,$$

where

$$\theta_0 = -\varepsilon\alpha \int \epsilon_{ijk} v_i E_j B_k, \quad \eta_0 = \varepsilon\alpha \int \epsilon_{ijk} v_{i,t} E_j B_k.$$

Next, from equation (4.27),

$$Q_2 = \int \rho_{,t} v_i v_i/2 = -\int J_{j,j} v_i v_i/2 = \int J_j v_i v_{i,j}$$

and, from (4.29),

$$Q_3 = -\int B_{k,t} b_k = -\int \epsilon_{kji} \frac{E_{i,j}}{\alpha} b_k = -\int E_i \epsilon_{ijk} \frac{b_{k,j}}{\alpha}.$$

So

$$Q = Q_1 + Q_2 + Q_3 = \int v_{i,j} (-\int \xi_i \xi_j f d\xi + B_i B_j + \varepsilon E_i E_j) + \int J_j v_i v_{i,j} + \eta_0 + \frac{d}{dt}\theta_0$$

$$+r_0$$

where

$$r_0 = -\int E_i (\epsilon_{ijk} \frac{b_{k,j}}{\alpha} - v_i).$$

(Notice that $r_0 = 0$ as $(b, v, e)$ satisfies the e-MHD system.) We also have

$$L_3 = -\int (B-b)_i b_{i,t} = -\int (B-b)_i \frac{\epsilon_{ijk} e_{k,j}}{\alpha} + r_1,$$

where

$$r_1 = -\int (B-b)_i (b_{i,t} - \frac{\epsilon_{ijk} e_{k,j}}{\alpha}).$$

(Notice that $r_1 = 0$ as $(b, v, e)$ satisfies the e-MHD system.) Thus

$$L_3 = \int e_k \epsilon_{ijk} \frac{(B-b)_{i,j}}{\alpha} + r_1$$

$$= -\int e_k(J_k - \varepsilon E_{k,t} - v_k) + r_1 + r_2,$$

(because of equation (4.30)), where

$$r_2 = -\int e_k(v_k - \frac{\epsilon_{kji} b_{i,j}}{\alpha}).$$

(Notice that $r_2 = 0$ if $(b, v, e)$ is solution to the e-MHD system.) So

$$L_3 = -\int e_k(J_k - \rho v_k) - \int e_k(\rho - 1)v_k + \frac{d}{dt}\varepsilon \int E_k e_k - \varepsilon \int E_k e_{k,t} + r_1 + r_2$$

$$= -\int e_k(J_k - \rho v_k) + \eta_1 + \eta_2 + \frac{d}{dt}\theta_1 + r_1 + r_2$$

(using $\rho = 1 - \varepsilon E_{i,i}$) where

$$\eta_1 = -\varepsilon \int (e_k v_k)_{,i} E_i,$$

$$\eta_2 = -\varepsilon \int E_k e_{k,t}$$

$$\theta_1 = \varepsilon \int E_k e_k.$$

It follows, using definition (4.35), that

$$L = L_1 + L_2 + L_3 = -\int (v_{k,t} + e_k)(J_k - \rho v_k) + \eta_1 + \eta_2 + \frac{d}{dt}\theta_1 + r_1 + r_2$$

Thus

$$L = \int v_{i,j}(v_j J_i - \rho v_j v_i) + L_4 + \eta_1 + \eta_2 + \frac{d}{dt}\theta_1 + r_1 + r_2 + r_3$$

where

$$r_3 = -\int (v_{i,t} + v_j v_{i,j} + e_i - \alpha \epsilon_{ijk} v_j b_k)(J_i - \rho v_i)$$

(which vanishes if $(b, v, e)$ is a solution to the e-MHD system), and

$$L_4 = -\int \alpha \epsilon_{ijk} v_j b_k (J_i - \rho v_i)$$

$$= -\int \alpha \epsilon_{ijk} v_j b_k J_i = -\varepsilon\alpha \int \epsilon_{ijk} v_j b_k E_{i,t} - \int \epsilon_{ijk} v_j b_k \epsilon_{ipq} B_{q,p}$$

(because of (4.30))

$$= \eta_3 + \frac{d}{dt}\theta_2 + L_5$$

where

$$\eta_3 = \varepsilon\alpha \int \epsilon_{ijk} (v_j b_k)_{,t} E_i,$$

$$\theta_2 = -\varepsilon\alpha \int \epsilon_{ijk} v_j b_k E_i,$$

$$L_5 = -\int v_j b_k (B_{k,j} - B_{j,k}).$$

We have

$$L_5 = \int v_j b_{k,j} B_k - \int v_{j,k} b_k B_j$$

(since both $v$ and $b$ are divergence free)

$$= -\int v_{j,k}(b_k B_j + B_k b_j) + \int v_j (b_{k,j} - b_{j,k}) B_k$$

$$= -\int v_{j,k}(b_k B_j + B_k b_j) + \int \epsilon_{ijk}\epsilon_{ipq} b_{q,p} v_j B_k$$

$$= -\int v_{j,k}(b_k B_j + B_k b_j) + r_4,$$

where

$$r_4 = \int \epsilon_{ijk}(\epsilon_{ipq} b_{q,p} - \alpha v_i) v_j B_k$$

(and vanishes if $(b, v, e)$ is solution to the e-MHD system). So

$$L_4 = \eta_3 + \frac{d}{dt}\theta_2 - \int v_{j,k}(b_k B_j + B_k b_j) + r_4$$

and

$$L = \int v_{i,j}(v_j J_i - \rho v_j v_i - b_i B_j - B_i b_j)$$

$$+\eta_1 + \eta_2 + \eta_3 + \frac{d}{dt}(\theta_1 + \theta_2) + r_1 + r_2 + r_3 + r_4$$

It follows, using definition (4.34), that

$$L + Q = Q' + \eta_0 + \eta_1 + \eta_2 + \eta_3 + \frac{d}{dt}(\theta_0 + \theta_1 + \theta_2) + r_0 + r_1 + r_2 + r_3 + r_4,$$

where

$$Q' = \int v_{i,j}((B-b)_j (B-b)_i + \varepsilon E_j E_i - \int (\xi_i - v_i)(\xi_j - v_j) f d\xi).$$

So, we have finally obtained

$$\frac{d}{dt}(H_{b,v} - \theta) = Q' + \eta + r,$$

where

$$\eta = \eta_0 + \eta_1 + \eta_2 + \eta_3 = \varepsilon \int \epsilon_{ijk} \alpha v_{i,t} E_j B_k - \varepsilon \int E_i (e_{i,t} + (e_k v_k)_{,i} - \alpha \epsilon_{ijk}(v_j b_k), t),$$

$$\theta = \theta_0 + \theta_1 + \theta_2 = -\varepsilon\alpha \int \epsilon_{ijk} v_i E_j B_k, + \varepsilon \int E_i (e_i - \alpha \epsilon_{ijk} v_j b_k)$$

and

$$r = r_0 + r_1 + r_2 + r_3 + r_4 = -\int E_i (\epsilon_{ijk} \frac{b_{k,j}}{\alpha} - v_i) - \int (B-b)_i (b_{i,t} - \frac{\epsilon_{ijk} e_{k,j}}{\alpha})$$

$$- \int e_k (v_k - \frac{\epsilon_{kji} b_{i,j}}{\alpha}) - \int (v_{i,t} + v_j v_{i,j} + e_i - \alpha \epsilon_{ijk} v_j b_k)(J_i - \rho v_i)$$

$$+ \int \epsilon_{ijk}(\epsilon_{ipq} b_{q,p} - \alpha v_i) v_j B_k$$

$$= \int (e_i - E_i + \epsilon_{ipq} \alpha v_p B_q)(\epsilon_{ijk} \frac{b_{k,j}}{\alpha} - v_i)$$

$$- \int (B-b)_i (b_{i,t} - \frac{\epsilon_{ijk} e_{k,j}}{\alpha}) - \int (v_{i,t} + v_j v_{i,j} + e_i - \alpha \epsilon_{ijk} v_j b_k)(J_i - \rho v_i).$$

This completes the proof of Proposition 4.2.2.

## 4.3 Convergence to the Euler equation

### 4.3.1 Result

In this section, we consider the Vlasov-Maxwell system (4.1), (4.2), (4.3), (4.4), with a different scaling where both $\alpha$ and $\varepsilon$ go to zero. The limit system is now the usual Euler equations for incompressible fluids

$$\begin{gathered} \partial_t v + (v \cdot \nabla) v + e = 0 \\ \text{with } \nabla \wedge e = 0, \text{ and } \nabla \cdot v = 0. \end{gathered} \tag{4.36}$$

Let us recall that, by introducing $\omega = \nabla \wedge v$, we get the standard vorticity formulation of the Euler equations, namely :

$$\partial_t \omega + v \cdot \nabla \omega = \omega \cdot \nabla v, \text{ with } -\Delta v = \nabla \wedge \omega.$$

Let us now state our second main result

*Theorem* 4.3.1. Let $(f(t,x,\xi),(B,E)(t,x))$ be a global smooth solution to the VM system for $x \in \mathbb{R}^3$, $\xi \in \mathbb{R}^3$, $t \geq 0$, with sufficient decay of $f$ as $\xi \to \infty$. Let $(v,e)(t,x)$ be a global smooth solution to the Euler equations for $x \in \mathbb{R}^3$, $t \geq 0$, such that $v$ as zero mean on $\mathbb{R}^3$. Let $\beta(t,x)$ a smooth vector field such that

$$\nabla \wedge \beta = v, \quad \nabla . \beta = 0.$$

Let us fix $T > 0$ and assume the initial conditions of the VM system to satisfy

$$\int f(0,x,\xi) dx d\xi = 1, \tag{4.37}$$

$$\int f(0,x,\xi) |\xi|^2 dx d\xi \leq C, \tag{4.38}$$

$$\varepsilon \int |E(0,x)|^2 dx \leq (\alpha + \sqrt{\varepsilon}) C, \tag{4.39}$$

$$\int |B(0,x) - \alpha \beta(0,x)|^2 dx \leq (\alpha + \sqrt{\varepsilon}) C, \tag{4.40}$$

$$\int |\xi - v(0,x)|^2 f(0,x,\xi) dx d\xi \leq (\alpha + \sqrt{\varepsilon}) C \tag{4.41}$$

where $C$ does not depend on $\varepsilon$. Then

$$\sup_{0 \leq t \leq T} \Big( \int |\xi - v(t,x)|^2 f(t,x,\xi) d\xi dx + \int |B(t,x) - \alpha \beta(t,x)|^2 dx \Big) \leq (\alpha + \sqrt{\epsilon}) C',$$

where $C'$ depends only on $T$, $C$ and $(v,e)$.

### 4.3.2 Proof of Theorem 4.3.1

We use again Proposition 4.2.2, but with a different choice for $(v, b, e)$. We assume $(v, e)$ to be the smooth solution of the Euler equation considered in the assumptions of Theorem 4.3.1, and define

$$b(t,x) = \alpha\beta(t,x), \quad \nabla \wedge \beta = v, \quad \nabla.\beta = 0.$$

This implies that

$$\theta = \varepsilon \int E \cdot (e + \alpha v \wedge (B - \alpha\beta))$$

$$\eta = -\varepsilon \int E \cdot (\partial_t e + \nabla(e \cdot v) - \alpha^2 \partial_t (v \wedge \beta) + \alpha \partial_t v \wedge B),$$

and

$$r = -\int (B - \alpha\beta) \cdot \alpha \partial_t \beta + \int \alpha^2 v \wedge \beta \cdot (J - \rho v).$$

(using that $\nabla \wedge \beta = v$ and $(v, e)$ is a solution to the Euler equations which in particular implies $\nabla \wedge e = 0$). Thus, $\eta$, $\theta$ and $r$ are uniformly bounded by $(\alpha + \sqrt{\epsilon})C''$ for some constant $C''$ depending only on $T$, $(b, v)$ and the total energy $H$. Then, the proof of Theorem 4.3.1 immediately follows as in the previous section for Theorem 4.2.1.

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# Chapitre 5

# Quasineutral limit for the relativistic Vlasov-Maxwell system

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## 5.1 Introduction

### 5.1.1 The Vlasov-Maxwell system

The modelling of plasmas, i.e. of rarefied ionized gases, leads naturally to the Vlasov-Maxwell system :

$$\begin{aligned}
&\partial_t g + \frac{c\xi}{\sqrt{\xi^2+c^2}}\cdot\nabla_x g + \frac{Ze}{m_i}(E + \frac{c\xi}{\sqrt{\xi^2+c^2}}\wedge B)\cdot\nabla_\xi g = 0,\\
&\partial_t f + \frac{c\xi}{\sqrt{\xi^2+c^2}}\cdot\nabla_x f - \frac{e}{m_e}(E + \frac{c\xi}{\sqrt{\xi^2+c^2}}\wedge B)\cdot\nabla_\xi f = 0,\\
&\mathrm{div}_x B = 0, \quad \mathrm{div}_x E = \frac{e}{\epsilon_0}\int (Zg - f)d\xi,\\
&\partial_t B + \ \mathrm{curl}_x E = 0, \quad -\frac{1}{c^2}\partial_t E + \ \mathrm{curl}_x B = e\mu_0 \int \frac{c\xi}{\sqrt{\xi^2+c^2}}(Zg - f)d\xi,
\end{aligned} \tag{5.1}$$

where $c$ denotes the light speed, $-e$ and $Ze$ the respective charges of the electron and ion, $\mu_0$ and $\epsilon_0 = 1/(c^2\mu_0)$ the permeability and the permittivity of vacuum. In those equations, $f$ (resp. $g$) is a positive function which represents the probability for an electron (resp. for an ion) to be at time $t$ at the point $x \in \Omega \subset \mathbf{R}^3$ with the momentum $\xi \in \mathbf{R}^3$.

The first two equations, the so-called Vlasov equations, express that charged particles evolve under the electromagnetic force :

$$\begin{aligned}
\frac{dX}{dt} &= V,\\
\frac{dP}{dt} &= q(E + V \wedge B),
\end{aligned}$$

where $X$ and $P$ are the position and the momentum of the particle at time $t$, and $V = cP/\sqrt{(cm)^2 + P^2}$ its relativistic velocity.

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The Maxwell equations take into account the wave propagation of the electromagnetic field. They can be possibly replaced by the Poisson equation in electrostatic regime, but not in relativistic regimes when the particles velocities are close to the light speed.

The understanding of this model is still incomplete : as the Vlasov-Maxwell system is both hyperbolic and nonlinear, we expect that singularities arise in finite time, and then propagate with finite speed. The mathematical study of such a system requires then a notion of solution with low regularity. The notion of weak solution we will use in this paper is not completely satisfactory since it does not ensure neither the uniqueness of the solution nor the fundamental physical properties.

*Proposition* 5.1.1. Let $(f, g)$ be a smooth solution of (5.1) on $[0, T]$. Then

– the global charge and the global energy are conserved on $[0, T]$

$$\begin{aligned}
&\iint f(t,x,\xi)d\xi dx = Z\iint g(t,x,\xi)d\xi dx = \iint f_{in}(x,\xi)d\xi dx,\\
&\iint c\sqrt{c^2+\xi^2}(m_e f + m_i g)(t,x,\xi)d\xi dx + \frac{1}{2}\int(\epsilon_0|E|^2 + \frac{1}{\mu_0}|B|^2)(t,x)dx\\
&\qquad = \iint c\sqrt{c^2+\xi^2}(m_e f_{in} + m_i g_{in})(x,\xi)d\xi dx + \frac{1}{2}\int(\epsilon_0|E_{in}|^2 + \frac{1}{\mu_0}|B_{in}|^2)(x)dx
\end{aligned} \tag{5.2}$$

– the following local conservation laws hold on $[0, T] \times \Omega$

$$\begin{aligned}
&\partial_t\rho_e + \mathrm{div}_x\hat{j}_e = \partial_t\rho_i + \mathrm{div}_x\hat{j}_i = 0\\
&\partial_t j_e + \mathrm{div}_x\int f\frac{c\xi\otimes\xi}{\sqrt{c^2+\xi^2}}d\xi + \frac{e}{m_e}\rho_e E + \frac{e}{m_e}\hat{j}_e\wedge B = 0\\
&\partial_t j_i + \mathrm{div}_x\int g\frac{c\xi\otimes\xi}{\sqrt{c^2+\xi^2}}d\xi - \frac{Ze}{m_i}\rho_i E - \frac{Ze}{m_i}\hat{j}_i\wedge B = 0
\end{aligned} \tag{5.3}$$

with the notations $\rho_e = \int f d\xi$, $\rho_i = \int g d\xi$ for the densities of charges, $j_e = \int f\xi d\xi$, $j_i = \int f\xi d\xi$ for the momentum densities, and $\hat{j}_e = \int f\frac{c\xi}{\sqrt{c^2+\xi^2}}d\xi$, $\hat{j}_i = \int g\frac{c\xi}{\sqrt{c^2+\xi^2}}d\xi$ for the current densities.

### 5.1.2 Scaling for the quasineutral regime

In some regimes, we can actually obtain a correct description of the plasma behaviour through simplified models. Here we are interested in the derivation of such a model for a quasineutral plasma. This means that we consider the plasma on a space scale which is very large compared with the Debye length, which allows to neglect the plasma oscillations (created by the electric field). As a matter of fact, in order to isolate the phenomena which are directly linked with the quasineutrality constraint, we will assume in all the sequel that ions have infinite mass, and consider that the ion charge density $Z\rho_i = n(t, x)$ is fixed. To be completely rigorous, we have now to precise all the relations between the respective sizes of the various parameters.

The various scales arising in the equations are those listed below

– $T$ the observation time scale ;
– $T_o = L/c$ a characteristic time scale for the light propagation (where $L$ is the observation space scale) ;
– $T_p = \sqrt{m\epsilon_0/ne^2}$ the reciprocal plasma frequency ;
– $T_c = v^*\sqrt{ne^2\mu_0/m}$ the reciprocal gyrokinetic frequency (where $v^* = L/T$ is the characteristic bulk velocity).

The quasineutrality assumption reads

$$\frac{T_p}{T} = \epsilon << 1.$$

We assume moreover that the bulk motion is non-relativistic, i.e.

$$\frac{T_o}{T} = \gamma << 1.$$

Then we introduce a third nondimensional number

$$\alpha = \frac{T_c}{T} = \frac{\gamma}{\epsilon},$$

which measures whether or not the magnetic field is a relativistic effect.

Finally we come down to the study of the following nondimensional system

$$\begin{aligned} &\partial_t f + \frac{\xi}{\sqrt{1+\gamma^2\xi^2}}\cdot\nabla_x f - (E + \alpha\frac{\xi}{\sqrt{1+\gamma^2\xi^2}} \wedge B)\cdot\nabla_\xi f = 0, \\ &\mathrm{div}_x B = 0, \quad \epsilon^2 \mathrm{div}_x E = n - \int f d\xi, \\ &\alpha\partial_t B + \ \mathrm{curl}_x E = 0, \quad -\alpha\epsilon^2\partial_t E + \ \mathrm{curl}_x B = -\alpha \int \frac{\xi}{\sqrt{1+\gamma^2\xi^2}} f d\xi, \end{aligned} \tag{5.4}$$

for small values of the parameters $\epsilon$ and $\gamma$.

From a mathematical point of view, in order to understand the main features of the plasma behaviour for small values of $\epsilon$, we are going to study the limit $\epsilon \to 0$ : in this limiting process, some phenomena (such as oscillations) will be neglected but they should be only corrective terms.

In such a limit, we expect that the negative charge $\rho_e$ equals everywhere the positive charge $n$ (which justifies the terminolgy of quasineutral limit), with spatial oscillations on a characteristic length of order $\epsilon$.

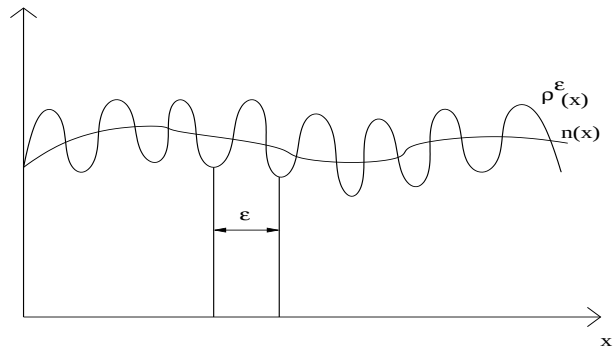


**figure 1**

The oscillations on the electron density generate of course oscillations on the electric and magnetic fields through the Maxwell equations.

The problem is to understand how these oscillations interfer, and in particular if the coupling by the nonlinear terms produces non-oscillating terms in the transport equation (which

corresponds to an energy loss or an energy winning on the non-oscillating part due to turbulent effects).

In this paper, we will actually consider only a small part of this problem, for technical reasons and especially because the asymptotic behaviour of the system depends strongly on many conditions. We start by giving our two main assumptions, and then try to explain their physical meaning as well as the phenomena we expect to be controlled or to disappear under such assumptions.

*(H1) the ion density of charge $n$ is homogeneous and constant, take for instance $n = 1$.*
This is of course a very particular case which could be extended a bit, assuming only that $n$ depends on $t$ and varies slowly with respect to $x$ (typically on lengthes of order $\epsilon^{-1}$).
If this assumption is not verified, the plasma will have a much more turbulent behaviour. Indeed, the singular perturbation

$$\mathrm{L}_n : (u, b, e) \mapsto \left( e, \frac{1}{\alpha} \operatorname{curl}_x e, -nu - \frac{1}{\alpha} \operatorname{curl}_x b \right)$$

does not preserve the Sobolev norms if $\nabla_x n \neq 0$. Then we expect that the regularity and compactness are lost instantaneously under this linear perturbation. Moreover, the kernel of such an operator $\mathrm{L}_n$ is much smaller if $n$ is not homogeneous, which means that the averaged fields (i.e. the projections of the fields on $\mathrm{Ker}(\mathrm{L}_n)$) contain few informations (see [7] for an analogous study on rotating fluids).

*(H2) the velocity profile is uniformly stable. More precisely, we assume that the initial data is almost monokinetic*

$$\iint |\xi - u^0(x)|^2 f^0(x, \xi) d\xi dx \to 0.$$

We could also weaken this assumption, and suppose that the initial density is close to any global thermodynamic equilibrium (for instance a Maxwellian with density $n$ and temperature 1). See [8] for a precise description of these equilibria.
If this assumption is not satisfied, instabilities will occur at the kinetic level (typically double-humped instabilities). These instabilities, which are still bad understood, cancel immediately the averaging process that we want to describe in this paper (see [5]).

### 5.1.3 Description of the asymptotics

Under assumptions $(H1)$ and $(H2)$, we are able to give a precise description of all oscillations, and of the possible coupling through the quadratic term of the transport equation (provided that there is no boundary).

The study of the asymptotics follows the following scheme. As the data are almost monokinetic, the density will remain close to a Dirac mass in $v$. We can prove that the moments behave as the solutions of

$$\begin{aligned}
&\rho = 1 - \epsilon \operatorname{div}_x(\epsilon E), \quad \operatorname{div}_x B = 0 \\
&\partial_t j + \operatorname{div}_x \frac{1}{\rho}(j \otimes j) + \frac{1}{\epsilon}(1 - \epsilon \operatorname{div}_x(\epsilon E))\epsilon E + \alpha j \wedge B = 0 \\
&\partial_t B + \frac{1}{\alpha\epsilon} \operatorname{curl}_x(\epsilon E) = 0 \\
&\partial_t(\epsilon E) - \frac{1}{\alpha\epsilon} \operatorname{curl}_x B = \frac{1}{\epsilon} j,
\end{aligned} \tag{5.5}$$

since the relativistic correction vanish in the limit.

The second part of the job is based on the study of the singular perturbation L

$$\mathrm{L} : (u, b, e) \mapsto \left( e, \frac{1}{\alpha}\ \mathrm{curl}_x e, -u - \frac{1}{\alpha}\ \mathrm{curl}_x b \right).$$

As all coefficients are constant, we can work with the Fourier formulation, what allows to determine easily the eigenvalues and eigenvectors of L. The limiting triplet $(u, e, b)$ belongs necessarily to Ker(L), so that

$$u = -\frac{1}{\alpha}\ \mathrm{curl}_x b, \quad e = 0.$$

Then we establish the limiting evolution equation by projecting the local conservation laws on the kernel of L. The difficulty consists in proving that there is no constructive resonance. The algebraic results we need here give conditions on the domains under which the convergence result holds.

Since there is no resonance, we recover for the mean field the limiting system involved in the case of well prepared data

$$\begin{cases} \partial_t v + v \cdot \nabla v + \underline{e} + \alpha v \wedge b = 0 \\ \partial_t b + \dfrac{1}{\alpha} \mathrm{curl}\ \underline{e} = 0 \\ nv + \dfrac{1}{\alpha}\ \mathrm{curl} b = 0 \end{cases} \tag{5.6}$$

Note that this system called the electron-MHD system had already been derived in [3, 4] in the case of well-prepared data for strong solutions of the Vlasov-Maxwell system.

### 5.1.4 Mathematical derivation

The relative entropy method used in this paper, and whose principle is described in the first paragraph of Section 4, has already allowed to solve many problems concerning the gyrokinetic and quasineutral limits in plasma physics (see [14, 3, 4, 8, 12]).

Let us recall the result previously obtained in [3, 4] for strong solutions of the non relativistic Vlasov-Maxwell system (with initial data depending only on two space coordinates with a perpendicular magnetic field) :

*Theorem* 5.1.1. [4] Let $(f(t,x,\xi), (B,E)(t,x))$ be a global smooth solution to the non relativistic Vlasov-Maxwell system for $x \in \mathbf{R}^3$, $\xi \in \mathbf{R}^3$, $t \geq 0$, with sufficient decay of $f$ as $\xi \to \infty$. Let $(v, b, e)(t, x)$ be a global smooth solution to the electron-MHD system for $x \in \mathbf{R}^3$, $t \geq 0$. Let us fix $T > 0$ and $\alpha > 0$ and assume the initial conditions of the non relativistic Vlasov-Maxwell system to satisfy

$$\int f(0,x,\xi) dx d\xi = 1, \quad \int f(0,x,\xi) |\xi|^2 dx d\xi \leq C,$$

$$\epsilon \int |E(0,x)|^2 dx \leq C\sqrt{\epsilon}, \quad \int |B(0,x) - b(0,x)|^2 dx \leq C\sqrt{\epsilon},$$

$$\int |\xi - v(0,x)|^2 f(0,x,\xi) dx d\xi \leq C\sqrt{\epsilon},$$

where $C$ does not depend on $\epsilon$. Then

$$\sup_{0 \leq t \leq T} \left( \int |\xi - v(t,x)|^2 f(t,x,\xi) dx d\xi + \int |B(t,x) - b(t,x)|^2 dx \right) \leq C'\sqrt{\epsilon},$$

where $C'$ depends only on $T$, $C$ and $(v, b, e)$.

Some questions remained nevertheless still open, essentially because of a lack of understanding of the Vlasov-Maxwell system. In particular,

- the global conservation of energy and the local conservation of momentum are not ensured for weak solutions of (5.4), i.e. for the only solutions of (5.4) that are known to exist globally in time;
- the relativistic correction has to be taken into account in the initial model (5.4) except in electrostatic regimes, but it does not appear in the limiting system;
- the oscillations generated by the Maxwell equations have a much more complicated structure than the ones created by the coupling with the Poisson equation.

The various restrictions imposed in [14, 3, 4, 8] concerning these three points are relaxed here, by analyzing precisely the analogy between the initial system (5.4) and the limiting system.

## 5.2 Main results

### 5.2.1 Weak solutions for the Vlasov-Maxwell system

We start by giving the mathematical framework of our study. We consider the spatial domain

$$\Omega = \frac{\mathbf{R}}{a_1\mathbf{Z}} \times \frac{\mathbf{R}}{a_2\mathbf{Z}} \times \frac{\mathbf{R}}{a_3\mathbf{Z}}$$

in order to avoid additional difficulties linked with the modelling of the boundary, in particular the appearance of boundary layers. We recall that the Cauchy problem (5.4) with initial data

$$f_{|t=0} = f_{in}, \quad E_{|t=0} = E_{in}, \quad B_{|t=0} = B_{in}, \tag{5.7}$$

satisfying the standard compatibility conditions

$$\epsilon^2 \mathrm{div}_x E_{in} = 1 - \int f_{in}(x,\xi)d\xi, \quad \mathrm{div}_x B_{in} = 0, \tag{5.8}$$

and the energy bound

$$\mathcal{E}_{in} = \frac{1}{\gamma^2}\iint \sqrt{1+\gamma^2\xi^2} f_{in} d\xi dx + \frac{1}{2}\int (\epsilon^2|E_{in}|^2 + |B_{in}|^2)dx < +\infty, \tag{5.9}$$

is not known (nor expected) to have global strong solutions. Then we will work with a very weak notion of solution.

A *weak solution* of (5.4–5.7) is a triplet

$$\begin{aligned}(f,E,B) \in L^\infty\big(\mathbf{R}^+, L^2(\Omega\times\mathbf{R}^3,\mathbf{R}^+)\times(L^2(\Omega))^2\big)\\ \cap C\big(\mathbf{R}^+; w\text{-}L^2(\Omega\times\mathbf{R}^3)\times(w\text{-}L^2(\Omega))^2\big)\end{aligned}$$

which satisfies

$$\frac{1}{\gamma^2}\iint f(t)\sqrt{1+\gamma^2\xi^2}d\xi dx + \frac{1}{2}\int(\epsilon^2|E(t)|^2+|B(t)|^2)dx \le \mathcal{E}_{in}\,, \quad \forall t>0\,, \tag{5.10}$$

and finally satisfies (5.4,5.7) in the sense of distributions.

The global existence of such weak solutions, as well as the local conservation of mass and the energy decay are established in [5]. Whether the local conservation of momentum holds in the sense of distributions on $\mathbf{R}_+^*\times\Omega$ is still unknown; this is one of the difficulties in

rigorously deriving hydrodynamic models from the Vlasov-Maxwell system. The construction of [5] yields actually a solution which satisfies in addition a conservation law for the momentum and a global energy equality with defect measures coming from a possible lack of compactness of the sequence of approximating solutions.

*Theorem* 5.2.1. For fixed $\epsilon, \alpha > 0$, let $f_{in}^{\epsilon,\alpha} \in L^1 \cap L^2(\Omega \times \mathbf{R}^3)$ be an ae. nonnegative function, and $E_{in}^{\epsilon,\alpha}, B_{in}^{\epsilon,\alpha} \in L^2(\Omega)$ be two vector fields satisfying the compatibility conditions (5.8) and the energy bound (5.9). Then there exists a weak solution $(f^{\epsilon,\alpha}, E^{\epsilon,\alpha}, B^{\epsilon,\alpha})$ to (5.4,5.7) which satisfies

– the local conservation of mass in the sense of distributions on $\mathbf{R}_+^* \times \Omega$

$$\partial_t \int f^{\epsilon,\alpha} d\xi + \mathrm{div}_x \int \frac{\xi}{\sqrt{1+\gamma^2\xi^2}} f^{\epsilon,\alpha} d\xi = 0\,, \tag{5.11}$$

– the local conservation of momentum with symmetric nonnegative matrix-valued defect measures $\mu_E, \mu_B \in L^\infty(\mathbf{R}^+, \mathcal{M}(\Omega))$, a vector-valued defect measure $\mu_{EB} \in L^\infty(\mathbf{R}^+, \mathcal{M}(\Omega))$, and a defect measure $m^{\epsilon,\alpha} \in L^\infty(\mathbf{R}^+, \mathcal{M}(\Omega \times S^2))$ (coming from the approximation scheme of the Vlasov-Maxwell equations)

$$\begin{aligned}
&\partial_t \left( \int \xi f^{\epsilon,\alpha}(t,x,\xi) d\xi + \gamma \int_{\sigma \in S^2} \sigma dm^{\epsilon,\alpha}(t,x,\sigma) \right) \\
&+\mathrm{div}_x \left( \int \frac{\xi \otimes \xi}{\sqrt{1+\gamma^2\xi^2}} f^{\epsilon,\alpha}(t,x,\xi) d\xi + \int_{\sigma \in S^2} \sigma \otimes \sigma dm^{\epsilon,\alpha}(t,x,\sigma) \right) \\
&+nE^{\epsilon,\alpha} - \mathrm{div}_x(\epsilon E^{\epsilon,\alpha} \otimes \epsilon E^{\epsilon,\alpha} - \frac{1}{2}|\epsilon E^{\epsilon,\alpha}|^2 Id) - \mathrm{div}_x(\mu_E^{\epsilon,\alpha} - \frac{1}{2} tr(\mu_E^{\epsilon,\alpha}) Id) \\
&-\mathrm{div}_x(B^{\epsilon,\alpha} \otimes B^{\epsilon,\alpha} - \frac{1}{2}|B^{\epsilon,\alpha}|^2 Id) - \mathrm{div}_x(\mu_B^{\epsilon,\alpha} - tr(\mu_B^{\epsilon,\alpha}) Id) \\
&+\epsilon\alpha \partial_t (\epsilon E^{\epsilon,\alpha} \wedge B^{\epsilon,\alpha}) + \epsilon\alpha \partial_t \mu_{EB}^{\epsilon,\alpha} = 0,
\end{aligned} \tag{5.12}$$

in the sense of distributions on $\mathbf{R}^+ \times \Omega$,

– the global conservation of energy with the defect measures $\mu_E^{\epsilon,\alpha}$, $\mu_B^{\epsilon,\alpha}$ and $m^{\epsilon,\alpha}$ for all $t > 0$

$$\begin{aligned}
&\frac{1}{\gamma^2} \iint \sqrt{1+\gamma^2\xi^2} f^{\epsilon,\alpha}(t,x,\xi) d\xi dx + \int\int_{\sigma \in S^2} dm^{\epsilon,\alpha}(t,x,\sigma) \\
&+\frac{1}{2} \int (\epsilon^2 |E^{\epsilon,\alpha}(t,x)|^2 + |B^{\epsilon,\alpha}(t,x)|^2) dx + \frac{1}{2} \int (tr(d\mu_E^{\epsilon,\alpha}) + tr(d\mu_B^{\epsilon,\alpha})) = \mathcal{E}_{in}^{\epsilon,\alpha}
\end{aligned} \tag{5.13}$$

as well as the inequality

$$\int d\mu_{EB}^{\epsilon,\alpha} g \le \frac{1}{2} \int g(tr(d\mu_E^{\epsilon,\alpha}) + tr(d\mu_B^{\epsilon,\alpha})), \tag{5.14}$$

for all positive test function $g(t,x)$.

### 5.2.2 Strong convergence for well-prepared initial data

In the case where the initial data are well prepared, which means that the initial velocity is essentially monokinetic, the initial fields converge strongly and the limiting fields belong to the kernel of the singular perturbation, we obtain the following strong convergence result.

*Theorem* 5.2.2. – Let $v_0, b_0$ be two divergence-free vector fields of $H^s(\Omega)$ $(s > 2 + \frac{3}{2})$ such that

$$nv_0 + \frac{1}{\alpha} \mathrm{curl}_x b_0 = 0,$$

for some homogeneous $n$. Then, there exist $T^* \in ]0, +\infty]$ and a unique $(v, b) \in L^\infty_{loc}([0, T^*[, H^s(\Omega))$ solution of

$$\begin{cases} \partial_t v + v \cdot \nabla v + \underline{e} + \alpha v \wedge b = 0 \\ \partial_t b + \dfrac{1}{\alpha} \operatorname{curl}_x \underline{e} = 0 \\ nv + \dfrac{1}{\alpha} \operatorname{curl}_x b = 0. \end{cases} \tag{5.15}$$

– Let $(f^{\epsilon,\alpha}_{in})_{\epsilon,\alpha}$ be a family of nonnegative functions of $L^1 \cap L^2(\Omega \times \mathbf{R}^3)$, and $(E^{\epsilon,\alpha}_{in})_{\epsilon,\alpha}, (B^{\epsilon,\alpha}_{in})_{\epsilon,\alpha}$ be two families of vector fields of $L^2(\Omega)$ satisfying the compatibility conditions (5.8) for the same constant $n$, as well as the convergence

$$\iint \frac{(\xi - v_0)^2}{1 + \sqrt{1 + \gamma^2 \xi^2}} f^{\epsilon,\alpha}_{in}(x, \xi) d\xi dx + \frac{1}{2} \int \epsilon^2 |E^{\epsilon,\alpha}_{in}(x)|^2 dx + \frac{1}{2} \int |B^{\epsilon,\alpha}_{in}(x) - b_0|^2 dx \to 0 \text{ as } \epsilon \to 0.$$

Let $(f^{\epsilon,\alpha}, E^{\epsilon,\alpha}, B^{\epsilon,\alpha})$ be, for every $\epsilon > 0, \alpha > 0$, a solution of the scaled Vlasov-Maxwell system (5.4) with initial condition (5.7). Then, for all $T < T^*$, the current density converges weakly

$$\hat{j}^{\epsilon,\alpha} = \int f^{\epsilon,\alpha} \xi / \sqrt{1 + \gamma^2 \xi^2} d\xi \rightharpoonup nv \text{ in } L^\infty([0, T], L^1(\Omega)),$$

the scaled electric field and the magnetic field converge strongly

$$\epsilon E^{\epsilon,\alpha} \to 0 \text{ and } B^{\epsilon,\alpha} \to b \text{ in } L^\infty([0, T], (L^2(\Omega))^2)$$

as $\epsilon \to 0, \alpha \to \overline{\alpha} < +\infty$, where $(v, b) \in L^\infty([0, T], H^s(\Omega))$ is the local strong solution of (5.15) with initial condition $(v_0, b_0)$.

### 5.2.3 Weak convergence for general initial data

In the case where we do not assume that the limit of the initial fields belong to the kernel of the singular perturbation, we obtain a weak convergence result.

*Theorem* 5.2.3. Let $(f^{\epsilon,\alpha}, E^{\epsilon,\alpha}, B^{\epsilon,\alpha})$ be, for every $\epsilon > 0, \alpha > 0$, a solution of the scaled Vlasov-Maxwell system (5.4) with initial condition (5.7) satisfying

$$\iint \frac{(\xi - v_0)^2}{1 + \sqrt{1 + \gamma^2 \xi^2}} f^{\epsilon,\alpha}_{in}(x, \xi) d\xi dx + \frac{1}{2} \int \epsilon^2 |E^{\epsilon,\alpha}_{in}(x)|^2 dx + \frac{1}{2} \int |B^{\epsilon,\alpha}_{in}(x) - b_0|^2 dx \to 0 \text{ as } \epsilon \to 0$$

for some sufficiently smooth vector fields $(v_0, e_0, b_0)$ such that $\operatorname{div}_x b_o = 0$.

Then, for all $T < T^*$, the current density

$$\hat{j}^{\epsilon,\alpha} = \int f^{\epsilon,\alpha} \xi / \sqrt{1 + \gamma^2 \xi^2} d\xi \rightharpoonup nv \text{ in the sense of measures,}$$

the scaled electric field and the magnetic field converge weakly

$$\epsilon E^{\epsilon,\alpha} \rightharpoonup 0 \text{ and } B^{\epsilon,\alpha} \rightharpoonup b \text{ in } L^\infty_{Loc}([0, T[, (L^2(\Omega))^2)$$

as $\epsilon \to 0, \alpha \to \overline{\alpha} < +\infty$, where $(v, b) \in L^\infty([0, T], H^s(\Omega))$ is the local strong solution of (5.15) with initial condition

$$\begin{cases} v_{in} = -\Delta_x (\alpha^2 \mathrm{Id} - \Delta_x)^{-1} v_0 + \alpha \operatorname{curl}_x (\alpha^2 \mathrm{Id} - \Delta_x)^{-1} b_0 \\ \\ b_{in} = -\alpha (\alpha^2 \mathrm{Id} - \Delta_x)^{-1} \operatorname{curl}_x v_0 + \alpha^2 (\alpha^2 \mathrm{Id} - \Delta_x)^{-1} b_0 \end{cases}$$

which is the projection of $(v_0, e_0, b_0)$ on the kernel of the operator L.

The proof of this theorem involves an oscillating quantity which describes the defect of strong convergence of the fields due to the more general initial data.

## 5.3 Refined theory for the Vlasov-Maxwell system

The aim of this section is to recall the main arguments used by DiPerna and Lions to prove the existence of renormalized solutions for the Vlasov-Maxwell system (5.4), and to show that their construction actually yields a solution which satisfies in addition a conservation law for the momentum (5.12) and a global energy equality (5.13) with defect measures. For the sake of simplicity, in this section, we will fix $\epsilon, \alpha > 0$ and drop the indices.

### 5.3.1 A priori estimates

Consider a sequence $(f^n, E^n, B^n)$ of approximate solutions of (5.4,5.7), for instance

$$\begin{aligned}
&\partial_t f^n + \frac{\xi}{\sqrt{1+\gamma^2\xi^2}}\cdot\nabla_x f^n - (E^{n-1} + \alpha\frac{\xi}{\sqrt{1+\gamma^2\xi^2}} \wedge B^{n-1})\cdot\nabla_\xi f^n = 0,\\
&\mathrm{div}_x B^n = 0, \quad \epsilon^2 \mathrm{div}_x E^n = n - \int f^n d\xi,\\
&\alpha\partial_t B^n + \ \mathrm{curl}_x E^n = 0, \quad -\alpha\epsilon^2\partial_t E^n + \ \mathrm{curl}_x B^n = -\alpha\int \frac{\xi}{\sqrt{1+\gamma^2\xi^2}} f^n d\xi,\\
&f^n_{|t=0} = f_{in}\,, \quad E^n_{|t=0} = E_{in} \text{ and } \ B^n_{|t=0} = B_{in}.
\end{aligned}$$

We have the following energy estimate

$$\sup_{t\in[0,T], n\in\mathbf{N}} \frac{1}{\gamma^2}\iint \sqrt{1+\gamma^2\xi^2} f^n(t,x,\xi)d\xi + \iint (\epsilon^2|E^n|^2(t,x) + |B^n|^2(t,x))dx < +\infty,$$

as well as the conservation of $L^p$ norms for $p \in [1,\infty]$

$$\sup_{t\in\mathbf{R}^+, n\in\mathbf{N}} \iint |f^n|^p(t,x,\xi)dxd\xi < +\infty.$$

Then, up to extraction of a subsequence (still denoted by $(f^n, E^n, B^n)$ for simplicity),

$$\begin{aligned}
f^n &\rightharpoonup f \text{ in } w^* - L^\infty(\mathbf{R}^+, L^2(\Omega\times\mathbf{R}^3)),\\
\epsilon E^n &\rightharpoonup \epsilon E \text{ in } w^* - L^\infty(\mathbf{R}^+, L^2(\Omega)),\\
B^n &\rightharpoonup B \text{ in } w^* - L^\infty(\mathbf{R}^+, L^2(\Omega)).
\end{aligned} \tag{5.16}$$

Of course, as the previous convergences hold only in weak spaces, we do not have

$$\|\epsilon E^n\|^2_{L^2(\Omega)} \to \|\epsilon E\|^2_{L^2(\Omega)}, \quad \|B^n\|^2_{L^2(\Omega)} \to \|B\|^2_{L^2(\Omega)},$$

but there exist matrix-valued defect measures $\mu_E$, $\mu_B$, a vector-valued defect measure $\mu_{EB}$ and a subsequence $(E^n, B^n)$ such that, for all $\varphi \in C^0(\Omega)$ and for all $i,j \in \{1,2,3\}$

$$\begin{aligned}
\int \epsilon^2 E^n_i E^n_j \varphi(x)dx &\to \int \epsilon^2 E_i E_j \varphi(x)dx + \int \varphi d(\mu_E)_{ij},\\
\int B^n_i B^n_j \varphi(x)dx &\to \int B_i B_j \varphi(x)dx + \int \varphi d(\mu_B)_{ij},\\
\int \epsilon E^n \wedge B^n \varphi(x)dx &\to \int \epsilon E \wedge B \varphi(x)dx + \int \varphi d(\mu_{EB}).
\end{aligned} \tag{5.17}$$

It is easy to check that $\mu_E$ and $\mu_B$ are nonnegative symmetric matrices. Moreover, from the Cauchy-Schwarz inequality, we deduce that

$$\int d\mu_{EB} \leq \frac{1}{2} \int tr(d\mu_E) + tr(d\mu_B).$$

On the other hand, the sequence $(f^n|\xi|)$ is bounded in $L^\infty(\mathbf{R}^+, \mathcal{M}(\Omega \times \mathbf{R}^3))$. Thus the sequence

$$\nu_n = \int_0^\infty r f^n(t,x,r\sigma) r^2 dr$$

of push-forwards of $f^n$ under the map $(t,x,\xi) \mapsto (t,x,\xi/|\xi|)$ is bounded in $L^\infty(\mathbf{R}^+, \mathcal{M}(\Omega \times S^2))$. Hence there exists a subsequence of $(f^n)$ such that $\nu^n$ converges to $\nu$ in $L^\infty(\mathbf{R}^+, \mathcal{M}(\Omega \times S^2))$ weak-*. We next define the defect measure associated to the subsequence $(f^n)$ by :

$$\iiint \psi(t,x,\sigma) dm(t,x,\sigma) = \iiint \psi(t,x,\sigma) d\nu(t,x,\sigma) - \iiint \psi(t,x,\frac{\xi}{|\xi|})|\xi| f(t,x,\xi) dt dx d\xi$$

for every $\psi \in C_c^0(\mathbf{R}^+ \times \Omega \times S^2)$. It is easy to check that $m$ is a positive measure.

Finally we are able to characterize the convergence defects of the approximation sequence through the various measures introduced above (for a more detailed review on these questions, we refer to [11]). This is the key argument which allows to obtain a conservation law for the momentum and a global energy equality, provided that we get a convenient formulation of these conservations for approximate solutions.

### 5.3.2 The local conservation of momentum

We have ssen that the following conservation law for the momentum holds formally :

$$\partial_t \int \xi f d\xi + \mathrm{div}_x \left( \int \frac{\xi \otimes \xi}{\sqrt{1+\gamma^2\xi^2}} f d\xi \right) + \rho E + \alpha \hat{j} \wedge B = 0.$$

In particular approximate solutions can be assumed sufficiently smooth to satisfy such a conservation. Nevertheless the weak compactness inferred from the a priori estimates does not allow to take limits in the quadratic terms ; the a priori bounds do not even allow to give sense to such terms since we have

$$\rho, \hat{j} \in L^\infty([0,T], L^1 \cap L^{4/3}(\Omega)),$$
$$E, B \in L^\infty([0,T], L^2(\Omega)).$$

The following formal computations (performed on approximate solutions) provide an equivalent form for the momentum conservation, which seems to be more adapted for our job. From Maxwell's equations we deduce that

$$\begin{aligned} \rho E = &\quad (n - \epsilon^2 \mathrm{div}_x E) E \\ \alpha \hat{j} \wedge B = &\quad (\alpha \epsilon^2 \partial_t E - \ \mathrm{curl}_x B) \wedge B \end{aligned}$$

At this point, we use the identity

$$u \mathrm{div}_x u = \mathrm{div}_x (u \otimes u) + u \wedge \ \mathrm{curl}_x u - \nabla_x \frac{|u|^2}{2},$$

which implies in particular

$$\begin{aligned}(\epsilon^2 \mathrm{div}_x E)E = &\quad \mathrm{div}_x\left((\epsilon E)^{\otimes 2} - \frac{1}{2}|\epsilon E|^2 Id\right) + \epsilon E \wedge \ \mathrm{curl}_x(\epsilon E)\\ = &\quad \mathrm{div}_x\left((\epsilon E)^{\otimes 2} - \frac{1}{2}|\epsilon E|^2 Id\right) + \epsilon E \wedge (-\alpha\epsilon\partial_t B)\end{aligned}$$

and

$$\mathrm{curl}_x B \wedge B = \mathrm{div}_x\left(B^{\otimes 2} - \frac{1}{2}|B|^2 Id\right)$$

Replacing in the previous identities, we obtain

$$\begin{aligned}\rho E + \alpha\hat{j} \wedge B = &\quad nE - \mathrm{div}_x\left((\epsilon E)^{\otimes 2} - \frac{1}{2}|\epsilon E|^2 Id\right)\\ &-\mathrm{div}_x\left(B^{\otimes 2} - \frac{1}{2}|B|^2 Id\right) + \epsilon\alpha\partial_t(\epsilon E \wedge B).\end{aligned}$$

Equipped with this identity, we can rewrite formally the local conservation of momentum

$$\begin{aligned}\partial_t \int \xi f d\xi + \mathrm{div}_x\left(\int \frac{\xi\otimes\xi}{\sqrt{1+\gamma^2\xi^2}} f d\xi\right) + nE - \mathrm{div}_x\left((\epsilon E)^{\otimes 2} - \frac{1}{2}|\epsilon E|^2 Id\right)\\ -\mathrm{div}_x\left(B^{\otimes 2} - \frac{1}{2}|B|^2 Id\right) + \epsilon\alpha\partial_t(\epsilon E \wedge B) = 0.\end{aligned} \tag{5.18}$$

### 5.3.3 Passage to the limit

Formally we have the local conservation of momentum (5.18) and the energy equality

$$\frac{1}{\gamma^2}\iint \frac{\xi^2}{\sqrt{1+\gamma^2\xi^2}} f d\xi dx + \frac{1}{2}\int(\epsilon^2|E|^2 + |B|^2)dx = \mathcal{E}_{in}$$

for all $t \in \mathbf{R}^+$, so that analogous statements hold rigorously for smooth approximate solutions. Taking limits in both equations and using the defect measures introduced in the previous paragraph, we get that the weak solutions constructed by DiPerna and Lions [5] satisfy (5.12) and (5.13).

## 5.4 The modulated energy method

### 5.4.1 Description of the method

The main results of this paper are obtained by a classical energy method. The principle is to modulate the energy of the system by test functions, and to obtain a stability inequality when these modulation functions are solutions of the limiting system.

The first step (paragraph 4.2) consists in computing the variation in time of the modulated energy defined as follows,

$$H_U^{\epsilon,\alpha}(t) = \iint \frac{|\xi - v|^2}{1+\sqrt{1+\gamma^2\xi^2}} f^{\epsilon,\alpha}(t,x,\xi)d\xi dx + \frac{1}{2}\int |\epsilon E^{\epsilon,\alpha} - e|^2 dx + \frac{1}{2}\int |B^{\epsilon,\alpha} - b|^2 dx. \tag{5.19}$$

for every test function $U = (v, e, b) \in C_c^\infty(\mathbf{R}^+ \times \Omega, (\mathbf{R}^3)^3)$ such that $\mathrm{div}_x b = 0$. Note that, in order to get a quantity which is uniformly bounded in the nonrelativistic limit, we do not

work with the energy but with a Lyapunov functional which is computed from the energy and the global mass

$$\begin{aligned} H^{\epsilon,\alpha}(t) = &\ \mathcal{E}^{\epsilon,\alpha}(t) - \frac{1}{\gamma^2}\iint f^{\epsilon,\alpha}(t,x,\xi)d\xi \\ = &\ \iint \frac{\xi^2}{1+\sqrt{1+\gamma^2\xi^2}} f^{\epsilon,\alpha}(t,x,\xi)d\xi dx + \frac{1}{2}\int |\epsilon E^{\epsilon,\alpha} - e|^2 dx + \frac{1}{2}\int |B^{\epsilon,\alpha} - b|^2 dx. \end{aligned}$$

Moreover, as we expect the limiting flow to be monokinetic and nonrelativistic, we just introduce a slight perturbation in the kinetic energy.

The second step (paragraph 4.3) consists in establishing a bound on this quantity involving an acceleration operator that is small when $(v,e,b)$ is a solution of

$$\begin{aligned} \partial_t v + v\cdot\nabla v + \frac{e}{\epsilon} + \alpha v \wedge b = 0 \\ \partial_t b + \frac{1}{\alpha\epsilon}\text{curl } e = 0 \\ \partial_t e - \frac{1}{\alpha\epsilon}\text{curl } b - n\frac{v}{\epsilon} + v\text{div } e = 0 \\ \text{div } b = 0. \end{aligned} \tag{5.20}$$

The last step in the convergence proof is then to study the asymptotic behaviour of the solutions of (5.20) as $\epsilon \to 0$ (depending on the size of $\alpha$). For some particular data, the so-called well-prepared data, the singular perturbation has little effect on the system, no oscillatory behaviour occurs and we will easily prove a strong convergence result (section 5). For general initial data, we have to describe precisely the oscillations and to study their coupling (section 6) in order to characterize the asymptotic behaviour (section 7).

### 5.4.2 Derivation of the modulated energy

*Proposition* 5.4.1. The modulated energy $H_U^{\epsilon,\alpha}(t)$ satisfies

$$\begin{aligned} H_U^{\epsilon,\alpha}(t) - H_U^{\epsilon,\alpha}(0) &+ \iint dm^{\epsilon,\alpha}(t,x,\sigma) + \frac{1}{2}\int tr(d\mu_E^{\epsilon,\alpha}) + tr(d\mu_B^{\epsilon,\alpha}) \\ &= \int_0^t \int D_x v : (h_U^{\epsilon,\alpha}(s,x)dx + d\nu^{\epsilon,\alpha}(s,x))ds \\ &+ \int A^{\epsilon,\alpha}(U)\cdot \begin{pmatrix} \rho^{\epsilon,\alpha}v - \hat{j}^{\epsilon,\alpha} \\ b - B^{\epsilon,\alpha} \\ e - \epsilon E^{\epsilon,\alpha} \end{pmatrix} dxds + R^{\epsilon,\alpha}(t) \end{aligned} \tag{5.21}$$

where the flux terms $h_U^{\epsilon,\alpha}$ and $\nu^{\epsilon,\alpha}$ are defined by

$$\begin{aligned} h_U^{\epsilon,\alpha} = &-\int f^{\epsilon,\alpha}\frac{(\xi-v)\otimes(\xi-v)}{\sqrt{1+\gamma^2\xi^2}}d\xi \\ &+[(\epsilon E^{\epsilon,\alpha} - e)\otimes(\epsilon E^{\epsilon,\alpha} - e) - \frac{1}{2}|\epsilon E^{\epsilon,\alpha} - e|^2 Id] \\ &+[(B^{\epsilon,\alpha} - b)\otimes(B^{\epsilon,\alpha} - b) - \frac{1}{2}|B^{\epsilon,\alpha} - b|^2 Id], \end{aligned}$$

$$\begin{aligned} d\nu^{\epsilon,\alpha} = &-\int_{\sigma\in S^2} \sigma\otimes\sigma dm^{\epsilon,\alpha} \\ &+[d\mu_E^{\epsilon,\alpha} - \frac{1}{2}tr(d\mu_E^{\epsilon,\alpha})Id] \\ &+[d\mu_B^{\epsilon,\alpha} - \frac{1}{2}tr(d\mu_B^{\epsilon,\alpha})Id] \end{aligned}$$

the acceleration term $A^{\epsilon,\alpha}(U)$ is given by

$$A^{\epsilon,\alpha}(U)=\left(\begin{array}{l}\partial_t v+v\cdot\nabla v+\dfrac{e}{\epsilon}+\alpha v\wedge b\\ \partial_t b+\dfrac{1}{\alpha\epsilon}\text{curl } e+(\alpha\epsilon\partial_t e-\text{curl } b)\wedge v\\ \partial_t e-\dfrac{1}{\alpha\epsilon}\text{curl } b-n\dfrac{v}{\epsilon}+v\text{div } e-(\alpha\epsilon\partial_t b+\text{curl } e)\wedge v\end{array}\right)$$

and all the correction terms are grouped in a remainder $R^{\epsilon,\alpha}$ as follows

$$\begin{aligned}R^{\epsilon,\alpha}(t)=&-\left[\iint v\xi(\frac{2}{1+\sqrt{1+\gamma^2\xi^2}}-1)f^{\epsilon,\alpha}d\xi dx\right]_0^t+\left[\iint v^2(\frac{1}{1+\sqrt{1+\gamma^2\xi^2}}-\frac{1}{2})f^{\epsilon,\alpha}d\xi dx\right]_0^t\\ &+\int_0^t\iint D_x v f^{\epsilon,\alpha}v\otimes v(\frac{1}{\sqrt{1+\gamma^2\xi^2}}-1)d\xi dxds+\int_0^t\iint\xi(-1+\frac{1}{\sqrt{1+\gamma^2\xi^2}})f^{\epsilon,\alpha}\partial_t v d\xi dxds\\ &+\left[\int\alpha\epsilon v\cdot\left[(\epsilon E^{\epsilon,\alpha}-e)\wedge(B^{\epsilon,\alpha}-b)dx+d\mu_{EB}^{\epsilon,\alpha}\right]\right]_0^t\\ &-\int_0^t\int\alpha\epsilon\partial_t v\cdot\left[(\epsilon E^{\epsilon,\alpha}-e)\wedge(B^{\epsilon,\alpha}-b)dx+d\mu_{EB}^{\epsilon,\alpha}\right]dxds\\ &+\gamma\left[\iint v.\sigma dm^{\epsilon,\alpha}\right]_0^t-\gamma\int_0^t\iint\partial_t v.\sigma dm^{\epsilon,\alpha}-\int_0^t\iint D_x v:\sigma\otimes\sigma dm^{\epsilon,\alpha}.\end{aligned}$$

*Démonstration.* The previous identity is obtained by a direct computation of the time derivative of the modulated energy. We first rewrite

$$\begin{aligned}H_U^{\epsilon,\alpha}(t)&=\iint\frac{\xi^2}{1+\sqrt{1+\gamma^2\xi^2}}f^{\epsilon,\alpha}d\xi dx-\iint\frac{2v\xi}{1+\sqrt{1+\gamma^2\xi^2}}f^{\epsilon,\alpha}d\xi dx\\ &\quad+\iint\frac{v^2}{1+\sqrt{1+\gamma^2\xi^2}}f^{\epsilon,\alpha}d\xi dx+\frac{1}{2}\int(|\epsilon E^{\epsilon,\alpha}-e|^2+|B^{\epsilon,\alpha}-b|^2)dx\\ &=H^{\epsilon,\alpha}(t)-\iint\frac{2v\xi}{1+\sqrt{1+\gamma^2\xi^2}}f^{\epsilon,\alpha}d\xi dx\\ &\quad+\iint\frac{v^2}{1+\sqrt{1+\gamma^2\xi^2}}f^{\epsilon,\alpha}d\xi dx+\frac{1}{2}\int((e^2-2\epsilon E^{\epsilon,\alpha}.e)+(b^2-2B^{\epsilon,\alpha}.b))dx\end{aligned}$$

By (5.13), it follows that

$$\begin{aligned}H_U^{\epsilon,\alpha}(t)+&\iint dm^{\epsilon,\alpha}(t,x,\sigma)+\frac{1}{2}\int(tr(d\mu_E^{\epsilon,\alpha}+tr(d\mu_B^{\epsilon,\alpha}))-H_U^{\epsilon,\alpha}(0)\\ &=-\int_0^t\frac{d}{dt}\iint v\xi f^{\epsilon,\alpha}d\xi dxds+\int_0^t\frac{d}{dt}\iint\frac{v^2}{2}f^{\epsilon,\alpha}d\xi dxds\\ &\quad-\int_0^t\frac{d}{dt}\iint v\xi(\frac{2}{1+\sqrt{1+\gamma^2\xi^2}}-1)f^{\epsilon,\alpha}d\xi dxds\\ &\quad+\int_0^t\frac{d}{dt}\iint v^2(\frac{1}{1+\sqrt{1+\gamma^2\xi^2}}-\frac{1}{2})f^{\epsilon,\alpha}d\xi dxds\\ &\quad+\int_0^t\frac{d}{dt}\int(-\epsilon E^{\epsilon,\alpha}e+\frac{e^2}{2}-B^{\epsilon,\alpha}b+\frac{b^2}{2})dxds.\end{aligned}$$

Denote by $C_1$ the first part of the remainder

$$C_1=-\left[\iint v\xi(\frac{2}{1+\sqrt{1+\gamma^2\xi^2}}-1)f^{\epsilon,\alpha}d\xi dx+\iint v^2(\frac{1}{1+\sqrt{1+\gamma^2\xi^2}}-\frac{1}{2})f^{\epsilon,\alpha}d\xi dx\right]_0^t$$

Recall that $\rho^{\epsilon,\alpha} = \int f^{\epsilon,\alpha} d\xi$, $j^{\epsilon,\alpha} = \int f^{\epsilon,\alpha} \xi d\xi$ and $\hat{j}^{\epsilon,\alpha} = \int f^{\epsilon,\alpha} \xi / \sqrt{1+\gamma^2\xi^2} d\xi$. Then,

$$\begin{aligned} H_U^{\epsilon,\alpha}(t) + & \iint dm^{\epsilon,\alpha}(t,x,\sigma) + \frac{1}{2}\int (tr(d\mu_E^{\epsilon,\alpha} + tr(d\mu_B^{\epsilon,\alpha})) - H_U^{\epsilon,\alpha}(0) \\ = & -\int_0^t \int (j^{\epsilon,\alpha}\partial_t v + v\partial_t j^{\epsilon,\alpha}) dxds + \int_0^t \int (\rho^{\epsilon,\alpha} v\partial_t v + \frac{v^2}{2}\partial_t \rho^{\epsilon,\alpha}) dxds + C_1 \\ & + \int_0^t \int (\partial_t e(e - \epsilon E^{\epsilon,\alpha}) + \partial_t b(b - B^{\epsilon,\alpha})) dxds - \int_0^t (e\partial_t(\epsilon E^{\epsilon,\alpha}) + b\partial_t B^{\epsilon,\alpha}) dxds. \end{aligned}$$

From the local conservations of mass and momentum (5.11,5.12), we deduce that

$$\begin{aligned} H_U^{\epsilon,\alpha}(t) + & \iint dm^{\epsilon,\alpha}(t,x,\sigma) + \frac{1}{2}\int (tr(d\mu_E^{\epsilon,\alpha} + tr(d\mu_B^{\epsilon,\alpha})) - H_U^{\epsilon,\alpha}(0) \\ = & -\int_0^t \int \hat{j}^{\epsilon,\alpha}\partial_t v dxds + \int_0^t \int (\rho^{\epsilon,\alpha} v\partial_t v - \frac{v^2}{2}\mathrm{div}_x \hat{j}^{\epsilon,\alpha}) dxds + C_2 + C_1 \\ & + \int_0^t \int v \left( \mathrm{div}_x \int \frac{\xi\otimes\xi}{\sqrt{1+\gamma^2\xi^2}} f^{\epsilon,\alpha} d\xi + F^{\epsilon,\alpha} \right) dxds \\ & + \int_0^t \int v \left( \gamma\partial_t \int \sigma dm^{\epsilon,\alpha} + \mathrm{div}_x \int \sigma\otimes\sigma dm^{\epsilon,\alpha} \right) \\ & + \int_0^t \int v \left( -\mathrm{div}_x(d\mu_E^{\epsilon,\alpha} - \frac{1}{2}tr(d\mu_E^{\epsilon,\alpha})Id) - \mathrm{div}_x(d\mu_B^{\epsilon,\alpha} - tr(d\mu_B^{\epsilon,\alpha})Id) + \epsilon\alpha\partial_t d\mu_{EB}^{\epsilon,\alpha} \right) \\ & + \int_0^t \int (\partial_t e(e - \epsilon E^{\epsilon,\alpha})e + \partial_t b(b - B^{\epsilon,\alpha})) dxds - \int_0^t (e\partial_t(\epsilon E^{\epsilon,\alpha}) + b\partial_t B^{\epsilon,\alpha}) dxds \\ = & -\int_0^t \int \hat{j}^{\epsilon,\alpha}\partial_t v dxds + \int_0^t \int (\rho^{\epsilon,\alpha} v\partial_t v + D_x v : v\otimes\hat{j}^{\epsilon,\alpha}) dxds + C_2 + C_1 \\ & - \int_0^t \iint D_x v : \int \frac{\xi\otimes\xi}{\sqrt{1+\gamma^2\xi^2}} f^{\epsilon,\alpha} d\xi dxds + \int_0^t \int v\cdot F^{\epsilon,\alpha} dxds \\ & + \gamma \left[ \iint v.\sigma dm^{\epsilon,\alpha} \right]_0^t - \gamma \int_0^t \iint \partial_t v.\sigma dm^{\epsilon,\alpha} - \int_0^t \iint D_x v : \sigma\otimes\sigma dm^{\epsilon,\alpha} \\ & + \int_0^t \int D_x v : \left( d\mu_E^{\epsilon,\alpha} + d\mu_B^{\epsilon,\alpha} - \frac{1}{2}tr(d\mu_E^{\epsilon,\alpha} + d\mu_B^{\epsilon,\alpha})Id \right) \\ & + \epsilon\alpha \left[ \int v.d\mu_{EB}^{\epsilon,\alpha} \right]_0^t - \epsilon\alpha \int_0^t \int \partial_t v d\mu_{EB}^{\epsilon,\alpha} \\ & + \int_0^t \int (\partial_t e(e - \epsilon E^{\epsilon,\alpha})e + \partial_t b(b - B^{\epsilon,\alpha})) dxds - \int_0^t (e\partial_t(\epsilon E^{\epsilon,\alpha}) + b\partial_t B^{\epsilon,\alpha}) dxds, \end{aligned}$$

where the force field $F^{\epsilon,\alpha}$ equals

$$F^{\epsilon,\alpha} = nE^{\epsilon,\alpha} - \mathrm{div}_x(\epsilon E^{\epsilon,\alpha}\otimes\epsilon E^{\epsilon,\alpha} - \frac{1}{2}|\epsilon E^{\epsilon,\alpha}|^2 Id) - \mathrm{div}_x(B^{\epsilon,\alpha}\otimes B^{\epsilon,\alpha} - \frac{1}{2}|B^{\epsilon,\alpha}|^2 Id) + \epsilon\alpha\partial_t(\epsilon E^{\epsilon,\alpha}\wedge B^{\epsilon,\alpha})$$

and the second part of the remainder $C_2$ is defined by

$$C_2 = -\int_0^t \int (j^{\epsilon,\alpha} - \hat{j}^{\epsilon,\alpha})\partial_t v dxds = -\int_0^t \iint \xi(1 - \frac{1}{\sqrt{1+\gamma^2\xi^2}}) f^{\epsilon,\alpha}\partial_t v d\xi dxds.$$

Then, by Maxwell's equations, we obtain

$$\begin{aligned}
H_U^{\epsilon,\alpha}(t)+ & \iint dm^{\epsilon,\alpha}(t,x,\sigma)+\frac{1}{2}\int (tr(d\mu_E^{\epsilon,\alpha}+tr(d\mu_B^{\epsilon,\alpha}))-H_U^{\epsilon,\alpha}(0)\\
= & -\int_0^t\int \hat{j}^{\epsilon,\alpha}\partial_t v dxds+\int_0^t\int (\rho^{\epsilon,\alpha}v\partial_t v+D_x v : v\otimes \hat{j}^{\epsilon,\alpha})dxds+C_2+C_1\\
& -\int_0^t\iint D_x v : \int \frac{\xi\otimes\xi}{\sqrt{1+\gamma^2\xi^2}}f^{\epsilon,\alpha}d\xi dxds+\int_0^t\int v\cdot F^{\epsilon,\alpha}dxds\\
& +\gamma\left[\iint v.\sigma dm^{\epsilon,\alpha}\right]_0^t-\gamma\int_0^t\iint \partial_t v.\sigma dm^{\epsilon,\alpha}-\int_0^t\iint D_x v:\sigma\otimes\sigma dm^{\epsilon,\alpha}\\
& +\int_0^t\int D_x v:\left(d\mu_E^{\epsilon,\alpha}+d\mu_B^{\epsilon,\alpha}-\frac{1}{2}tr(d\mu_E^{\epsilon,\alpha}+d\mu_B^{\epsilon,\alpha})Id\right)\\
& +\epsilon\alpha\left[\int v.d\mu_{EB}^{\epsilon,\alpha}\right]_0^t-\epsilon\alpha\int_0^t\int \partial_t v d\mu_{EB}^{\epsilon,\alpha}\\
& +\int_0^t\int (\partial_t e(e-\epsilon E^{\epsilon,\alpha})e+\partial_t b(b-B^{\epsilon,\alpha}))dxds\\
& -\int_0^t\left(e\left(\frac{1}{\alpha\epsilon}\ \mathrm{curl}_x B^{\epsilon,\alpha}+\frac{1}{\epsilon}\hat{j}^{\epsilon,\alpha}\right)+b\left(-\frac{1}{\alpha\epsilon}\ \mathrm{curl}_x\epsilon E^{\epsilon,\alpha}\right)\right)dxds,
\end{aligned}$$

which can be rewritten

$$\begin{aligned}
H_U^{\epsilon,\alpha}(t)+ & \iint dm^{\epsilon,\alpha}(t,x,\sigma)+\frac{1}{2}\int (tr(d\mu_E^{\epsilon,\alpha}+tr(d\mu_B^{\epsilon,\alpha}))-H_U^{\epsilon,\alpha}(0)\\
= & -\int_0^t\int \hat{j}^{\epsilon,\alpha}\partial_t v dxds+\int_0^t\int (\rho^{\epsilon,\alpha}v\partial_t v+D_x v : v\otimes \hat{j}^{\epsilon,\alpha})dxds+C_3+C_2+C_1\\
& -\int_0^t\int D_x v:\left(\int \frac{\xi\otimes\xi}{\sqrt{1+\gamma^2\xi^2}}f^{\epsilon,\alpha}d\xi dx+d\nu^{\epsilon,\alpha}\right)ds+\int_0^t\int v\cdot F^{\epsilon,\alpha}dxds\\
& +\int_0^t\int (\partial_t e(e-\epsilon E^{\epsilon,\alpha})e+\partial_t b(b-B^{\epsilon,\alpha}))dxds\\
& -\int_0^t\left(e\left(\frac{1}{\alpha\epsilon}\ \mathrm{curl}_x B^{\epsilon,\alpha}+\frac{1}{\epsilon}\hat{j}^{\epsilon,\alpha}\right)+b\left(-\frac{1}{\alpha\epsilon}\ \mathrm{curl}_x\epsilon E^{\epsilon,\alpha}\right)\right)dxds,
\end{aligned}\tag{5.22}$$

introducing the following groups of defect terms

$$d\nu^{\epsilon,\alpha}=\int_{\sigma\in S^2}\sigma\otimes\sigma dm^{\epsilon,\alpha}-\left(d\mu_E^{\epsilon,\alpha}+d\mu_B^{\epsilon,\alpha}-\frac{1}{2}tr(d\mu_E^{\epsilon,\alpha}+d\mu_B^{\epsilon,\alpha})Id\right)$$

and

$$\begin{aligned}
C_3=&\gamma\left[\iint v.\sigma dm^{\epsilon,\alpha}\right]_0^t-\gamma\int_0^t\iint\partial_t v.\sigma dm^{\epsilon,\alpha}\\
&+\epsilon\alpha\left[\int v.d\mu_{EB}^{\epsilon,\alpha}\right]_0^t-\epsilon\alpha\int_0^t\int\partial_t v d\mu_{EB}^{\epsilon,\alpha}
\end{aligned}$$

The second step of the proof consists in introducing systematically the modulations $\xi-v$, $\hat{j}^{\epsilon,\alpha}-\rho^{\epsilon,\alpha}v$, $\epsilon E^{\epsilon,\alpha}-e$ and $B^{\epsilon,\alpha}-b$ in order to get the structure of the right hand side in

(5.22).

$$
\begin{aligned}
H_U^{\epsilon,\alpha}(t)+ \quad & \iint dm^{\epsilon,\alpha}(t,x,\sigma) + \frac{1}{2}\int (tr(d\mu_E^{\epsilon,\alpha} + tr(d\mu_B^{\epsilon,\alpha})) - H_U^{\epsilon,\alpha}(0) \\
= \quad & -\int_0^t \int (\hat{j}^{\epsilon,\alpha} - \rho^{\epsilon,\alpha} v)\partial_t v dx ds + \int_0^t \int D_x v : v \otimes \hat{j}^{\epsilon,\alpha} dx ds + C_3 + C_2 + C_1 \\
& -\int_0^t \int D_x v : \left( \int \frac{(\xi - v + v)^{\otimes 2}}{\sqrt{1+\gamma^2\xi^2}} f^{\epsilon,\alpha} d\xi dx + d\nu^{\epsilon,\alpha} \right) ds + \int_0^t \int v \cdot F^{\epsilon,\alpha} dx ds \\
& + \int_0^t \int (\partial_t e(e - \epsilon E^{\epsilon,\alpha}) + \partial_t b(b - B^{\epsilon,\alpha})) dx ds \\
& - \int_0^t \int e \left( \frac{1}{\alpha\epsilon} \operatorname{curl}_x (B^{\epsilon,\alpha} - b + b) + \frac{1}{\epsilon}(\hat{j}^{\epsilon,\alpha} - \rho^{\epsilon,\alpha} v + \rho^{\epsilon,\alpha} v) \right) dx ds \\
& + \int_0^t \int b \left( \frac{1}{\alpha\epsilon} \operatorname{curl}_x (\epsilon E^{\epsilon,\alpha} - e + e) \right) dx ds,
\end{aligned}
$$

which can be rewritten

$$
\begin{aligned}
H_U^{\epsilon,\alpha}(t)+ \quad & \iint dm^{\epsilon,\alpha}(t,x,\sigma) + \frac{1}{2}\int (tr(d\mu_E^{\epsilon,\alpha} + tr(d\mu_B^{\epsilon,\alpha})) - H_U^{\epsilon,\alpha}(0) \\
= \quad & -\int_0^t \int (\hat{j}^{\epsilon,\alpha} - \rho^{\epsilon,\alpha} v) \left( \partial_t v + (v.\nabla_x)v + \frac{1}{\epsilon} e + \alpha v \wedge b \right) dx ds + C_3 + C_2 + C_1 \\
& -\int_0^t \int D_x v : \left( \int \frac{(\xi - v)^{\otimes 2}}{\sqrt{1+\gamma^2\xi^2}} f^{\epsilon,\alpha} d\xi dx + d\nu^{\epsilon,\alpha} \right) ds + C_4 \\
& + \int_0^t \int v \cdot \left( F^{\epsilon,\alpha} - \frac{1}{\epsilon}\rho^{\epsilon,\alpha} e - \alpha \hat{j}^{\epsilon,\alpha} \wedge b \right) dx ds \\
& + \int_0^t \int (e - \epsilon E^{\epsilon,\alpha}) \left( \partial_t e - \frac{1}{\alpha\epsilon} \operatorname{curl}_x b \right) dx ds \\
& + \int_0^t \int (b - B^{\epsilon,\alpha}) \left( \partial_t b + \frac{1}{\alpha\epsilon} \operatorname{curl}_x e \right) dx ds,
\end{aligned}
\tag{5.23}
$$

with

$$
C_4 = \int_0^t \iint D_x v : f^{\epsilon,\alpha} v \otimes v \left( \frac{1}{\sqrt{1+\gamma^2\xi^2}} - 1 \right) d\xi dx ds.
$$

It remains then to deal with the nonlinear term

$$
\begin{aligned}
& F^{\epsilon,\alpha} - \frac{1}{\epsilon}\rho^{\epsilon,\alpha} e - \alpha \hat{j}^{\epsilon,\alpha} \wedge b \\
& \quad = -\operatorname{div}_x (\epsilon E^{\epsilon,\alpha} \otimes \epsilon E^{\epsilon,\alpha} - \frac{1}{2}|\epsilon E^{\epsilon,\alpha}|^2 Id) - \operatorname{div}_x (B^{\epsilon,\alpha} \otimes B^{\epsilon,\alpha} - \frac{1}{2}|B^{\epsilon,\alpha}|^2 Id) \\
& \quad + \epsilon\alpha \partial_t (\epsilon E^{\epsilon,\alpha} \wedge B^{\epsilon,\alpha}) + n E^{\epsilon,\alpha} - \frac{1}{\epsilon}\rho^{\epsilon,\alpha} e - \alpha \hat{j}^{\epsilon,\alpha} \wedge b.
\end{aligned}
$$

It becomes when we introduce the modulation

$$
\begin{aligned}
&F^{\epsilon,\alpha} - \frac{1}{\epsilon}\rho^{\epsilon,\alpha}e - \alpha\hat{j}^{\epsilon,\alpha}\wedge b \\
&\quad = -\mathrm{div}_x((\epsilon E^{\epsilon,\alpha}-e)^{\otimes 2} - \frac{1}{2}|\epsilon E^{\epsilon,\alpha}-e|^2 Id) - \mathrm{div}_x((B^{\epsilon,\alpha}-b)^{\otimes 2} - \frac{1}{2}|B^{\epsilon,\alpha}-b|^2 Id) \\
&\quad + \epsilon\alpha\partial_t((\epsilon E^{\epsilon,\alpha}-e)\wedge(B^{\epsilon,\alpha}-b)) + \frac{1}{\epsilon}n(\epsilon E^{\epsilon,\alpha}-e) - \frac{1}{\epsilon}(\rho^{\epsilon,\alpha}-n)e - \alpha\hat{j}^{\epsilon,\alpha}\wedge b \\
&\quad - \mathrm{div}_x(\epsilon E^{\epsilon,\alpha}\otimes e + e\otimes \epsilon E^{\epsilon,\alpha} - e\cdot \epsilon E^{\epsilon,\alpha} Id) \\
&\quad - \mathrm{div}_x(B^{\epsilon,\alpha}\otimes b + b\otimes B^{\epsilon,\alpha} - b\cdot B^{\epsilon,\alpha} Id) \\
&\quad + \mathrm{div}_x(e^{\otimes 2} - \frac{1}{2}|e|^2 Id) + \mathrm{div}_x(b^{\otimes 2} - \frac{1}{2}|b|^2 Id) \\
&\quad + \epsilon\alpha\Big(\partial_t e\wedge(B^{\epsilon,\alpha}-b) + (\epsilon E^{\epsilon,\alpha}-e)\wedge\partial_t b + e\wedge\partial_t B^{\epsilon,\alpha} + \partial_t(\epsilon E^{\epsilon,\alpha})\wedge b\Big)
\end{aligned}
$$

Using the relation

$$
\mathrm{div}_x(a\otimes b + b\otimes a - a\cdot b Id) = \mathrm{div}_x(a)b + \mathrm{div}_x(b)a + \ \mathrm{curl}_x(a)\wedge b + \ \mathrm{curl}_x(b)\wedge a
$$

leads to

$$
\begin{aligned}
&F^{\epsilon,\alpha} - \frac{1}{\epsilon}\rho^{\epsilon,\alpha}e - \alpha\hat{j}^{\epsilon,\alpha}\wedge b \\
&\quad = -\mathrm{div}_x((\epsilon E^{\epsilon,\alpha}-e)^{\otimes 2} - \frac{1}{2}|\epsilon E^{\epsilon,\alpha}-e|^2 Id) - \mathrm{div}_x((B^{\epsilon,\alpha}-b)^{\otimes 2} - \frac{1}{2}|B^{\epsilon,\alpha}-b|^2 Id) \\
&\quad + \epsilon\alpha\partial_t((\epsilon E^{\epsilon,\alpha}-e)\wedge(B^{\epsilon,\alpha}-b)) + \frac{1}{\epsilon}n(\epsilon E^{\epsilon,\alpha}-e) - \frac{1}{\epsilon}(\rho^{\epsilon,\alpha}-n)e - \alpha\hat{j}^{\epsilon,\alpha}\wedge b \\
&\quad - \mathrm{div}_x(\epsilon E^{\epsilon,\alpha})e - \mathrm{div}_x(e)\epsilon E^{\epsilon,\alpha} - \ \mathrm{curl}_x(\epsilon E^{\epsilon,\alpha})\wedge e - \ \mathrm{curl}_x e\wedge \epsilon E^{\epsilon,\alpha} \\
&\quad - \ \mathrm{curl}_x(B^{\epsilon,\alpha})\wedge b - \ \mathrm{curl}_x b\wedge B^{\epsilon,\alpha} \\
&\quad + \epsilon\mathrm{div}_x e + \ \mathrm{curl}_x e\wedge e + \ \mathrm{curl}_x b\wedge b \\
&\quad + \epsilon\alpha\Big(\partial_t e\wedge(B^{\epsilon,\alpha}-b) + (\epsilon E^{\epsilon,\alpha}-e)\wedge\partial_t b) + e\wedge\partial_t B^{\epsilon,\alpha} + \partial_t(\epsilon E^{\epsilon,\alpha})\wedge b\Big)
\end{aligned}
$$

or equivalently

$$
\begin{aligned}
&F^{\epsilon,\alpha} - \frac{1}{\epsilon}\rho^{\epsilon,\alpha}e - \alpha\hat{j}^{\epsilon,\alpha}\wedge b \\
&\quad = -\mathrm{div}_x((\epsilon E^{\epsilon,\alpha}-e)^{\otimes 2} - \frac{1}{2}|\epsilon E^{\epsilon,\alpha}-e|^2 Id) - \mathrm{div}_x((B^{\epsilon,\alpha}-b)^{\otimes 2} - \frac{1}{2}|B^{\epsilon,\alpha}-b|^2 Id) \\
&\quad + \epsilon\alpha\partial_t((\epsilon E^{\epsilon,\alpha}-e)\wedge(B^{\epsilon,\alpha}-b)) + \frac{1}{\epsilon}n(\epsilon E^{\epsilon,\alpha}-e) - \frac{1}{\epsilon}\Big(\rho^{\epsilon,\alpha}-n+\epsilon^2\mathrm{div}_x E^{\epsilon,\alpha}\Big)e \\
&\quad - \mathrm{div}_x(e)(\epsilon E^{\epsilon,\alpha}-e) - \Big(\ \mathrm{curl}_x(\epsilon E^{\epsilon,\alpha}) + \epsilon\alpha\partial_t B^{\epsilon,\alpha}\Big)\wedge e - \ \mathrm{curl}_x e\wedge(\epsilon E^{\epsilon,\alpha}-e) \\
&\quad + \Big(-\ \mathrm{curl}_x(B^{\epsilon,\alpha}) + \partial_t(\epsilon^2\alpha E^{\epsilon,\alpha}) - \alpha\hat{j}^{\epsilon,\alpha}\Big)\wedge b - \ \mathrm{curl}_x b\wedge(B^{\epsilon,\alpha}-b) \\
&\quad + \epsilon\alpha\Big(\partial_t e\wedge(B^{\epsilon,\alpha}-b) + (\epsilon E^{\epsilon,\alpha}-e)\wedge\partial_t b\Big)
\end{aligned}
$$

By Maxwell's equations, we finally obtain

$$
\begin{aligned}
&F^{\epsilon,\alpha} - \frac{1}{\epsilon}\rho^{\epsilon,\alpha}e - \alpha\hat{j}^{\epsilon,\alpha}\wedge b \\
&\quad = -\mathrm{div}_x((\epsilon E^{\epsilon,\alpha}-e)^{\otimes 2} - \frac{1}{2}|\epsilon E^{\epsilon,\alpha}-e|^2 Id) - \mathrm{div}_x((B^{\epsilon,\alpha}-b)^{\otimes 2} - \frac{1}{2}|B^{\epsilon,\alpha}-b|^2 Id) \\
&\quad + \epsilon\alpha\partial_t((\epsilon E^{\epsilon,\alpha}-e)\wedge(B^{\epsilon,\alpha}-b)) + \frac{1}{\epsilon}n(\epsilon E^{\epsilon,\alpha}-e) \\
&\quad - \mathrm{div}_x(e)(\epsilon E^{\epsilon,\alpha}-e) - \ \mathrm{curl}_x e\wedge(\epsilon E^{\epsilon,\alpha}-e) - \ \mathrm{curl}_x b\wedge(B^{\epsilon,\alpha}-b) \\
&\quad + \epsilon\alpha\left(\partial_t e\wedge(B^{\epsilon,\alpha}-b) + (\epsilon E^{\epsilon,\alpha}-e)\wedge\partial_t b\right)
\end{aligned}
\tag{5.24}
$$

Plugging (5.24) into (5.23) provides then

$$
\begin{aligned}
& H_U^{\epsilon,\alpha}(t) + \iint dm^{\epsilon,\alpha}(t,x,\sigma) + \frac{1}{2}\int (tr(d\mu_E^{\epsilon,\alpha} + tr(d\mu_B^{\epsilon,\alpha}) - H_U^{\epsilon,\alpha}(0) \\
= & -\int_0^t \int (\hat{j}^{\epsilon,\alpha} - \rho^{\epsilon,\alpha} v)\left(\partial_t v + (v.\nabla_x)v + \frac{1}{\epsilon}e + \alpha v \wedge b\right) dxds + C_3 + C_2 + C_1 \\
& -\int_0^t \int D_x v : \left(\int \frac{(\xi - v)^{\otimes 2}}{\sqrt{1+\gamma^2\xi^2}} f^{\epsilon,\alpha} d\xi dx + d\nu^{\epsilon,\alpha}\right) ds + C_4 + C_5 \\
& +\int_0^t \int D_x v : \left((\epsilon E^{\epsilon,\alpha} - e)^{\otimes 2} - \frac{1}{2}|\epsilon E^{\epsilon,\alpha} - e|^2 Id) + (B^{\epsilon,\alpha} - b)^{\otimes 2} - \frac{1}{2}|B^{\epsilon,\alpha} - b|^2 Id\right) dxds \\
& +\int_0^t \int (e - \epsilon E^{\epsilon,\alpha})\left(\partial_t e - \frac{1}{\alpha\epsilon}\ \mathrm{curl}_x b - \frac{1}{\epsilon} n v + v \mathrm{div}_x e + v \wedge\ \mathrm{curl}_x e + \alpha\epsilon v \wedge \partial_t b\right) dxds \\
& +\int_0^t \int (b - B^{\epsilon,\alpha})\left(\partial_t b + \frac{1}{\alpha\epsilon}\ \mathrm{curl}_x e + v \wedge\ \mathrm{curl}_x b - \epsilon\alpha v \wedge \partial_t e\right) dxds,
\end{aligned}
\tag{5.25}
$$

where

$$
\begin{aligned}
C_5 &= \epsilon\alpha \int_0^t \int v.\partial_t((\epsilon E^{\epsilon,\alpha} - e) \wedge (B^{\epsilon,\alpha} - b)) dxds \\
&= \epsilon\alpha \left[\int v((\epsilon E^{\epsilon,\alpha} - e) \wedge (B^{\epsilon,\alpha} - b)) dx\right]_0^t - \epsilon\alpha \int_0^t \int \partial_t v.((\epsilon E^{\epsilon,\alpha} - e) \wedge (B^{\epsilon,\alpha} - b)) dxds.
\end{aligned}
$$

Checking that $R^{\epsilon,\alpha} = C_1 + C_2 + C_3 + C_4 + C_5$ concludes the proof. □

### 5.4.3 The stability inequality

The particular structure of the identity established in the previous Proposition (5.4.1) allows to prove the following stability inequality.

*Corollary* 5.4.1. The modulated energy $H_U^{\epsilon,\alpha}(t)$ satisfies the Gronwall inequality

$$
\begin{aligned}
H_U^{\epsilon,\alpha}(t) & + \iint dm^{\epsilon,\alpha}(t,x,\sigma) + \frac{1}{2}\int (tr(d\mu_E^{\epsilon,\alpha} + tr(d\mu_B^{\epsilon,\alpha}) \\
& \leq H_U^{\epsilon,\alpha}(0) \exp\left(6\int_0^t \|D_x v\|_{L^\infty} d\tau\right) + O(\gamma) + O(\epsilon) \\
& + \int_0^t \exp\left(6\int_0^s \|D_x v\|_{L^\infty} d\tau\right) \int A^{\epsilon,\alpha}(U) \cdot \begin{pmatrix} \rho^{\epsilon,\alpha} v - \hat{j}^{\epsilon,\alpha} \\ b - B^{\epsilon,\alpha} \\ e - \epsilon E^{\epsilon,\alpha} \end{pmatrix} dxds.
\end{aligned}
\tag{5.26}
$$

*Démonstration.* This corollary is based essentially on the control of the relativistic corrections which are expected to converge to 0 as $\gamma \to 0$. With the notations of the previous paragraph, we have

$$
\begin{aligned}
|C_1| \ \leq\ & ||v||_{L^\infty} \int\int \frac{\gamma|\xi|^2}{1 + \sqrt{1+\gamma^2|\xi|^2}} f^{\epsilon,\alpha} dxd\xi \\
& + ||v||_{L^\infty}^2 \int\int \frac{\gamma|\xi|}{2(1+\sqrt{1+\gamma^2|\xi|^2})} f^{\epsilon,\alpha} dxd\xi
\end{aligned}
$$

because

$$
\left|\frac{2}{1+\sqrt{1+\gamma^2\xi^2}} - 1\right| = \left|\frac{1 - \sqrt{1+\gamma^2\xi^2}}{1+\sqrt{1+\gamma^2\xi^2}}\right| \leq \frac{\gamma|\xi|}{1+\sqrt{1+\gamma^2\xi^2}}.
$$

Since the global mass and the energy $H^{\epsilon,\alpha}$ are conserved, we get by Cauchy Schwarz inequality

$$|C_1| \leq C_{in}\gamma(\|v\|_{L^\infty} + \|v\|_{L^\infty}^2).$$

In the same way, we obtain for all $t \in [0,T]$

$$|C_2| \leq ||\partial_t v||_{L^\infty} \int_0^t \iint \frac{2\gamma|\xi|^2}{1+\sqrt{1+\gamma^2|\xi|^2}} f^{\epsilon,\alpha} dx d\xi ds \leq 2TC_{in}\gamma||\partial_t v||_{L^\infty},$$

and finally

$$|C_4| \leq ||D_x v||_{L^\infty}||v||_{L^\infty}^2 \int_0^t \iint \frac{2\gamma|\xi|}{1+\sqrt{1+\gamma^2|\xi|^2}} f^{\epsilon,\alpha} dx d\xi ds \leq 2TC_{in}\gamma||\nabla v||_{L^\infty}||v||_{L^\infty}^2.$$

Then this part of the remainder $R^{\epsilon,\alpha}$ satisfies

$$|C_1| + |C_2| + |C_4| = O(\gamma)$$

where the constants depend only on the initial mass and energy, and on the $W^{1,\infty}(\mathbf{R}^+ \times \Omega)$ norm of $v$.

The second step of the proof consists in estimating the other terms in the remainder $R^{\epsilon,\alpha}$. By the energy inequality,

$$\begin{aligned}
|C_3| &\leq 2\gamma m^{\epsilon,\alpha}(\Omega \times S^2)||v||_{L^\infty} + \gamma T m^{\epsilon,\alpha}(\Omega \times S^2)||\partial_t v||_{L^\infty} \\
&+2\epsilon\alpha|\mu_{EB}|(\Omega)||v||_{L^\infty} + \epsilon\alpha T|\mu_{EB}|(\Omega)||\partial_t v||_{L^\infty} \\
&\leq 4C_{in}(\epsilon\alpha + \gamma)\left(||v||_{L^\infty} + T||\partial_t v||_{L^\infty}\right),
\end{aligned}$$

$$\begin{aligned}
|C_5| &\leq 2\alpha\epsilon||\epsilon E^{\epsilon,\alpha} - e||_{L^2}||B^{\epsilon,\alpha} - b||_{L^2}||v||_{L^\infty} + \alpha\epsilon T||\epsilon E^{\epsilon,\alpha} - e||_{L^2}||B^{\epsilon,\alpha} - b||_{L^2}||\partial_t v||_{L^\infty} \\
&\leq 4C_{in}\epsilon\left(||v||_{L^\infty} + T||\partial_t v||_{L^\infty}\right),
\end{aligned}$$

from which we deduce that

$$|C_3| + |C_5| = O(\gamma) + O(\epsilon)$$

where the constants depend only on the initial mass and energy, and on the $W^{1,\infty}(\mathbf{R}^+ \times \Omega)$ norm of $v$.

Then, in order to apply Gronwall's lemma, it remains to prove that the flux terms $h_U^{\epsilon,\alpha}$ and $\nu^{\epsilon,\alpha}$ can be controlled in terms of the modulated energy

$$H_U^{\epsilon,\alpha} + \iint dm^{\epsilon,\alpha} + \frac{1}{2}\int tr(d\mu_E^{\epsilon,\alpha} + d\mu_B^{\epsilon,\alpha}).$$

We have clearly

$$\left\| \int f^{\epsilon,\alpha} \frac{(\xi - v)^{\otimes 2}}{\sqrt{1+\gamma^2\xi^2}} d\xi \right\|_{L^1} \leq 6 \iint f^{\epsilon,\alpha} \frac{|\xi - v|^2}{1+\sqrt{1+\gamma^2\xi^2}} d\xi dx,$$

$$\|(\epsilon E^{\epsilon,\alpha} - e)^{\otimes 2} - \frac{1}{2}|\epsilon E^{\epsilon,\alpha} - e|^2 Id\|_{L^1} \leq 3\int |\epsilon E^{\epsilon,\alpha} - e|^2 dx,$$

and

$$\|(B^{\epsilon,\alpha} - b)^{\otimes 2} - \frac{1}{2}|B^{\epsilon,\alpha} - b|^2 Id\|_{L^1} \leq 3\int |B^{\epsilon,\alpha} - b|^2 dx,$$

from which we deduce that

$$\|h_U^{\epsilon,\alpha}\|_{L^1} \leq 6H_U^{\epsilon,\alpha}.$$

In the same way, as $\mu_E$ and $\mu_B$ are non negative symetric matrix

$$\iint |\sigma \otimes \sigma dm^{\epsilon,\alpha}| \leq 6m^{\epsilon,\alpha}(\Omega \times S^2),$$

$$\int |\mu_E^{\epsilon,\alpha} - \frac{1}{2} tr(d\mu_E^{\epsilon,\alpha} Id) \leq 3tr(\mu_E)(\Omega),$$

$$\int |d\mu_B^{\epsilon,\alpha} - \frac{1}{2} tr(d\mu_B^{\epsilon,\alpha}) Id| \leq 3tr(\mu_B)(\Omega),$$

so that

$$|\nu^{\epsilon,\alpha}|(\Omega) \leq 6 \left( \iint dm^{\epsilon,\alpha} + \frac{1}{2} \int tr(d\mu_E^{\epsilon,\alpha} + d\mu_B^{\epsilon,\alpha}) \right).$$

Thus, applying Gronwall's lemma concludes the proof. □

## 5.5 Strong convergence for well prepared initial data

In this section we restrict our attention to initial data for which solutions of (5.20) do not present any fast oscillation (nonnegligible in norm).

Indeed, we suppose that the initial data $(v_0, e_0, b_0)$ belongs to the kernel of the singular perturbation, i.e.

$$e_0 = 0, \quad nv_0 + \frac{1}{\alpha} \operatorname{curl}_x b_0 = 0.$$

Since $e_0 = 0$, we modulate $H^{\epsilon,\alpha}(t)$ with $U_\epsilon = (v, \epsilon \underline{e}, b)$ instead of $U = (v, 0, b)$. We obtain a similar result but in a more precise form wich will be useful in the sequel.

Then we expect that any solution $(v, e, b)$ of (5.20) with initial data $(v_0, 0, b_0)$ behaves asymptotically as $\epsilon \to 0$ as the solution $(v, 0, b)$ of

$$\begin{cases} \partial_t v + v \cdot \nabla v + \underline{e} + \alpha v \wedge b = 0 \\ \partial_t b + \dfrac{1}{\alpha} \operatorname{curl} \underline{e} = 0 \\ nv + \dfrac{1}{\alpha} \operatorname{curl} b = 0 \end{cases} \tag{5.27}$$

where $\underline{e} \sim e/\epsilon$ is the Lagrange multiplier associated to the constraint $nv + \frac{1}{\alpha} \operatorname{curl} b = 0$.

In particular, if the following convergence holds

$$\begin{aligned} \iint \frac{(\xi - v_0)^2}{1 + \sqrt{1 + \gamma^2 \xi^2}} f_{in}^{\epsilon,\alpha}(x,\xi) d\xi dx + \frac{1}{2} \int \epsilon^2 |E_{in}^{\epsilon,\alpha}(x)|^2 dx + \frac{1}{2} \int |B_{in}^{\epsilon,\alpha}(x) - b_0|^2 dx \\ = H_U^{\epsilon,\alpha}(0) \overset{\epsilon \to 0}{\longrightarrow} 0, \end{aligned} \tag{5.28}$$

which means that

- the initial velocity profile is essentially monokinetic ;
- the initial fields converge strongly ;
- and the limiting fields belong to the kernel of the singular perturbation ;

then we expect that

$$\hat{j}^{\epsilon,\alpha} \sim \rho^{\epsilon,\alpha} v_{\epsilon,\alpha} \to nv;$$

$$\epsilon E^{\epsilon,\alpha} \sim e_{\epsilon,\alpha} \to 0;$$

$$B^{\epsilon,\alpha} \sim b_{\epsilon,\alpha} \to b;$$

where $\rho^{\epsilon,\alpha}$, $\hat{j}^{\epsilon,\alpha}$ are the density and momentum associated with any solution of (5.4), $(v_{\epsilon,\alpha}, e_{\epsilon,\alpha}, b_{\epsilon,\alpha})$ is the solution of the asymptotic system (5.20) and $(v, b)$ satisfy the mean field equation (5.27). The mathematical formulation of this result is given in the following

*Proposition* 5.5.1. Let $(f_{in}^{\epsilon,\alpha})_{\epsilon,\alpha}$ be a family of nonnegative functions of $L^1 \cap L^2(\Omega \times \mathbf{R}^3)$, and $(E_{in}^{\epsilon,\alpha})_{\epsilon,\alpha}, (B_{in}^{\epsilon,\alpha})_{\epsilon,\alpha}$ be two families of vector fields of $L^2(\Omega)$ satisfying the compatibility conditions (5.8) for some constant $n$ and the uniform energy bound

$$\sup_{\epsilon,\alpha} H^{\epsilon,\alpha}(0) < +\infty.$$

Assume that there exist two divergence-free vector fields $v_0, b_0 \in H^s(\Omega)$ (for $s > 2 + 3/2$) such that

$$nv_0 + \frac{1}{\alpha} \operatorname{curl}_x b_0 = 0,$$

$$\iint \frac{(\xi - v_0)^2}{1 + \sqrt{1 + \gamma^2 \xi^2}} f_{in}^{\epsilon,\alpha}(x,\xi) d\xi dx + \frac{1}{2} \int \epsilon^2 |E_{in}^{\epsilon,\alpha}(x)|^2 dx + \frac{1}{2} \int |B_{in}^{\epsilon,\alpha}(x) - b_0|^2 dx \to 0 \text{ as } \epsilon \to 0.$$

Let $(f^{\epsilon,\alpha}, E^{\epsilon,\alpha}, B^{\epsilon,\alpha})$ be, for every $\epsilon > 0, \alpha > 0$, a solution of the scaled Vlasov-Maxwell system (5.4) with initial condition (5.7). Then, there exists $T > 0$ such that the current density $\hat{j}^{\epsilon,\alpha} = \int f^{\epsilon,\alpha} \xi / \sqrt{1 + \gamma^2 \xi^2} d\xi$ converges weakly in $L^\infty([0,T], L^1(\Omega))$, the scaled electric field $\epsilon E^{\epsilon,\alpha}$ and the magnetic field $B^{\epsilon,\alpha}$ converge strongly in $L^\infty_{Loc}([0,T[, (L^2(\Omega))^2)$ to $(nv, 0, b)$ as $\epsilon \to 0, \alpha \to \overline{\alpha} < +\infty$, where $(v, b) \in L^\infty([0,T], H^s(\Omega))$ is the local strong solution of (5.27) with initial condition $(v_0, b_0)$.

In other words, this result shows that

- the space $\mathrm{Ker}(\mathrm{L}_n)$ is invariant under the flow of (5.20) (up to corrections of higher order in $\epsilon$) : if the initial data belongs to $\mathrm{Ker}(\mathrm{L}_n)$, then the solution of (5.20) is close to $\mathrm{Ker}(\mathrm{L}_n)$ ;
- the mean field equation corresponding to (5.20) (i.e. its projection on $\mathrm{Ker}(\mathrm{L}_n)$) is (5.27) ;
- for well-prepared initial data, the strong convergence of the fields and the monokinetic profile are preserved : more precisely a convergence result such as (5.28) holds for all $t > 0$.

The proof of this result is based on the Gronwall's inequality obtained in the previous section, and on the study of the limiting system. Indeed, if we were able to exhibit a smooth solution $U_{\epsilon,\alpha} = (v_{\epsilon,\alpha}, e_{\epsilon,\alpha}, b_{\epsilon,\alpha})$ of (5.20) for all $\epsilon, \alpha > 0$, then we would have

$$A^{\epsilon,\alpha}(U_{\epsilon,\alpha}) = 0$$

and by corollary 5.4.1, $U_{\epsilon,\alpha}$ would give a good approximation of $(\hat{j}^{\epsilon,\alpha}/\rho^{\epsilon,\alpha}, \epsilon E^{\epsilon,\alpha}, B^{\epsilon,\alpha})$.

### 5.5.1 Theory for the limiting system

The first step consists then in characterizing the asymptotic system (5.20), and more precisely the associated mean field system (5.27) (since initial data are well-prepared).

Let us prove a result stated in [4] in the particular case when the magnetic field is perpendicular.

*Lemma* 5.5.1. Let $v_0, b_0$ be two divergence-free vector fields of $H^s(\Omega)$ $(s > 1 + \frac{3}{2})$ such that

$$nv_0 + \frac{1}{\alpha} \operatorname{curl}_x b_0 = 0,$$

for some homogeneous $n$. Then, there exist $T^* \in ]0, +\infty]$ and a unique $(v, b) \in L^\infty_{loc}([0, T^*[, H^s)$ solution of (5.27).

*Démonstration.* If $n$ is homogeneous, because of the constraint,

$$\operatorname{div}_x v = -\frac{1}{\alpha n} \operatorname{div}_x(\operatorname{curl}_x b).$$

Then,

$$\partial_t \operatorname{curl}_x v + (v \cdot \nabla_x) \operatorname{curl}_x v - (\operatorname{curl}_x v \cdot \nabla_x) v + \operatorname{curl}_x \underline{e} + \alpha \operatorname{curl}_x (v \wedge b) = 0,$$

and

$$\operatorname{curl}_x (v \wedge b) = (b \cdot \nabla_x) v - (v \cdot \nabla_x) b.$$

Let us introduce

$$w = \operatorname{curl}_x v - \alpha b.$$

The system (5.27) can be rewritten

$$\begin{aligned} &\partial_t w + v \cdot \nabla w - w \cdot \nabla v = 0 \\ &-\Delta v + \alpha^2 n v = \operatorname{curl}_x w \\ &nv + \frac{1}{\alpha} \operatorname{curl} b = 0, \operatorname{div} b = 0. \end{aligned}$$

Consider the mapping

$$M : v \in H^s \mapsto \tilde{v} = M(v) \in H^s$$

where $\tilde{v}$ is defined as the solution of the elliptic equation

$$-\Delta_x \tilde{v} + \alpha^2 n \tilde{v} = \tilde{w}$$

and $\tilde{w}$ satisfies the following transport equation

$$\partial_t \tilde{w} + v \cdot \nabla_x \tilde{w} - \tilde{w} \cdot \nabla_x v = 0.$$

For $s > 1 + 3/2$, and for $T$ sufficiently small, we can prove that $M$ is a contraction. Therefore, the existence results are the same as for the incompressible Euler equations. □

If $n$ is not homogeneous, the limiting system is much more complicated and we do not have any existence result for strong solutions. However, we should have a weak convergence result if we introduce a notion of dissipative solutions for the limiting system. From now on, we assume that $n$ is homogeneous.

### 5.5.2 Weak compactness results

In order to get the asymptotic behaviour of the macroscopic quantities $(\hat{j}^{\epsilon,\alpha}, \epsilon E^{\epsilon,\alpha}, B^{\epsilon,\alpha})$, we have first to establish the relative compactness of the families.

From the energy inequality we deduce that

$$\sup_{\epsilon,\alpha} \|\epsilon E^{\epsilon,\alpha}\|^2_{L^\infty(\mathbf{R}^+,L^2(\Omega))} < +\infty,$$

$$\sup_{\epsilon,\alpha} \|B^{\epsilon,\alpha}\|^2_{L^\infty(\mathbf{R}^+,L^2(\Omega))} < +\infty,$$

$$\sup_{\epsilon,\alpha} \sup_{t\in\mathbf{R}^+} \iint \frac{\xi^2}{1+\sqrt{1+\gamma^2\xi^2}} f^{\epsilon,\alpha} dx d\xi < +\infty,$$

while the conservation of mass provides

$$\iint f^{\epsilon,\alpha} dx d\xi = n|\Omega|.$$

Then we obtain a uniform bound on $\hat{j}^{\epsilon,\alpha}$ by interpolation

$$\begin{aligned}
|\hat{j}^{\epsilon,\alpha}| &\leq \left(\int f^{\epsilon,\alpha} dx d\xi\right)^{1/2} \left(\int \frac{\xi^2}{1+\gamma^2\xi^2} f^{\epsilon,\alpha} dx d\xi\right)^{1/2} \\
&\leq \left(\int f^{\epsilon,\alpha} dx d\xi\right)^{1/2} \left(\int \frac{2\xi^2}{1+\sqrt{1+\gamma^2\xi^2}} f^{\epsilon,\alpha} dx d\xi\right)^{1/2}
\end{aligned}$$

which implies in particular that

$$\sup_{\epsilon,\alpha} \|\hat{j}^{\epsilon,\alpha}\|_{L^\infty(\mathbf{R}^+,L^1(\Omega))} < +\infty.$$

Up to extraction of a subsequence, we then have

$$\epsilon E^{\epsilon,\alpha} \rightharpoonup E, \quad B^{\epsilon,\alpha} \rightharpoonup B \text{ in } w^* - L^\infty(\mathbf{R}^+, L^2(\Omega))$$

$$\rho^{\epsilon,\alpha} \rightharpoonup \rho, \quad \hat{j}^{\epsilon,\alpha} \rightharpoonup j \text{ in the sense of measures.}$$

Moreover, using the Poisson equation, we obtain that

$$\rho = n.$$

### 5.5.3 Convergence results

Let $(v, b)$ be the solution of (5.27) with initial condition $(v_0, b_0)$. Comparing both systems (5.27) and (5.20) shows that $U_\epsilon = (v, \epsilon\underline{e}, b)$ is an $\epsilon$-approximate solution of (5.20), and more precisely that it satisfies

$$\begin{aligned}
A^{\epsilon,\alpha}(U_\epsilon) &= \begin{pmatrix}
\partial_t v + v\cdot\nabla v + \underline{e} + \alpha v \wedge b \\
\partial_t b + \dfrac{1}{\alpha}\text{curl } \underline{e} + (\alpha\epsilon^2\partial_t\underline{e} - \text{curl } b)\wedge v \\
\epsilon\partial_t\underline{e} - \dfrac{1}{\alpha\epsilon}\text{curl } b - n\dfrac{v}{\epsilon} + \epsilon v \text{div } \underline{e} - (\alpha\epsilon\partial_t b + \epsilon\text{curl } \underline{e})\wedge v
\end{pmatrix} \\
&= \begin{pmatrix}
0 \\
(\alpha\epsilon^2\partial_t\underline{e})\wedge v \\
\epsilon\partial_t\underline{e} + \epsilon v\text{div } \underline{e}
\end{pmatrix}.
\end{aligned}$$

As $\underline{e} = -(\partial_t v + v \cdot \nabla_x v + \alpha v \wedge b)$, it is uniformly bounded in $W^{1,\infty}([0,T] \times \Omega)$. Then, since $\iint dm^{\epsilon,\alpha}(t,x,\sigma) + \frac{1}{2}\int tr(d\mu_E^{\epsilon,\alpha}) + tr(d\mu_B^{\epsilon,\alpha}) \geq 0$, plugging this last identity in the stability inequality (5.26) leads to

$$H_{U_\epsilon}^{\epsilon,\alpha}(t) \leq H_{U_\epsilon}^{\epsilon,\alpha}(0) \exp(6 \int_0^t \|D_x v(s)\|_{L^\infty} ds) + O(\gamma) + O(\epsilon).$$

The assumption on the initial data gives exactly

$$H_{U_\epsilon}^{\epsilon,\alpha}(0) = H_U^{\epsilon,\alpha}(0) + O(\epsilon) \to 0,$$

with $U = (v, 0, b)$, from which we deduce

$$H_{U_\epsilon}^{\epsilon,\alpha}(t) \to 0 \text{ as } \epsilon \to 0,$$

or equivalently

$$H_U^{\epsilon,\alpha}(t) \to 0 \text{ as } \epsilon \to 0$$

for all $t < T^*$ where $T^*$ is the lifespan of the strong solution of (5.27). In particular,

$$\epsilon E^{\epsilon,\alpha} \to 0, \quad B^{\epsilon,\alpha} \to b \text{ strongly in } L^\infty([0,T], L^2(\Omega)).$$

To get the convergence result on the current, we use a - by now standard - argument of convexity. Define

$$h_v^{\epsilon,\alpha}(t) = K(\hat{j}^{\epsilon,\alpha} - v\hat{\rho}^{\epsilon,\alpha}; \hat{\rho}^{\epsilon,\alpha}) = \int \frac{|\hat{j}^{\epsilon,\alpha} - v\hat{\rho}^{\epsilon,\alpha}|^2}{\hat{\rho}^{\epsilon,\alpha}} dx$$

with

$$K(m,\sigma) = \int \sup_{b \in \mathcal{D}([0,T]\times\Omega)} \frac{1}{2} < \sigma(t,x); |b(t,x)|^2 > + < m(t,x); b(t,x) > dxdt,$$

and

$$\hat{\rho}^{\epsilon,\alpha} = \int \frac{1}{\sqrt{1+\gamma^2\xi^2}} f^{\epsilon,\alpha} d\xi.$$

From the Cauchy-Schwarz inequality we deduce that

$$|\hat{j}^{\epsilon,\alpha} - v\hat{\rho}^{\epsilon,\alpha}|^2 \leq \left( \int \frac{(\xi - v)^2}{1 + \sqrt{1+\gamma^2\xi^2}} f^{\epsilon,\alpha} d\xi \right) \left( \int \frac{1 + \sqrt{1+\gamma^2\xi^2}}{1+\gamma^2\xi^2} f^{\epsilon,\alpha} d\xi \right)$$

and consequently

$$h_v^{\epsilon,\alpha}(t) \leq 2H_{U_\epsilon}^{\epsilon,\alpha}(t).$$

Note in particular that $\hat{j}^{\epsilon,\alpha} - \hat{\rho}^{\epsilon,\alpha} v \to 0$ strongly in $L^\infty([0,T], L^1(\Omega))$.

On the other hand, it is easy to check that

$$\hat{\rho}^{\epsilon,\alpha} \to n$$

in the sense of measures. Indeed, the relativistic correction $\rho^{\epsilon,\alpha} - \hat{\rho}^{\epsilon,\alpha}$ can be estimated as the remainder $C_4$ in section 4.3. Then since $K$ is lower semi-continuous for the convergence in the vague sense of measures

$$\int \frac{|j - nv|^2}{n} dx \leq \lim h_v^{\epsilon,\alpha}(t) \leq \lim H_U^{\epsilon,\alpha}(t) = 0$$

which proves that $j = nv$.

The uniqueness of the limit point ensures that the whole family $(\hat{j}^{\epsilon,\alpha}, \epsilon E^{\epsilon,\alpha}, B^{\epsilon,\alpha})$ converges strongly to $(nv, 0, b)$.

## 5.6 Study of the singular perturbation

In view of the previous results, for general initial data, we expect $(\hat{j}^{\epsilon,\alpha}, \epsilon E^{\epsilon,\alpha}, B^{\epsilon,\alpha})$ to behave as the solution of the following singular system

$$\begin{aligned} \partial_t v + v\cdot\nabla v + \frac{e}{\epsilon} + \alpha v\wedge b = 0 \\ \partial_t b + \frac{1}{\alpha\epsilon}\text{curl } e = 0 \\ \partial_t e - \frac{1}{\alpha\epsilon}\text{curl } b - \frac{v}{\epsilon} + v\text{div } e = 0 \\ \text{div } b = 0, \end{aligned} \tag{5.29}$$

that is to oscillate under the linear penalization L around a mean state satisfying (5.27). (Recall that we restrict our attention to the case where $n$ is constant, and for simplicity $n \equiv 1$.)

In order to obtain a complete description of the asymptotic, we will first describe the oscillations generated by the singular perturbation and then, we will check that generically the oscillations do not bring any contribution in the evolution of the mean field.

### 5.6.1 Description of the oscillations

Define the linear operator $S$

$$S : \begin{pmatrix} \text{div}_x v \\ \text{div}_x e \\ \text{curl}_x v \\ \text{curl}_x e \\ (-\Delta_x)^{1/2} b \end{pmatrix} \mapsto \begin{pmatrix} \text{div}_x e \\ -\text{div}_x v \\ \text{curl}_x e \\ \frac{1}{\alpha}\Delta_x b - \text{curl}_x v \\ \frac{1}{\alpha}(-\Delta_x)^{1/2}\text{curl}_x e \end{pmatrix} \tag{5.30}$$

that is a variant of L which has a simpler spectral theory. Describing the oscillations generated by the linear penalization requires a good understanding of the structure of $S$ and in particular a precise description of its kernel.

*Lemma* 5.6.1. Consider the bounded antiselfadjoint operator $S$ defined on $L^2(\Omega)$ by (5.30). Then,

– for all $k \in \mathbf{Z}^3$, the symbol $S_k$ of $S$ admits the following purely imaginar eigenvalues

$$\begin{aligned} &\lambda_1(k) = i, \quad \lambda_2(k) = -i, \\ &\lambda_3(k) = 0, \quad \lambda_4(k) = i\sqrt{\frac{|k|^2}{\alpha^2}+1}, \quad \lambda_5(k) = -i\sqrt{\frac{|k|^2}{\alpha^2}+1}, \end{aligned} \tag{5.31}$$

with the notation $|k|^2 = \frac{k_1^2}{a_1^2} + \frac{k_2^2}{a_2^2} + \frac{k_3^2}{a_3^2}$.

– the projections $\Pi_{k,j}$ $(j \in \{1,..,5\})$ on the eigenspaces of $S_k$ are bounded uniformly in $k \in \mathbf{Z}^3$. In particular, the projection on the kernel of $S$ is the pseudo-differential

operator $\mathcal{P}$ of order 0 defined by

$$\mathcal{P} = \begin{pmatrix} 0 & 0 & & & \\ 0 & 0 & & & \\ & & (-\Delta_x)(\alpha^2 Id - \Delta_x)^{-1} & 0\,Id & -\alpha(-\Delta_x)^{1/2}(\alpha^2 Id - \Delta_x)^{-1} \\ & & 0\,Id & 0\,Id & 0\,Id \\ & & -\alpha(-\Delta_x)^{1/2}(\alpha^2 Id - \Delta_x)^{-1} & 0\,Id & \alpha^2(\alpha^2 Id - \Delta_x)^{-1} \end{pmatrix};$$

– the group of isometries $\mathcal{S}$ generated by $S$ preserves all Sobolev norms.

*Démonstration.* As $S$ is a linear pseudo-differential operator with constant coefficients, a natural idea to study its spectral properties is to consider its symbol, i.e. the following block diagonal matrix

$$S_k = \begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & 0\,Id & Id & 0\,Id \\ & & -Id & 0\,Id & -\frac{|k|}{\alpha} Id \\ & & 0\,Id & \frac{|k|}{\alpha} Id & 0\,Id \end{pmatrix}$$

where $Id$ denotes the identity of $\mathbf{C}^3$.

• In order to describe the oscillating modes, it is enough to consider separately each block. The eigenvalues of

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

are $i$ and $-i$ which we denote by $\lambda_1(k)$ and $\lambda_2(k)$. The eigenvalues of

$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & -\frac{|k|}{\alpha} \\ 0 & \frac{|k|}{\alpha} & 0 \end{pmatrix}$$

are $\lambda_3(k) = 0$, $\lambda_4(k) = i\sqrt{\frac{|k|^2}{\alpha^2} + 1}$ and $\lambda_5(k) = -i\sqrt{\frac{|k|^2}{\alpha^2} + 1}$. As all eigenvalues are purely imaginar, it is easy to check that $S_k$ generates a group of isometries $(\exp(tS_k)_{t\in\mathbf{R}})$.

• To obtain the projections $\Pi_{k,j}$ $(j \in \{1, .., 5\})$ on the eigenspaces of $S_k$, we have to compute the eigenvectors and the transfer matrices, which is a technical step performed in Appendix A. Then the explicit formula for $\Pi_{k,j}$ shows that its norm is uniformly bounded with respect to $j$ and $k$ : in particular, the projections are pseudo-differential operators of order 0.

- The projection on the kernel of $S$ satisfies for all $k \in \mathbf{Z}^3$

$$\mathcal{P}_k = \Pi_{k,3} = \begin{pmatrix} 0 \\ 0 \\ \frac{-|k|}{\alpha\lambda}\, Id \\ 0\, Id \\ \frac{1}{\lambda}\, Id \end{pmatrix} \left(0, 0, \frac{-|k|}{\alpha\lambda}\, Id, 0\, Id, \frac{1}{\lambda}\, Id\right),$$

Then,

$$\mathcal{P} = \begin{pmatrix} 0 \\ 0 \\ -(-\Delta_x)^{1/2}(\alpha^2 Id - \Delta_x)^{-1/2} \\ 0\, Id \\ \alpha(\alpha^2 Id - \Delta_x)^{-1/2} \end{pmatrix} \left(0, 0, -(-\Delta_x)^{1/2}(\alpha^2 Id - \Delta_x)^{-1/2}, 0\, Id, \alpha(\alpha^2 Id - \Delta_x)^{-1/2}\right)$$

which concludes the proof. □

Lemma 5.6.1 shows that the linear part of the system (5.29) generates a group of isometries $\mathcal{S}$. Conjugating the equations (5.29) by $\mathcal{S}$ allows then to remove the fast temporal oscillations. Indeed we introduce the new variable

$$W_\epsilon = \mathcal{S}(\frac{t}{\epsilon})U_\epsilon \tag{5.32}$$

which is expected to have uniformly bounded time derivatives.

*Lemma* 5.6.2. Let $U^\epsilon = (\mathrm{div}_x v^\epsilon, \mathrm{div}_x e^\epsilon,\ \mathrm{curl}_x v^\epsilon,\ \mathrm{curl}_x e^\epsilon, (-\Delta_x)^{1/2} b^\epsilon)$ where $(v^\epsilon, e^\epsilon, b^\epsilon)$ is a solution of (5.29). Then the vector field $W^\epsilon$ defined by (5.32) satisfies the non-autonomous system of nonlinear equations

$$\partial_t W^\epsilon + Q(\frac{t}{\epsilon}, W^\epsilon, W^\epsilon) = 0 \tag{5.33}$$

where the operator $Q$ is defined by its Fourier coefficients

$$\forall k \in \mathbf{Z}^3 \quad Q_k(t, V, W) = \sum_{l+m=k} \sum_{\eta \in [[1;5]]^3} \exp(t\omega_\eta(k;l;m)) s_\eta(k;l;m)) V_l W_m$$

with

$$\omega_\eta(k;l;m) = \lambda_{\eta_1}(k) - \lambda_{\eta_2}(k) - \lambda_{\eta_3}(k). \tag{5.34}$$

Moreover, the tensor $s_\eta(l+m;l;m)$ satisfies

$$|s_\eta(l+m;l;m)| \leq C(|l| + |m|)$$

for some non negative constant $C$ independant of $l$ and $m$.

*Démonstration.* We can rewrite the equation satisfied by $U^\epsilon$ in the following way

$$\partial_t U^\epsilon + \frac{1}{\epsilon} S U^\epsilon + R(U^\epsilon; U^\epsilon) = 0 \tag{5.35}$$

where $R$ is the symmetric bilinear operator obtained by polarization of the quadratic forms

$$R(U,U) = \begin{pmatrix} \mathrm{div}_x(v \cdot \nabla v + \alpha v \wedge b) \\ \mathrm{div}_x(v \mathrm{div}_x e) \\ \mathrm{curl}_x(v \cdot \nabla v + \alpha v \wedge b) \\ \mathrm{curl}_x(v \mathrm{div}_x e) \\ 0 \end{pmatrix} \tag{5.36}$$

where $U$ is the vector field of coordinates $(\mathrm{div}_x v, \mathrm{div}_x e,\ \mathrm{curl}_x v,\ \mathrm{curl}_x e, (-\Delta_x)^{1/2} b)$, and $S$ is the linear penalization defined in Lemma 5.6.1.

Conjugating (5.35) by the group $\mathcal{S}$ provides

$$\mathcal{S}(\frac{t}{\epsilon})\partial_t U^\epsilon + \frac{1}{\epsilon}\mathcal{S}(\frac{t}{\epsilon}) S U^\epsilon + \mathcal{S}(\frac{t}{\epsilon}) R(U^\epsilon; U^\epsilon) = 0,$$

or equivalently

$$\partial_t(\mathcal{S}(\frac{t}{\epsilon})U^\epsilon) + \mathcal{S}(\frac{t}{\epsilon}) R(U^\epsilon; U^\epsilon) = 0.$$

Then

$$\partial_t W^\epsilon + \mathcal{S}(\frac{t}{\epsilon}) R(\mathcal{S}(\frac{-t}{\epsilon})W^\epsilon; \mathcal{S}(\frac{-t}{\epsilon})W^\epsilon) = 0,$$

which can be rewritten on the Fourier side

$$\partial_t W_k^\epsilon + \mathcal{S}_k(\frac{t}{\epsilon}) \sum_{l+m=k} \tilde{R}_{lm}[\mathcal{S}_l(\frac{-t}{\epsilon})W_l^\epsilon; \mathcal{S}_m(\frac{-t}{\epsilon})W_m^\epsilon] = 0$$

where $R_{lm}$ are the Fourier coefficients of $R$. By definition (5.36) of $R$

$$|R_{lm}| \leq C(|l| + |m|).$$

In order to describe explicitly the dependence with respect to time, we decompose the penalization according to the different modes. From Lemma 5.6.1 we deduce that

$$\mathcal{S}_k(t) = P_k^{-1} \begin{pmatrix} \lambda_1(k) & 0 & 0 & 0 & 0 \\ 0 & \lambda_2(k) & 0 & 0 & 0 \\ 0 & 0 & \lambda_3(k) Id & 0 & 0 \\ 0 & 0 & 0 & \lambda_4(k) Id & 0 \\ 0 & 0 & 0 & 0 & \lambda_5(k) Id \end{pmatrix} P_k$$

where $Id$ denotes the identity of $\mathbf{C}^3$. Then

$$\partial_t W_k^\epsilon + \sum_{l+m=k} \sum_{\eta \in [[1;5]]^3} \exp(\frac{t}{\epsilon}\omega_\eta(k,l,m)) s_\eta(k,l,m)[W_l^\epsilon; W_m^\epsilon] = 0$$

where the phase $\omega$ is defined by

$$\omega_\eta(k,l,m) = \lambda_{\eta_1}(k) - \lambda_{\eta_2}(l) - \lambda_{\eta_3}(m)$$

and the tensor $s$ satisfies

$$s_\eta(k,l,m)[W_l;W_m] = \Pi_{k,\eta_1} R_{lm}[\Pi_{l,\eta_2} W_l; \Pi_{m,\eta_3} W_m].$$

Since $||\Pi_{k,j}|| \leq C$ and $|R_{lm}| \leq C(|l|+|m|)$, we check that $|s_\eta(k,l,m)| \leq C(|l|+|m|)$. □

Lemma 5.6.2 seems to indicate that $W^\epsilon$ as well as its time derivatives should be uniformly bounded with respect to $\epsilon$. Such a result would imply in particular that (up to extraction of a subsequence) the family $W^\epsilon$ converges strongly to some $W$, which is supposed to solve the following limit system

$$\partial_t W + \overline{Q}(W,W) = 0 \tag{5.37}$$

where the autonomous bilinear operator $(V,W) \mapsto \overline{Q}(V,W)$ is defined by its Fourier coefficients

$$\forall k \in \mathbf{Z}^3, \quad \overline{Q}_k(V,W) = \sum_{\substack{l+m=k \\ \omega_\eta(k,l,m)=0}} s_\eta(k,l,m)[V_l;W_m], \tag{5.38}$$

the other terms converging weakly to 0 because of fast oscillations (formal time averaging).

Of course, we are not able to justify directly such an asymptotic. The idea consists then in describing precisely the formal limiting system (5.37) and then in constructing approximate solutions to the original system (5.33). A stability result will then allow to conclude.

### 5.6.2 Description of the resonances

In order to characterize and to obtain precise estimates on the solution $W$ of the system (5.37), the first step is to understand its structure, in particular the type of coupling produced by the quadratic term. We start by a precise description of the resonances.

*Lemma* 5.6.3. For all $\eta \in \{1,..,5\}^3$ and all $k,l,m \in \mathbf{Z}^3$, define $\omega_\eta(k,l,m)$ by (5.34). Then there exists a set $\mathcal{A} \subset (\mathbf{R}_*^+)^3$ of Lebesgue measure zero such that for all $(a_1,a_2,a_3) \in (\mathbf{R}_*^+)^3 \backslash \mathcal{A}$,

- $\omega_\eta(l+m,l,m) = 0$ implies that $\eta_1 = 3$ or $\eta_2 = 3$ or $\eta_3 = 3$
- $\omega_\eta(l+m,l,m) = 0$ with $\eta_1 = 3 \Rightarrow \{\eta_2,\eta_3\} = \{1,2\}$ or $\{\eta_2,\eta_3\} = \{4,5\}$ or $\eta_2 = \eta_3 = 3$ ;
- $\omega_\eta(l+m,l,m) = 0$ with $\eta_2 = 3 \Rightarrow \eta_1 = \eta_3$ ;
- $\omega_\eta(l+m,l,m) = 0$ with $\eta_3 = 3 \Rightarrow \eta_1 = \eta_2$.

Moreover, if $\omega_\eta(l+m,l,m) \neq 0$,

$$|\omega_\eta(l+m,l,m)|^{-1} \leq C(1+|l|)^s(1+|m|)^s, \tag{5.39}$$

for some nonnegative constants $C$ and $s$ depending only on $(a_1,a_2,a_3)$.

*Démonstration.* The results concerning the resonances, i.e. the solutions of the dispersion equation

$$\omega_\eta(l+m,l,m) = 0$$

come from algebraic properties of the functions $k \mapsto \lambda_j(k)$ defined by (5.31) : the main argument is the small divisor estimate stated in Appendix B.

We expect all resonances to involve generically at least one zero eigenvalue. To establish such a claim, we consider

$$q(l,m) = \Pi_{\eta \in \{1,2,4,5\}^3} \omega_\eta(l+m,l,m)$$

By (5.34), $q(l,m)$ is a polynomial with respect to the variables $\lambda_j(k)$ ( for $j \in \{1,2,4,5\}$ and $k \in \{l, m, l+m\}$). Considerations of symmetry ensure that it is actually a polynomial with respect to $\sigma_j(k)$ ($j \in [\![1,4]\!], k \in \{l, m, l+m\}$), where $(\sigma_j)_{j\in[\![1,4]\!]}$ are the symmetrical functions associated to $(\lambda_j)_{j\in\{1,2,4,5\}}$. Computing these elementary symmetrical functions shows that $q(l,m)$ is a polynomial with respect to $|l|^2$, $|m|^2$ and $|l+m|^2$. Then there exists a polynomial $P$ such that

$$q(l,m) = P\left(\frac{l_1}{a_1}, \frac{l_2}{a_2}, \frac{l_3}{a_3}, \frac{m_1}{a_1}, \frac{m_2}{a_2}, \frac{m_3}{a_3}\right)$$

By Proposition 5.9.1, there exist a set $\mathcal{A} \subset (\mathbf{R}_*^+)^3$ of Lebesgue measure zero and $\Omega \subset \mathbf{Z}^6$ such that for all $(a_1,a_2,a_3) \in (\mathbf{R}_+^*)^3 \setminus \mathcal{A}$, $\exists (C,s)$,

$$\begin{aligned} &\forall (l,m) \in \Omega,\ q(l,m) \equiv 0 \\ &\forall (l,m) \in \mathbf{Z}^6 \setminus \Omega,\ |q(l,m)|^{-1} \leq C(1+|l|)^s(1+|m|)^s \end{aligned} \tag{5.40}$$

In particular, as $|\omega_\eta(l+m,l,m)| \leq 3\left(1+\frac{2}{\alpha}\right)(1+|l|)(1+|m|)$ for all $\eta \in \{1,2,4,5\}^3$ and all $l,m \in \mathbf{Z}^3$, this implies

$$\forall (l,m) \in \mathbf{Z}^6 \setminus \Omega,\ \forall \eta \in \{1,2,4,5\}^3,\ |\omega_\eta(l+m,l,m)|^{-1} \leq C'(1+|l|)^{s'}(1+|m|)^{s'}$$

for some nonnegative constants $C'$ and $s'$.

It remains then to prove that $\Omega = \emptyset$. Consider $(l,m) \in \mathbf{Z}^6$. If $\eta_1$, $\eta_2$ and $\eta_3$ are not equal to 3, $\lambda_{\eta_1}(l+m)$, $\lambda_{\eta_2}(l)$ and $\lambda_{\eta_3}(m)$ tend to 1 or -1 as $a_1,a_2,a_3 \to +\infty$. Then, for all $\eta$, $\omega_\eta(l+m,l,m)$ converges to an odd number as $a_1,a_2,a_3 \to +\infty$. Thus $q(l,m) \not\equiv 0$ and $(l,m) \notin \Omega$. Then (5.40) provides the first assertion in Lemma 5.6.3, i.e. a necessary condition for having a resonance.

In order to complete our characterization of the resonances, we have then to consider the cases where (at least) one of the eigenvalues is zero. By a symmetry argument, it is enough to study one case, for instance $\eta_1 = 3$.

Then, $\omega_\eta(l+m,l,m) = 0$ if and only if

$$\lambda_{\eta_2}(l) + \lambda_{\eta_3}(m) = 0,$$

which is obviously equivalent to the following

- $\eta_2 = \eta_3 = 3$ ;
- $\{\eta_2,\eta_3\} = \{1,2\}$ ;
- $\{\eta_2,\eta_3\} = \{4,5\}$ and $|l| = |m|$.

In order to obtain an estimate for $|\omega_\eta(l+m,l,m)|^{-1}$ when $\lambda_{\eta_2}(l)+\lambda_{\eta_3}(m) \neq 0$, we compute it. If $\lambda_{\eta_2}(l)$ and $\lambda_{\eta_3}(m)$ have the same sign, $|\omega_\eta(l+m,l,m)| \geq 2$. If they have opposite signs,

- either $|\omega_\eta(l+m,l,m)| = \left|\sqrt{1+\frac{|l|^2}{\alpha^2}} - 1\right| \geq C$ ;
- or $|\omega_\eta(l+m,l,m)| = \left|\sqrt{1+\frac{|m|^2}{\alpha^2}} - 1\right| \geq C$ ;
- or $|\omega_\eta(l+m,l,m)| = \left|\sqrt{1+\frac{|l|^2}{\alpha^2}} - \sqrt{1+\frac{|m|^2}{\alpha^2}}\right| \geq C(1+|l|)^{-1}(1+|m|)^{-1}$ ;

where $C$ depends only on $a_1,a_2,a_3$. □

Note that the mean-field, that is the projection of $W$ on the kernel of $S$, seems to play a particular role. In the sequel, we will denote $\overline{W}$ this projection and $W_{osc} = W - \overline{W}$. Of

course, $W_{osc}$ does not depend on $\epsilon$, but we call it the oscillating part of $W$ since it gives the oscillating part of $U$ :

$$\overline{U} = \mathcal{S}(-\frac{t}{\epsilon})\overline{W} = \overline{W},$$

$$U_{osc} = \mathcal{S}(-\frac{t}{\epsilon})W_{osc},$$

should be respectively the weak limit of $U^\epsilon$, and the oscillating part of $U^\epsilon$ which describes exactly the defect of strong convergence of $U^\epsilon$ to $\overline{U}$.

### 5.6.3 Structure of the limiting system

Equipped with the previous result on the resonances, we are now able to analyze the structure of the limiting system. More precisely, we will prove the following

*Proposition* 5.6.1. There exists a set $\mathcal{A} \subset (\mathbf{R}_*^+)^3$ of Lebesgue measure zero such that for all $(a_1, a_2, a_3) \in (\mathbf{R}_*^+)^3 \setminus \mathcal{A}$, any solution $W$ of (5.37) can be decomposed in

$$W = \overline{W} + W_{osc}$$

where $\overline{W}$ solves a locally well-posed nonlinear autonomous equation (which is more or less equivalent to the limiting equation (5.27) under a convenient change of variables), and $W_{osc}$ is governed by a linear system of equations whose coefficients depend on $\overline{W}$.

#### The mean-field equation

The mean-field $\overline{W}$ is the projection of $W$ on the kernel $\mathrm{Ker}(S)$, which means that

$$\overline{W}_k = \Pi_{k,3} W_k.$$

Then from (5.37) we deduce that

$$\partial_t \overline{W}_k + \Pi_{k,3} \sum_{\substack{l+m=k, \eta \in [[1;5]]^3, \\ \omega_\eta(k,l,m)=0,}} s_\eta(k;l;m)[W_l; W_m] = 0$$

or equivalently

$$\partial_t \overline{W}_k + \sum_{\substack{l+m=k, \eta \in [[1;5]]^3, \\ \omega_\eta(k,l,m)=0, \eta_1=3}} s_\eta(k;l;m)[W_l; W_m] = 0 \tag{5.41}$$

By Lemma 5.6.3, we know that

$$\omega_\eta(k,l,m) = 0 \text{ with } \lambda_{\eta_1}(k) = 0$$

if and only if $\{\eta_2, \eta_3\} = \{1,2\}$, or $\eta_2 = \eta_3 = 3$, or $\{\eta_2, \eta_3\} = \{4,5\}$ with $|l| = |m|$.

Then (5.41) can be rewritten

$$\begin{aligned} &\partial_t \overline{W}_k + \sum_{l+m=k} s_{3,3,3}(k;l;m)[\overline{W}_l; \overline{W}_m] \\ &= - \sum_{\substack{l+m=k, \\ \{\eta_2,\eta_3\}=\{1,2\}, \eta_1=3}} s_\eta(k;l;m)[W_l; W_m] - \sum_{\substack{l+m=k, |l|=|m| \\ \{\eta_2,\eta_3\}=\{4,5\}, \eta_1=3}} s_\eta(k;l;m)[W_l; W_m] \end{aligned}$$

Let us then introduce the coordinates of $W_k$ in the following basis of eigenvectors

$$W_k = \begin{pmatrix} 1 \\ i \\ 0 \\ 0 \\ 0 \end{pmatrix} \mu_k + \begin{pmatrix} 1 \\ -i \\ 0 \\ 0 \\ 0 \end{pmatrix} \nu_k + \begin{pmatrix} 0 \\ 0 \\ \frac{-|k|}{\alpha\lambda_k} Id \\ 0\, Id \\ \frac{1}{\lambda_k} Id \end{pmatrix} \phi_k + \begin{pmatrix} 0 \\ 0 \\ \frac{1}{\lambda_k} Id \\ -i\, Id \\ \frac{-|k|}{\alpha\lambda_k} Id \end{pmatrix} \chi_k + \begin{pmatrix} 0 \\ 0 \\ \frac{1}{\lambda_k} Id \\ i\, Id \\ \frac{-|k|}{\alpha\lambda_k} Id \end{pmatrix} \psi_k,$$

where $\mu_k, \nu_k \in \mathbf{C}$ and $\phi_k, \chi_k, \psi_k \in \mathbf{C}^3$. Note that, since $W$ is real-valued, $W_k^* = W_{-k}$ and

$$\mu_k^* = \nu_{-k}, \quad \psi_k^* = \chi_{-k} \quad \text{and} \quad \phi_k^* = \phi_{-k}.$$

Moreover, with these notations, we have

$$\overline{W}_k = \begin{pmatrix} 0 \\ 0 \\ \frac{-|k|}{\alpha\lambda_k} Id \\ 0\, Id \\ \frac{1}{\lambda_k} Id \end{pmatrix} \phi_k.$$

The problem is now to determine the equation for the vector-valued coordinate $\phi_k$ :

$$\phi_k = \left(0, 0, \frac{-|k|}{\alpha\lambda_k} Id, 0\, Id, \frac{1}{\lambda_k} Id\right) W_k = \left(0, 0, \frac{-|k|}{\alpha\lambda_k} Id, 0\, Id, \frac{1}{\lambda_k} Id\right) \overline{W}_k.$$

By definition of $s_\eta(k,l,m)$, we have

$$\begin{aligned} &\partial_t \phi_k + \left(0, 0, \frac{-|k|}{\alpha\lambda_k} Id, 0\, Id, \frac{1}{\lambda_k} Id\right) \sum_{l+m=k} R_{lm}[\overline{W}_l, \overline{W}_m] \\ &= -\left(0, 0, \frac{-|k|}{\alpha\lambda_k} Id, 0\, Id, \frac{1}{\lambda_k} Id\right) \sum_{\substack{l+m=k, \\ \{\eta_2,\eta_3\}=\{1,2\}}} R_{lm}[\Pi_{l,\eta_2} W_l, \Pi_{m,\eta_3} W_m] \\ &\quad -\left(0, 0, \frac{-|k|}{\alpha\lambda_k} Id, 0\, Id, \frac{1}{\lambda_k} Id\right) \sum_{\substack{l+m=k, \\ \{\eta_2,\eta_3\}=\{4,5\}, \quad |l|=|m|}} R_{lm}[\Pi_{l,\eta_2} W_l, \Pi_{m,\eta_3} W_m] \end{aligned}$$

which can be rewritten

$$\partial_t \phi_k + \left(0, 0, \frac{-|k|}{\alpha\lambda_k} Id, 0\, Id, \frac{1}{\lambda_k} Id\right) \sum_{k=l+m} R_{lm} \left[ \begin{pmatrix} 0 \\ 0 \\ \frac{-|l|}{\alpha\lambda_l} Id \\ 0\, Id \\ \frac{1}{\lambda_l} Id \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \frac{-|m|}{\alpha\lambda_m} Id \\ 0\, Id \\ \frac{1}{\lambda_m} Id \end{pmatrix} \right] \phi_l \otimes \phi_m = -T_k^{\mu\nu} - T_k^{\chi\psi}$$

where

$$T_k^{\mu\nu} = 2\left(0,0,\frac{-|k|}{\alpha\lambda_k}\,Id, 0\,Id, \frac{1}{\lambda_k}\,Id\right)\sum_{k=l+m} R_{lm}\left[\begin{pmatrix}1\\ i\\ 0\,Id\\ 0\,Id\\ 0\,Id\end{pmatrix},\begin{pmatrix}1\\ -i\\ 0\,Id\\ 0\,Id\\ 0\,Id\end{pmatrix}\right]\mu_l\nu_m$$

$$T_k^{\chi\psi} = 2\left(0,0,\frac{-|k|}{\alpha\lambda_k}\,Id, 0\,Id, \frac{1}{\lambda_k}\,Id\right)\sum_{k=l+m} R_{lm}\left[\begin{pmatrix}0\\ 0\\ \frac{1}{\lambda_l}\,Id\\ -i\,Id\\ \frac{|k|}{\alpha\lambda_l}\,Id\end{pmatrix},\begin{pmatrix}0\\ 0\\ \frac{1}{\lambda_m}\,Id\\ i\,Id\\ \frac{|k|}{\alpha\lambda_m}\,Id\end{pmatrix}\right]\chi_l\otimes\psi_m.$$

Note that we only need to compute two (vector-valued) components of $R_{lm}$.

A symmetry argument (developped in Appendix C) allows actually to prove that both terms $T_k^{\mu\nu}$ and $T_k^{\chi\psi}$ vanish, so that $\phi$ satisfies an autonomous equation.

Keeping only the terms involving $\phi$, we get

$$\partial_t\phi_k + \frac{-|k|}{\alpha\lambda_k} ik\wedge \sum_{k=l+m}\left[\left(\frac{-il\wedge\phi_l}{\alpha|l|\lambda_l}\right).im\left(\frac{-im\wedge\phi_m}{\alpha|m|\lambda_m}\right) + \alpha\left(\frac{-il\wedge\phi_l}{\alpha|l|\lambda_l}\right)\wedge\frac{\phi_m}{|m|\lambda_m}\right] = 0$$

From this equation, we can deduce the equation satisfied by the mean field

$$\overline{U} = \overline{W} = \begin{pmatrix}0\\ 0\\ -(-\Delta_x)^{1/2}(\alpha^2 Id-\Delta_x)^{-1/2}\phi\\ 0\\ \alpha(\alpha^2 Id-\Delta_x)^{-1/2}\phi\end{pmatrix} = \begin{pmatrix}0\\ 0\\ \mathrm{curl}_x\overline{v}\\ 0\\ (-\Delta_x)^{1/2}\overline{b}\end{pmatrix}$$

which is exactly the limiting system (5.27) obtained for well-prepared initial data :

$$\partial_t(\ \mathrm{curl}_x\overline{v} - \alpha\overline{b}) + \ \mathrm{curl}_x(\overline{v}.\nabla_x\overline{v} + \alpha\overline{v}\wedge\overline{b}) = 0 \tag{5.42}$$

since $(\ \mathrm{curl}_x\overline{v} - \alpha\overline{b})_k = -\frac{\lambda\alpha}{|k|}\phi_k$.

**The equation for the oscillating part**

By definition, the oscillating part $W_{osc}$ of $W$ satisfies $W_{osc} = W - \overline{W}$ where $\overline{W} = \mathcal{P}W$ is the projection of $W$ on $\mathrm{Ker}(S)$ : in other words, $W_{osc}$ is the sum of the projections of $W$ on the other eigenspaces of $S$. The goal of this paragraph is then to determine the equations governing the coordinates $\mu_k$, $\nu_k$, $\psi_k$ and $\chi_k$.

From (5.37) we deduce that

$$\partial_t \Pi_{k,1} W_k + \Pi_{k,1} \sum_{\substack{l+m=k, \eta\in[[1;5]]^3, \\ \omega_\eta(k,l,m)=0,}} s_\eta(k;l;m)[W_l;W_m] = 0$$

or equivalently

$$\partial_t \mu_k + \left(\frac{1}{2}, -\frac{i}{2}, 0\,Id, 0\,Id, 0\,Id\right) \sum_{\substack{l+m=k, \eta\in[[1;5]]^3, \\ \omega_\eta(k,l,m)=0, \eta_1=1}} s_\eta(k;l;m)[W_l;W_m] = 0 \tag{5.43}$$

By Lemma 5.6.3, we know that

$$\omega_\eta(k,l,m) = 0 \text{ with } \eta_1 = 1$$

if and only if $\eta_2 = 3$ and $\eta_3 = 1$, or $\eta_2 = 1$ and $\eta_3 = 3$. Then (5.43) can be rewritten

$$\partial_t \mu_k + 2\left(\frac{1}{2}, -\frac{i}{2}, 0\,Id, 0\,Id, 0\,Id\right) \sum_{l+m=k} R_{lm} \left[ \begin{pmatrix} 0 \\ 0 \\ \frac{-|l|}{\alpha\lambda_l}\,Id \\ 0\,Id \\ \frac{1}{\lambda_l}\,Id \end{pmatrix}, \begin{pmatrix} 1 \\ i \\ 0\,Id \\ 0\,Id \\ 0\,Id \end{pmatrix} \right] \phi_l \mu_m = 0$$

or in explicit form

$$\begin{aligned} \partial_t \mu_k + ik \cdot \sum_{k=l+m} \Bigg[ & \left(\frac{il}{\alpha\lambda_l|l|} \wedge (-|l|)\phi_l\right) \cdot im \left(\frac{-im}{|m|^2}\mu_m\right) + \left(\frac{-im}{|m|^2}\mu_m\right) \cdot il \left(\frac{il}{\alpha\lambda_l|l|} \wedge (-|l|)\phi_l\right) \\ & + \alpha \left(\frac{-im}{|m|^2}\mu_m \wedge \frac{\phi_l}{|l|\lambda_l}\right) - i \left(\frac{il}{\alpha\lambda_l|l|} \wedge (-|l|)\phi_l\right) i\mu_m \Bigg] = 0 \end{aligned} \tag{5.44}$$

In the same way,

$$\begin{aligned} \partial_t \nu_k + ik \cdot \sum_{k=l+m} \Bigg[ & \left(\frac{il}{\alpha\lambda_l|l|} \wedge (-|l|)\phi_l\right) \cdot im \left(\frac{-im}{|m|^2}\nu_m\right) + \left(\frac{-im}{|m|^2}\nu_m\right) \cdot il \left(\frac{il}{\alpha\lambda_l|l|} \wedge (-|l|)\phi_l\right) \\ & + \alpha \left(\frac{-im}{|m|^2}\nu_m \wedge \frac{\phi_l}{|l|\lambda_l}\right) + i \left(\frac{il}{\alpha\lambda_l|l|} \wedge (-|l|)\phi_l\right) (-i\nu_m) \Bigg] = 0 \end{aligned} \tag{5.45}$$

From (5.37) we also deduce that

$$\partial_t \chi_k + \left(0, 0, \frac{1}{2\lambda_k}\,Id, \frac{i}{2}\,Id, \frac{|k|}{2\alpha\lambda_k}\,Id\right) \sum_{\substack{l+m=k, \eta\in[[1;5]]^3, \\ \omega_\eta(k,l,m)=0,}} s_\eta(k;l;m))[W_l;W_m] = 0.$$

By Lemma 5.6.3, we know that

$$\omega_\eta(k,l,m) = 0 \text{ with } \eta_1 = 4$$

if and only if $\eta_2 = 3,\ \eta_3 = 4$ and $|k| = |m|$, or $\eta_2 = 4,\ \eta_3 = 3$ and $|k| = |l|$. Then

$$\partial_t \chi_k + 2\left(0, 0, \frac{1}{2\lambda_k}\, Id, \frac{i}{2}\, Id, \frac{|k|}{2\alpha\lambda_k}\, Id\right) \sum_{\substack{l+m=k,\\ |k|=|m|}} R_{lm} \left[ \begin{pmatrix} 0 \\ 0 \\ \frac{-|l|}{\alpha\lambda_l}\, Id \\ 0\, Id \\ \frac{1}{\lambda_l}\, Id \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \frac{1}{\lambda_m}\, Id \\ -i\, Id \\ \frac{|m|}{\alpha\lambda_m}\, Id \end{pmatrix} \right] \phi_l \chi_m = 0,$$

which can be rewritten

$$\begin{aligned} \partial_t \chi_k + \frac{1}{\lambda_k}\, ik \wedge \sum_{\substack{l+m=k,\\ |k|=|m|}} & \left[ \left( \frac{-il \wedge \phi_l}{\alpha |l| \lambda_l} \right) \cdot im \left( \frac{im \wedge \chi_m}{|m|^2 \lambda_m} \right) + \alpha \left( \frac{-il \wedge \phi_l}{\alpha |l| \lambda_l} \right) \wedge \frac{\chi_m}{\alpha \lambda_m} \right. \\ & \left. + \left( \frac{im \wedge \chi_m}{|m|^2 \lambda_m} \right) \cdot il \left( \frac{-il \wedge \phi_l}{\alpha |l| \lambda_l} \right) + \alpha \left( \frac{im \wedge \chi_m}{|m|^2 \lambda_m} \right) \wedge \frac{\phi_l}{|l| \lambda_l} \right] = 0. \end{aligned} \tag{5.46}$$

And in the same way,

$$\begin{aligned} \partial_t \psi_k + \frac{1}{\lambda_k}\, ik \wedge \sum_{\substack{l+m=k,\\ |k|=|m|}} & \left[ \left( \frac{-il \wedge \phi_l}{\alpha |l| \lambda_l} \right) \cdot im \left( \frac{im \wedge \psi_m}{|m|^2 \lambda_m} \right) + \alpha \left( \frac{-il \wedge \phi_l}{\alpha |l| \lambda_l} \right) \wedge \frac{\psi_m}{\alpha \lambda_m} \right. \\ & \left. + \left( \frac{im \wedge \psi_m}{|m|^2 \lambda_m} \right) \cdot il \left( \frac{-il \wedge \phi_l}{\alpha |l| \lambda_l} \right) + \alpha \left( \frac{im \wedge \psi_m}{|m|^2 \lambda_m} \right) \wedge \frac{\phi_l}{|l| \lambda_l} \right] = 0. \end{aligned} \tag{5.47}$$

In particular, the equations governing $\mu$, $\nu$, $\chi$ and $\psi$ are linear pseudo-differential equations whose coefficients depend (linearly) on $\phi$ and its first derivatives, which concludes the proof of Proposition 5.6.1.

**Theory for the asymptotic system**

The previous analysis of the structure of the limiting system (5.37) allows to prove that the Cauchy problem is well-posed for smooth initial data.

*Proposition* 5.6.2. Let $(a_1, a_2, a_3) \in (\mathbf{R}_*^+)^3 \setminus \mathcal{A}$ where $\mathcal{A}$ is the set of Lebesgue measure zero defined in Proposition 5.6.1, and $(W^{in}) \in C^{r+1}(\Omega)$ with $r > 2$. Then there exists $T^* > 0$ such that the system (5.37) admits a unique strong solution $W \in L^\infty_{loc}([0, T^*[, H^r(\Omega))$ with initial data $W^{in}$.

*Démonstration.* By Proposition 5.6.1, for all $(a_1, a_2, a_3) \in (\mathbf{R}_*^+)^3 \setminus \mathcal{A}$, any solution $W$ of (5.37) can be decomposed in

$$W = \overline{W} + W_{osc}$$

where $\overline{W}$ solves a well-posed nonlinear autonomous equation (which is more or less equivalent to the limiting equation (5.27) under a convenient change of variables), and $W_{osc}$ is governed by a linear system of equations whose coefficients depend on $\overline{W}$.

If $W^{in} \in C^{r+1}(\Omega)$, then $\overline{W}^{in} = \mathcal{P} W^{in} \in C^{r+1}(\Omega)$ because $\mathcal{P}$ is a pseudo-differential operator of order 0. Moreover, we have identified the autonomous equation for $\overline{W} = (0, 0,\ \mathrm{curl}_x \overline{v}, 0, (-\Delta_x)^{1/2} \overline{b})^T$ to be exactly the system (5.27) for $(\overline{v}, \overline{b})$. We have seen in Lemma

5.5.1 that (5.27) can be studied exactly as the 3D incompressible Euler equation. Then there exists $T^* > 0$ such that there exists $\overline{W}$ solution of (5.41) on $[0, T^*[$ and for all $T < T^*$,

$$\|\overline{W}\|_{L^\infty([0,T],C^{r+1})} + \left\|\partial_t \overline{W}\right\|_{L^\infty([0,T],C^r)} \leq \|(\overline{v}, \overline{b})\|_{L^\infty([0,T],C^{r+2})} \leq C,$$

where $C$ depends on $T$ and on $\|(\overline{v}^{in}, \overline{b}^{in})\|_{C^{r+2}} \leq \|W^{in}\|_{C^{r+1}}$.

The equations governing $\mu$, $\nu$, $\chi$ and $\psi$ are linear equations of the form

$$\partial_t W_{osc} + \alpha(\phi).\nabla_x W_{osc} = \beta(W_{osc}, \phi)$$

where $\alpha$ is a linear pseudodifferential operator of order 0, $\beta$ is a bilinear pseudo-differential operator of order 0 with respect to $W_{osc}$ and of order 1 with respect to $\phi$, and such that

$$\mathrm{div}_x \alpha(\phi) = 0.$$

Then it is easy to check that

$$\frac{d}{dt}\|W_{osc}\|^2_{H^s} \leq C\|W_{osc}\|^2_{H^s}\|\phi\|_{C^{s+1}},$$

and by Gronwall's inequality

$$\|W_{osc}\|_{L^\infty([0,T],H^s)} \leq \|W^{in}\|^2_{H^s} \exp(CT\|\overline{W}\|_{L^\infty([0,T],C^{s+1})}).$$

Then, for all $T < T^*$, there exists $C_T > 0$ such that

$$\|W\|_{L^\infty([0,T],H^r)} + \|\partial_t W\|_{L^\infty([0,T],H^{r-1})} \leq C_T. \tag{5.48}$$

□

## 5.7 Weak convergence for general initial data

### 5.7.1 Error estimates

In this section, we want to prove that the solution $W$ of (5.37) provides a good estimate of any solution $W^\epsilon$ of (5.33). A natural idea is to use a stability argument of Gronwall type to prove that

$$\|W^\epsilon - W\|_{L^\infty(\Omega)} \to 0.$$

For instance, if

$$\partial_t W + Q(\frac{t}{\epsilon}, W, W) \to 0 \tag{5.49}$$

the following would hold

$$\partial_t(W - W^\epsilon) + Q(\frac{t}{\epsilon}, W - W^\epsilon, W + W^\epsilon) \to 0$$

and then, by definition of $Q$,

$$\frac{d}{dt}||W - W^\epsilon||_{L^\infty} \leq C||W - W^\epsilon||_{L^\infty}||W||_{W^{1\infty}} + \mathrm{O}(\epsilon),$$

which would give the expected convergence result by the Gronwall Lemma, since $W$ satisfies the strong estimate $||W||_{W^{1,\infty}} \leq C$.

Nevertheless the convergence (5.49) does not hold in norm but only in a weak sense (recall that oscillating terms have been neglected in order to obtain the asymptotic system). A standard method to study singular perturbation consists then in introducing a small quantity $\epsilon y_\epsilon$ with $\|y_\epsilon\| = O(1)$ in a convenient norm, and such that

$$\partial_t(W+\epsilon y^\epsilon) + Q(\frac{t}{\epsilon}, (W+\epsilon y^\epsilon), (W+\epsilon y^\epsilon)) \to 0. \tag{5.50}$$

The previous stability argument allows then to conclude since $W+\epsilon y^\epsilon$ obviously has the same behaviour as $W$.

*Lemma* 5.7.1. Let $(W^{in}) \in C^{r+1}(\Omega)$ with $r > 3$. Denote by $W \in L^\infty([0,T], H^r)$ the solution of (5.37) with initial data $W^{in}$ ; define $y^\epsilon$ by its Fourier coefficients

$$\forall k \in \mathbf{Z}^3,\ y_k^\epsilon = - \sum_{\substack{l+m=k,\ |l|+|m|\le |\log\epsilon|,\\ \eta\in[\![1,5]\!]^3,\ \omega_\eta(k,l,m)\neq 0}} \frac{\exp\left(\frac{t}{\epsilon}\omega_\eta(k,l,m)\right)}{\omega_\eta(k,l,m)} s_\eta(k,l,m) W_l W_m \tag{5.51}$$

Then,

– there exists a nonnegative constant $C$ such that

$$\|y^\epsilon\|_{L^\infty([0,T],H^r)} \le C|\log\epsilon|^{2s+1}$$

where $s$ depends only on $(a_1,a_2,a_3) \in (\mathbf{R}_*^+)^3 \setminus \mathcal{A}$.

– there exists $\delta \in C(\mathbf{R}^+)$ with $\delta(0) = 0$ such that

$$\left\| \partial_t(W+\epsilon y^\epsilon) + Q\left(\frac{t}{\epsilon}, (W+\epsilon y^\epsilon), (W+\epsilon y^\epsilon)\right)\right\|_{L^\infty([0,T],H^2)} \le \delta(\epsilon)\,.$$

*Démonstration.* By Proposition 5.6.2, the solution $W$ of (5.37) with initial data $W^{in} \in C^{r+1}(\Omega)$ satisfies the following regularity estimate

$$\|W\|_{L^\infty([0,T],H^r)} + \|\partial_t W\|_{L^\infty([0,T],H^{r-1})} \le C_T$$

for some nonnegative constant $C_T$, as soon as $T$ is strictly less than the lifespan $T^*$.

On the other hand, by Lemma 5.6.3, there exists nonnegative constants $(C,s)$ such that

$$\forall (l,m,\eta),\ \omega_\eta(l+m,l,m) \neq 0 \Rightarrow |\omega_\eta(l+m,l,m)|^{-1} \le C(1+|l|)^s(1+|m|)^s\,.$$

By Lemma 5.6.2,

$$|s_\eta(l+m,l,m)| \le C(|l|+|m|)$$

Combining these last two estimates leads to

$$\begin{aligned} |y_k^\epsilon| &\le C \sum_{l+m=k,\ |l|+|m|\le|\log\epsilon|} (1+|l|)^s(1+|m|)^s(|l|+|m|)|W_l||W_m| \\ &\le C|\log\epsilon|^{2s+1} \sum_{l+m=k} |W_l||W_m|\,. \end{aligned}$$

Then

$$(1+|k|^2)^{r/2}|y_k^\epsilon| \le C|\log\epsilon|^{2s+1} \sum_{l+m=k} (1+|l|^2)^{r/2}|W_l|(1+|m|^2)^{r/2}|W_m|$$

which, together with (5.48) gives the expected bound on $\|y^\epsilon\|_{L^\infty([0,T],H^r)}$.

In the next step we check that $W+\epsilon y^\epsilon$ approximately verifies (5.33). By (5.37) and (5.51),

$$\begin{aligned}\partial_t(W+\epsilon y^\epsilon)_k = &- \sum_{\substack{l+m=k,\\ \omega_\eta(k,l,m)=0}} s_\eta(k,l,m)[W_l,W_m] \\ &- \sum_{\substack{l+m=k,\,|l|+|m|\le|\log\epsilon|,\\ \omega_\eta(k,l,m)\neq 0}} \exp\left(\frac{t}{\epsilon}\omega_\eta(k,l,m)\right) s_\eta(k,l,m)[W_l,W_m] \\ &- \epsilon \sum_{\substack{l+m=k,\,|l|+|m|\le|\log\epsilon|,\\ \omega_\eta(k,l,m)\neq 0}} \frac{\exp\left(\frac{t}{\epsilon}\omega_\eta(k,l,m)\right)}{\omega_\eta(k,l,m)} s_\eta(k,l,m)\partial_t[W_l,W_m]\end{aligned}$$

from which we deduce that

$$\begin{aligned}&\partial_t(W+\epsilon y^\epsilon)_k + Q_k(\frac{t}{\epsilon}, W+\epsilon y^\epsilon, W+\epsilon y^\epsilon) \\ &= \sum_{\substack{l+m=k,|l|+|m|>|\log\epsilon|\\ \omega_\eta(k,l,m)\neq 0}} \exp\left(\frac{t}{\epsilon}\omega_\eta(k,l,m)\right) s_\eta(k,l,m)[W_l,W_m] + \epsilon Q_k\left(\frac{t}{\epsilon}, y^\epsilon, 2W+\epsilon y^\epsilon\right) \\ &\quad - \epsilon \sum_{\substack{l+m=k,\,|l|+|m|\le|\log\epsilon|,\\ \omega_\eta(k,l,m)\neq 0}} \frac{\exp\left(\frac{t}{\epsilon}\omega_\eta(k,l,m)\right)}{\omega_\eta(k,l,m)} s_\eta(k,l,m)\partial_t[W_l,W_m]\,.\end{aligned}$$

The estimates on $s_\eta(l+m,l,m)$ and $\omega_\eta(l+m,l,m)$ give then

$$\begin{aligned}(1+|k|^2)&\left|\partial_t(W+\epsilon y^\epsilon)_k + Q_k\left(\frac{t}{\epsilon}, W+\epsilon y^\epsilon, W+\epsilon y^\epsilon\right)\right| \\ &\le C \sum_{l+m=k,|l|+|m|>|\log\epsilon|} (|l|+|m|)^{-(r-3)}(1+|l|^2)^{r/2}|W_l|(1+|m|^2)^{r/2}|W_m| \\ &+ C\epsilon \sum_{l+m=k} (1+|l|+|m|)^3|y^\epsilon_l||(2W+\epsilon y^\epsilon)_m| \\ &+ C\epsilon \sum_{\substack{l+m=k,\,|l|+|m|\le|\log\epsilon|,\\ \omega_\eta(k,l,m)\neq 0}} (1+|l|)^s(1+|m|)^s(1+|l|+|m|)^3|\partial_t W_l||W_m|\end{aligned}$$

which can be rewritten

$$\begin{aligned}\left\|\partial_t(W+\epsilon y^\epsilon)_k + Q_k\left(\frac{t}{\epsilon}, W+\epsilon y^\epsilon, W+\epsilon y^\epsilon\right)\right\|_{H^2} &\le C|\log\epsilon|^{-(r-3)}\|W\|^2_{H^r} \\ &+ C\epsilon\|y^\epsilon\|_{H^3}\|2W+\epsilon y^\epsilon\|_{H^3} \\ &+ C\epsilon|\log\epsilon|^{2s+3}\|\partial_t W\|_{L^2}\|W\|_{L^2}\,.\end{aligned}$$

Using the a priori estimates on $W$ and $y^\epsilon$ leads to the expected result. □

### 5.7.2 Convergence result

The weak compactness results established in the well-prepared case are still valid : up to extraction of a subsequence, we then have

$$\epsilon E^{\epsilon,\alpha} \rightharpoonup e, \quad B^{\epsilon,\alpha} \rightharpoonup b \text{ in } w^* - L^\infty(\mathbf{R}^+, L^2(\Omega))$$

$$\rho^{\epsilon,\alpha} \rightharpoonup \rho, \quad \hat{j}^{\epsilon,\alpha} \rightharpoonup j = \rho v \text{ in the sense of measures.}$$

Moreover, using the Poisson equation, we obtain that

$$\rho = 1.$$

It remains then to identify the weak limits $v$, $e$ and $b$, and more precisely to establish that $e = 0$, and $(v, b)$ is a solution of (5.27), or in other words that

$$\begin{pmatrix} 0 \\ 0 \\ \operatorname{curl}_x v \\ 0 \\ (-\Delta_x)^{1/2} b \end{pmatrix} = \overline{W} = \mathcal{P}W$$

where $W$ is the solution of (5.37).

As the stability method used here can only provide strong convergence results, we actually need a more precise description of the asymptotic behaviour. As $W + \epsilon y^\epsilon$ approximately verifies the macroscopic system (5.33), we expect $U_\epsilon = \mathcal{S}\left(-\frac{t}{\epsilon}\right)(W + \epsilon y^\epsilon)$ to provide a good approximation of the macroscopic quantities

$$(\operatorname{div}_x \hat{j}^{\epsilon,\alpha}, \operatorname{div}_x(\epsilon E^{\epsilon,\alpha}),\ \operatorname{curl}_x \hat{j}^{\epsilon,\alpha},\ \operatorname{curl}_x(\epsilon E^{\epsilon,\alpha}), (-\Delta_x)^{1/2} B^{\epsilon,\alpha})$$

defined from the Vlasov-Maxwell system. Indeed we will prove that

$$\begin{aligned} \hat{j}^{\epsilon,\alpha} &\sim u_{\epsilon,\alpha} \equiv -\nabla_x(-\Delta_x)^{-1} U_{\epsilon,1} + \operatorname{curl}_x(-\Delta_x)^{-1} U_{\epsilon,3}, \\ \epsilon E^{\epsilon,\alpha} &\sim e_{\epsilon,\alpha} \equiv -\nabla_x(-\Delta_x)^{-1} U_{\epsilon,2} + \operatorname{curl}_x(-\Delta_x)^{-1} U_{\epsilon,4}, \\ B^{\epsilon,\alpha} &\sim b_{\epsilon,\alpha} \equiv (-\Delta_x)^{-1/2} U_{\epsilon,5}. \end{aligned}$$

By Lemma 5.7.1, we know that there exists $\delta \in C(\mathbf{R}^+)$ with $\delta(0) = 0$ such that

$$\left\| \partial_t (W + \epsilon y^\epsilon) + Q\left(\frac{t}{\epsilon}, (W + \epsilon y^\epsilon), (W + \epsilon y^\epsilon)\right) \right\|_{L^\infty([0,T], H^2)} \leq \delta(\epsilon)\,,$$

from which we deduce that

$$\left\| \partial_t U_\epsilon + \frac{1}{\epsilon} S U_\epsilon + R(U_\epsilon, U_\epsilon) \right\|_{L^\infty([0,T], H^2)} \leq \delta(\epsilon)\,.$$

Comparing both systems (5.35) and (5.20) shows that

$$A^{\epsilon,\alpha}(u_{\epsilon,\alpha}, e_{\epsilon,\alpha}, b_{\epsilon,\alpha}) = O(\delta(\epsilon))$$

in $L^\infty([0,T] \times \Omega)$. Then plugging this last identity in the stability inequality (5.26) leads to

$$H^{\epsilon,\alpha}_{u_{\epsilon,\alpha}, e_{\epsilon,\alpha}, b_{\epsilon,\alpha}}(t) \leq H^{\epsilon,\alpha}_{u_{\epsilon,\alpha}, e_{\epsilon,\alpha}, b_{\epsilon,\alpha}}(0) \exp(6 \int_0^t \|D_x v(s)\|_{L^\infty} ds) + O(\gamma) + O(\delta(\epsilon)).$$

The assumption on the initial data gives exactly

$$H^{\epsilon,\alpha}_{u_{\epsilon,\alpha}, e_{\epsilon,\alpha}, b_{\epsilon,\alpha}}(0) = H^{\epsilon,\alpha}_{u,e,b}(0) + O(\epsilon) \to 0$$

with

$$u \equiv -\nabla_x(-\Delta_x)^{-1}U_1^{in} + \ \mathrm{curl}_x(-\Delta_x)^{-1}U_3^{in},$$
$$e \equiv -\nabla_x(-\Delta_x)^{-1}U_2^{in} + \ \mathrm{curl}_x(-\Delta_x)^{-1}U_4^{in},$$
$$b \equiv (-\Delta_x)^{-1/2}U_5^{in} \text{ where } U^{in} = W^{in}.$$

We then deduce

$$H^{\epsilon,\alpha}_{u_{\epsilon,\alpha},e_{\epsilon,\alpha},b_{\epsilon,\alpha}}(t) \to 0 \text{ as } \epsilon, \gamma \to 0,$$

for all $t < T^*$ where $T^*$ is the lifespan of the strong solution of (5.27). In particular,

$$\epsilon E^{\epsilon,\alpha} \rightharpoonup 0, \quad B^{\epsilon,\alpha} \rightharpoonup b \text{ in } w - L^\infty([0,T], L^2(\Omega)).$$

To get the convergence result on the current, a natural idea is to use the following estimate

$$\int \frac{|\hat{j}^{\epsilon,\alpha} - \hat{\rho}^{\epsilon,\alpha}u_{\epsilon,\alpha}|^2}{\hat{\rho}^{\epsilon,\alpha}} \leq 2H^{\epsilon,\alpha}_{u_{\epsilon,\alpha},e_{\epsilon,\alpha},b_{\epsilon,\alpha}}$$

as in the well-prepared case. Nevertheless, since $u_{\epsilon,\alpha}$ has an oscillating part, we should need some regularity with respect to time on $\hat{\rho}^{\epsilon,\alpha}$ and $\sqrt{\hat{\rho}^{\epsilon,\alpha}}$ to conclude directly, which seems to be a difficult issue. Here we will rather use an argument of weak convergence. Taking weak limits in the Ampere equation (the last equation in (5.4)) shows that the limiting macroscopic quantities $(v, e, b)$ belongs to the kernel of the penalization $L$, in particular

$$\mathrm{curl}_x b = -\alpha v.$$

We have like this identified the weak limits $v$, $e$ and $b$, which concludes the proof of theorem 5.2.3.

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## 5.8 Appendix A The singular perturbation

The study of the asymptotic behaviour of a singular system such as (5.29) requires a good understanding of the linear perturbation. This first appendix is devoted to technical computations leading to the precise description of the oscillating modes of the linear operator

$S$ with symbol

$$S_k = \begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & 0Id & Id & 0Id \\ & & -Id & 0Id & -\frac{|k|}{\alpha}Id \\ & & 0Id & \frac{|k|}{\alpha}Id & 0Id \end{pmatrix}.$$

- We are first interested in the diagonalization of the first block

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

which is actually independent of $k$.

We have seen that its eigenvalues are $i$ and $-i$. Correponding eigenvectors are $(1, i)$ and $(1, -i)$, from which we deduce that the transfer matrices are

$$P = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \quad \text{and} \quad P^{-1} = \begin{pmatrix} 1/2 & -i/2 \\ 1/2 & i/2 \end{pmatrix}$$

and that the projections $\Pi_{k,1}$ and $\Pi_{k,2}$ can be written

$$\Pi_{k,1} = \begin{pmatrix} 1 \\ i \\ 0\,Id \\ 0\,Id \\ 0\,Id \end{pmatrix} \left(\frac{1}{2}, -\frac{i}{2}, 0\,Id, 0\,Id, 0\,Id\right),$$

$$\Pi_{k,2} = \begin{pmatrix} 1 \\ -i \\ 0\,Id \\ 0\,Id \\ 0\,Id \end{pmatrix} \left(\frac{1}{2}, \frac{i}{2}, 0\,Id, 0\,Id, 0\,Id\right).$$

- The second block can be tensorized as follows

$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & -\frac{|k|}{\alpha} \\ 0 & \frac{|k|}{\alpha} & 0 \end{pmatrix} \otimes Id$$

where $Id$ denotes the identity of $\mathbf{C}^3$.

The eigenvalues of such a matrix are $\lambda_3(k) = 0$, $\lambda_4(k) = i\sqrt{\frac{|k|^2}{\alpha^2}+1}$ and $\lambda_5(k) = -i\sqrt{\frac{|k|^2}{\alpha^2}+1}$. The transfer matrices diagonalizing the first factor can be written

$$P_k = \begin{pmatrix} \frac{-|k|}{\alpha\lambda} & \frac{1}{\lambda} & \frac{1}{\lambda} \\ 0 & -i & i \\ \frac{1}{\lambda} & \frac{|k|}{\alpha\lambda} & \frac{|k|}{\alpha\lambda} \end{pmatrix} \quad \text{and} \quad P_k^{-1} = \begin{pmatrix} \frac{-|k|}{\alpha\lambda} & 0 & \frac{1}{\lambda} \\ \frac{1}{2\lambda} & \frac{i}{2} & \frac{|k|}{2\alpha\lambda} \\ \frac{1}{2\lambda} & -\frac{i}{2} & \frac{|k|}{2\alpha\lambda} \end{pmatrix},$$

with the notation $\lambda = \sqrt{1+|k|^2/\alpha^2}$ from which we deduce the explicit formula for the projections $\Pi_{k,3}$, $\Pi_{k,4}$ and $\Pi_{k,5}$ :

$$\Pi_{k,3} = \begin{pmatrix} 0 \\ 0 \\ \frac{-|k|}{\alpha\lambda}\, Id \\ 0\, Id \\ \frac{1}{\lambda}\, Id \end{pmatrix} \left(0, 0, \frac{-|k|}{\alpha\lambda}\, Id, 0\, Id, \frac{1}{\lambda}\, Id\right),$$

$$\Pi_{k,4} = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{\lambda}\, Id \\ -i\, Id \\ \frac{|k|}{\alpha\lambda}\, Id \end{pmatrix} \left(0, 0, \frac{1}{2\lambda}\, Id, \frac{i}{2}\, Id, \frac{|k|}{2\alpha\lambda}\, Id\right),$$

$$\Pi_{k,5} = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{\lambda}\, Id \\ i\, Id \\ \frac{|k|}{\alpha\lambda}\, Id \end{pmatrix} \left(0, 0, \frac{1}{2\lambda}\, Id, -\frac{i}{2}\, Id, \frac{|k|}{2\alpha\lambda}\, Id\right).$$

## 5.9 Appendix B Small divisor estimate

In order to describe the coupling between the various oscillating components through the nonlinear terms, a basic tool is the study of the resonances, i.e. of the solutions $(l, m, \eta) \in (\mathbf{Z}^3)^2 \times [\![1,5]\!]^3$ of the dispersion equation

$$\omega_\eta(l+m, l, m) = \lambda_{\eta_1}(l+m) - \lambda_{\eta_2}(l) - \lambda_{\eta_3}(m) = 0,$$

where $(\lambda_j(k))_{j\in[\![1,5]\!]}$ denote the eigenvalues of $R_k$ for all $k \in \mathbf{Z}^3$. As these eigenvalues are the roots of a polynom with polynomial coefficients in $k_1/a_1$, $k_2/a_2$ and $k_3/a_3$, the following small divisor estimate [6] plays a crucial role to obtain the structure of the limiting equation.

*Proposition* 5.9.1. Let $P(l,m)$ be a polynomial in $a_3^{-1}$ with coefficients that are polynomials in $l, m$. Then, there exist $\mathcal{A} \subset \mathbf{R}_*^+$ of Lebesgue measure zero and $\Omega \subset \mathbf{Z}^6$ such that

$$\forall (l,m) \in \Omega,\ P(l,m) \equiv 0$$

$$\forall a_3 \in \mathbf{R}_+^* \setminus \mathcal{A},\ \exists (C,s),\ \forall (l,m) \in \mathbf{Z}^6 \setminus \Omega,\ |P(l,m)|^{-1} \leq C(1+|l|)^s (1+|m|)^s$$

## 5.10 Appendix C
## Symmetry properties of the bilinear operator $R$

In order to prove that the mean field satisfies an autonomous equation, we have to establish that there does not exist any constructive coupling between oscillating terms, which is due to the particular form of the nonlinear term.

The explicit formula for $R_{lm}$ shows that

$$\begin{aligned}
&\left(0,0,\frac{-|k|}{\alpha\lambda_k}\,Id, 0\,Id, \frac{1}{\lambda}\,Id\right) R_{l,m}[X_l, Y_m] \\
&= \frac{-|k|}{2\alpha\lambda_k} ik \wedge \left( \left( \frac{-il}{|l|^2} X_{l,1} + \frac{il}{|l|^2} \wedge X_{l,3} \right) .im \left( \frac{-im}{|m|^2} Y_{m,1} + \frac{im}{|m|^2} \wedge Y_{m,3} \right) \right. \\
&\left. + \alpha \left( \frac{-il}{|l|^2} X_{l,1} + \frac{il}{|l|^2} \wedge X_{l,3} \right) \wedge \frac{1}{|m|} Y_{m,5} \right) \\
&+ \frac{-|k|}{2\alpha\lambda_k} ik \wedge \left( \left( \frac{-im}{|m|^2} Y_{m,1} + \frac{im}{|m|^2} \wedge Y_{m,3} \right) .il \left( \frac{-il}{|l|^2} X_{l,1} + \frac{il}{|l|^2} \wedge X_{l,3} \right) \right. \\
&\left. + \alpha \left( \frac{-im}{|m|^2} Y_{m,1} + \frac{im}{|m|^2} \wedge Y_{m,3} \right) \wedge \frac{1}{|l|} X_{l,5} \right)
\end{aligned}$$

Plugging $X_l = (\mu_l, i\mu_l, 0, 0, 0)^T$ and $Y_m = (\nu_m, -i\nu_m, 0, 0, 0)^T$ into the previous identity leads to

$$\begin{aligned}
T_k^{\mu\nu} &= \frac{-|k|}{2\alpha\lambda_k} ik \wedge \sum_{k=l+m} \mu_l \nu_m \left( \frac{il}{|l|^2} .im \left( \frac{im}{|m|^2} \right) + \frac{im}{|m|^2} .il \left( \frac{il}{|l|^2} \right) \right) \\
&= + \frac{|k|}{2\alpha\lambda_k} ik \wedge \sum_{k=l+m} \mu_l \nu_m \frac{il.m}{|l|^2 |m|^2} k = 0
\end{aligned}$$

In the same way, for $X_l = (0, 0, \frac{1}{\lambda_l}\chi_l, -i\chi_l, \frac{|l|}{\alpha\lambda_l}\chi_l)^T$ and $Y_m = (0, 0, \frac{1}{\lambda_m}\psi_m, i\psi_m, \frac{|m|}{\alpha\lambda_m}\psi_m)^T$ with $|l| = |m|$, we obtain

$$\begin{aligned}
&\left(0,0,\frac{-|k|}{\alpha\lambda_k}\,Id, 0\,Id, \frac{1}{\lambda_k}\,Id\right) R_{l,m}[X_l, Y_m] \\
&= \frac{-|k|}{2\alpha\lambda_k} ik \wedge \left( \left( \frac{il}{\lambda|l|^2} \wedge \chi_l \right) .im \left( \frac{im}{\lambda|m|^2} \wedge \psi_m \right) + \alpha \left( \frac{il}{\lambda|l|^2} \wedge \chi_l \right) \wedge \frac{1}{\alpha\lambda} \psi_m \right) \\
&+ \frac{-|k|}{2\alpha\lambda_k} ik \wedge \left( \left( \frac{im}{\lambda|m|^2} \wedge \psi_m \right) .il \left( \frac{il}{\lambda|l|^2} \wedge \chi_l \right) + \alpha \left( \frac{im}{\lambda|m|^2} \wedge \psi_m \right) \wedge \frac{1}{\alpha\lambda} \chi_l \right)
\end{aligned}$$

where $\lambda = \lambda_l = \lambda_m$, from which we deduce

$$\begin{aligned}
T_k^{\chi\psi} &= \frac{-|k|}{2\alpha\lambda_k} k \wedge \sum_{k=l+m,\,|l|=|m|} \frac{1}{\lambda|l|^4} \left((l \wedge \chi_l).m\,(m \wedge \psi_m) + (m \wedge \psi_m).l\,(l \wedge \chi_l)\right) \\
&+ \frac{|k|}{2\alpha\lambda_k} k \wedge \sum_{k=l+m,\,|l|=|m|} \frac{1}{\lambda|l|^2} \left((l \wedge \chi_l) \wedge \psi_m + (m \wedge \psi_m) \wedge \chi_l\right) \\
&= \frac{-|k|}{2\alpha\lambda_k} k \wedge \sum_{k=l+m,\,|l|=|m|} \frac{1}{\lambda|l|^4} \left((l \wedge \chi_l).m\,(m \wedge \psi_m) + (m \wedge \psi_m).l\,(l \wedge \chi_l)\right) \\
&+ \frac{|k|}{2\alpha\lambda_k} k \wedge \sum_{k=l+m,\,|l|=|m|} \frac{1}{\lambda|l|^4} \left((l \wedge \chi_l) \wedge ((m \wedge \psi_m) \wedge m) + (m \wedge \psi_m) \wedge ((l \wedge \chi_l) \wedge l)\right)
\end{aligned}$$

Indeed, we have

$$(m \wedge \psi_m) \wedge m = |m|^2 \psi_m \text{ and } (l \wedge \chi_l) \wedge l = |l|^2 \chi_l$$

by the divergence-free conditions $m.\psi_m = l.\chi_l = 0$. Then, developping the vectorial product leads to

$$\begin{aligned}
T_k^{\chi\psi} &= \frac{-|k|}{2\alpha\lambda_k} k \wedge \sum_{k=l+m,\,|l|=|m|} \frac{1}{\lambda|l|^4} \left((l \wedge \chi_l).m\,(m \wedge \psi_m) + (m \wedge \psi_m).l\,(l \wedge \chi_l)\right) \\
&+ \frac{|k|}{2\alpha\lambda_k} k \wedge \sum_{k=l+m,\,|l|=|m|} \frac{1}{\lambda|l|^4} \left((l \wedge \chi_l).m(m \wedge \psi_m) + (m \wedge \psi_m).l(l \wedge \chi_l)\right) \\
&- \frac{|k|}{2\alpha\lambda_k} k \wedge \sum_{k=l+m,\,|l|=|m|} \frac{1}{\lambda|l|^4} \left((l \wedge \chi_l).(m \wedge \psi_m)m + (m \wedge \psi_m).(l \wedge \chi_l)l\right) \\
&= -\frac{|k|}{2\alpha\lambda_k} k \wedge \sum_{k=l+m,\,|l|=|m|} \frac{1}{\lambda|l|^4} (l \wedge \chi_l).(m \wedge \psi_m)k = 0
\end{aligned}$$

# Chapitre 6

# On a Relaxation Approximation of the Incompressible Navier-Stokes Equations

Yann Brenier [1], Roberto Natalini [2], Marjolaine Puel [3]

**abstract**

We consider an hyperbolic singular perturbation of the incompressible Navier Stokes equations in two space dimensions. The approximating system under consideration, arises as a diffusive rescaled version of a standard relaxation approximation for the incompressible Euler equations. The aim of this work is to give a rigorous justification of its asymptotic limit toward the Navier Stokes equations using the modulated energy method.

## 6.1 Introduction

Let us consider the incompressible Euler equations, namely

$$\begin{cases} \partial_t u + \nabla \cdot (u \otimes u) = \nabla \phi, \\ \nabla \cdot u = 0, \\ u(0,x) = u_0(x), \end{cases} \tag{6.1}$$

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for $(t,x) \in [0,T] \times \mathbf{T}^2$, where $\mathbf{T}^2$ is the unit periodic square $\mathbf{R}^2/\mathbf{Z}^2$. This system describes a perfect incompressible fluid, the unknowns $u$ and $\phi$ corresponding respectively to the velocity, which is valued in $\mathbf{R}^2$, and to the pressure of the fluid.

To approximate these equations, most in the spirit of [13], we introduce its relaxed version, which is obtained by a singular perturbation of the nonlinear term $(u \otimes u)$, through a supplementary matrix valued variable $V : \mathbf{T}^2 \to \mathbf{R}^4$. This leads to the following system

$$\begin{cases} \partial_t u + \nabla \cdot (V) = \nabla \phi, \\ \\ \partial_t V + a \nabla u = -\dfrac{1}{\eta}(V - u \otimes u), \\ \\ \nabla \cdot u = 0, \\ \\ u(0,x) = u_0(x), V(0,x) = V_0(x). \end{cases} \tag{6.2}$$

Let us notice that, as $\eta$ goes to zero, we formally recover system (6.1).

Let us consider now a diffusive scaling, namely, for $\varepsilon > 0$, we set

$$\begin{cases} u^\varepsilon(t,x) & := \dfrac{1}{\sqrt{\varepsilon}} u\left(\dfrac{x}{\sqrt{\varepsilon}}, \dfrac{t}{\varepsilon}\right), \\ \\ V^\varepsilon(x,t) & := \dfrac{1}{\varepsilon} V\left(\dfrac{x}{\sqrt{\varepsilon}}, \dfrac{t}{\varepsilon}\right), \\ \\ \phi^\varepsilon(x,t) & := \dfrac{1}{\varepsilon} \phi\left(\dfrac{x}{\sqrt{\varepsilon}}, \dfrac{t}{\varepsilon}\right). \end{cases} \tag{6.3}$$

Therefore system (6.2) becomes, setting from now $\eta = 1$,

$$\begin{cases} \partial_t u^\varepsilon + \nabla \cdot (V^\varepsilon) = \nabla \phi^\varepsilon, \\ \\ \sqrt{\varepsilon} \partial_t V^\varepsilon + \dfrac{a}{\sqrt{\varepsilon}} \nabla u^\varepsilon = -\dfrac{1}{\sqrt{\varepsilon}}(V^\varepsilon - u^\varepsilon \otimes u^\varepsilon), \\ \\ \nabla \cdot u^\varepsilon = 0, \\ \\ u^\varepsilon(0,x) = u_0^\varepsilon(x), V^\varepsilon(0,x) = V_0^\varepsilon(x). \end{cases} \tag{6.4}$$

In this paper we shall prove that, under some suitable assumptions, the solutions to (6.4) converge, when $\varepsilon$ goes to 0, to the (smooth) solutions of the incompressible Navier Stokes

equations

$$\begin{cases} \partial_t U + \nabla \cdot (U \otimes U) - a\Delta U = \nabla \phi, \\ \\ \nabla \cdot U = 0, \\ \\ U(0,x) = U_0(x). \end{cases} \tag{6.5}$$

This result could be promptly recovered, at least at a formal level, if we assume that, in some (weak) topologies, not only $u^\varepsilon \to U$, but also $\varepsilon V^\varepsilon \to 0$ and $u^\varepsilon \otimes u^\varepsilon \to U \otimes U$. The aim of this paper is to show how to obtain this result in a different (and simpler) way by using the modulated energy method [3], leading to a direct error estimate in the strong $L^\infty([0,T], L^2(\mathbf{T}^2))$ norm, for all finite positive $T$.

Let us recall that the diffusive scaling $(\frac{x}{\sqrt{\varepsilon}}, \frac{t}{\varepsilon})$ has been largely investigated in the framework of hydrodynamic limits of the Boltzmann equations, see for instance [7] and references therein. Starting from the works about the diffusive limit of the Carleman equations by Kurtz [10] and McKean [19], this scaling has also been systematically used in the analysis of hyperbolic-parabolic relaxation limits for weak solutions of hyperbolic systems of balance laws with strongly diffusive source terms by means of compensated compactness techniques by Marcati and collaborators [17, 16, 18, 6]. For other diffusive kinetic models and approximations, we refer to [14, 12, 11]. A general class of kinetic approximations for (possibly degenerate) parabolic equations in multi-D has been considered in [4, 1]. Let us also point out that the same scaling was used in [15] to analyze the time-asymptotic limit of the Jin-Xin relaxation model [13], towards the fundamental solution of the diffusive Burgers equation.

Finally let us remark that our scaling can be considered as a hyperbolic perturbation of the Navier Stokes equations, which is similar to the Cattaneo *hyperbolic heat equation* [5], just eliminating the unknown $V$ in equations (6.4)

$$\begin{cases} \partial_t u^\varepsilon + P(\nabla \cdot (u^\varepsilon \otimes u^\varepsilon)) - a\Delta u^\varepsilon + \varepsilon \partial_{tt} u^\varepsilon = 0, \\ \\ \nabla \cdot u^\varepsilon = 0, \end{cases} \tag{6.6}$$

where $P$ represents the projection on the divergence free vectors. In this regard, we mention that some quite different hyperbolic perturbations of the Navier Stokes equations has been investigated in [20], by considering incompressible viscoelastic fluids of Oldroyd type. We also point out that a similar approximation has been also recently proposed in [2] for numerical purposes, as a reduced kinetic model.

Concerning the method of the modulated energy, let us recall that it has been used by Brenier in [3] to prove the convergence in a quasi-neutral limit of the current involved in the Vlasov-Poisson system toward a dissipative solution of the incompressible Euler equations. The method consists in estimating, through its time derivative, a suitable modification of the the standard energy functional, which is obtained by introducing in the energy a modulation by a well-adapted test function, in practice the (smooth) solution to the limit equation. This method has connections with the relative entropy method used by Yau [22], and the modulated hamiltonian method introduced by Grenier [8] to solve boundary layer problems. Here we can use some special energy functionals, most in the spirit of Tzavaras estimates for the Jin–Xin relaxation model [21].

The paper is organized as follows. In Section 2 we give some analytical backgrounds and state our main result. Estimates and proofs are given in Section 3.

## 6.2 Analytical backgrounds and statements

First we shall state the existence of smooth local solutions for system (6.4).

*Theorem* 6.2.1. Suppose the initial data $(u_0^\varepsilon(x), V_0^\varepsilon(x))$ are smooth functions belonging to $H^s$ for $s \geq 2$. Then, there exists a positive time $T^\varepsilon$, which depends only on the initial data, and a solution $(u^\varepsilon, V^\varepsilon, \phi^\varepsilon) \in C([0,T];(H^s)^3)$ to system (6.4). Moreover, if $T^\varepsilon < \infty$, then

$$\lim_{t \to T_-^\varepsilon} ||(u^\varepsilon, V^\varepsilon)||_{H^2} \to \infty. \tag{6.7}$$

The proof follows easily by arguing as for the classical wave equation, by using energy estimates and the Gagliardo–Niremberg inequalities, see for instance [9], and it is omitted.

In the following we shall use the norm

$$|u|_{H^2(\mathbf{T}^2)} = ||u||_{L^2(\mathbf{T}^2)} + ||\text{curl } u||_{L^2(\mathbf{T}^2)} + ||\nabla(\text{curl } u)||_{L^2(\mathbf{T}^2)}.$$

Let us recall that, since $\nabla \cdot u = 0$, this norm is equivalent to the $H^2$ norm. Moreover we shall denote by $C_0$ a given positive constant such that $C_0 < \sqrt{a}$. Finally $K_s$ is the constant which appears in the Sobolev inequality in two space dimensions, under the norm $|\cdot|_{H^2(\mathbf{T}^2)}$.

The study of the asymptotic behavior of the sequence $u^\varepsilon$, as $\varepsilon$ goes to zero, leads to the statement of our main result.

*Theorem* 6.2.2. Let $T \geq 0$ and $U^0$ be a smooth divergence free vector field on $\mathbf{T}^2$. Let also $(u_0^\varepsilon, V_0^\varepsilon)$ be a sequence of smooth initial data on $\mathbf{T}^2$ for problem (6.4). Assume moreover that there exists a constant $C$ independent of $\varepsilon$ such that

$$||u_0^\varepsilon||_{H^1(\mathbf{T}^2)} \leq C \tag{6.8}$$

$$||\partial_t u^\varepsilon(0,\cdot)||_{L^2(\mathbf{T}^2)} \leq C \tag{6.9}$$

$$|u_0^\varepsilon|_{H^2(\mathbf{T}^2)} < \frac{C_0}{K_s\sqrt{\varepsilon}} \tag{6.10}$$

$$\int_{\mathbf{T}^2} |u_0^\varepsilon(x) - U^0(x)|^2 dx \leq C\sqrt{\varepsilon}. \tag{6.11}$$

Then, $u^\varepsilon$ is a global solution of the relaxed system (6.4) and converges, as $\varepsilon \to 0$, in $L^\infty([0,T], L^2(\mathbf{T}^2))$ towards the (unique smooth) solution $U$ of the incompressible Navier Stokes equations (6.5) with $U^0$ as initial data. In addition

$$\sup_{t\in[0,T]} \int_{\mathbf{T}^2} |u^\varepsilon - U|^2 dx \leq C_T\sqrt{\varepsilon},$$

where $C_T$ depends only on $T$, $U$, $C$ and $C_0$.

## 6.3 Proof of the theorem

### 6.3.1 Preliminaries

First, we shall prove some energy estimates under an a priori assumption on the $L^\infty$ norm of $u^\varepsilon$. Therefore we shall verify that this assumption holds actually true.

**The energy estimate**

Let us give our basic energy estimate.

*Proposition.* Assume that there exists $T > 0$ such that $||u^\varepsilon||_{L^\infty} \leq \sqrt{\frac{a}{\varepsilon}}$ for all $t \leq T$. Then, setting $w^\varepsilon := \text{curl } u^\varepsilon$, we have the following estimates

$$\frac{d}{dt}\int (\frac{1}{2}|u^\varepsilon + \varepsilon\partial_t u^\varepsilon|^2 + \varepsilon^2|\partial_t u^\varepsilon|^2 + \varepsilon a|\nabla u^\varepsilon|^2)dx \leq 0, \tag{6.12}$$

and

$$\frac{d}{dt}\int (\frac{1}{2}|w^\varepsilon + \varepsilon\partial_t w^\varepsilon|^2 + \varepsilon^2|\partial_t w^\varepsilon|^2 + \varepsilon a|\nabla w^\varepsilon|^2)dx \leq 0, \tag{6.13}$$

for all $t \leq T$.

*Démonstration.* Let us multiply equation (6.6) by $(u^\varepsilon + 2\varepsilon\partial_t u^\varepsilon)$ to obtain, after integration by parts in space and writing $\partial_t u\partial_{tt} u = \partial_t(u\partial_t u) - (\partial_t u)^2$,

$$\begin{aligned}&\frac{d}{dt}\int (\frac{1}{2}|u^\varepsilon + \varepsilon\partial_t u^\varepsilon|^2 + \varepsilon^2|\partial_t u^\varepsilon|^2 + \varepsilon a|\nabla u^\varepsilon|^2)dx \\ &+\varepsilon\int |\partial_t u^\varepsilon + \nabla\cdot(u^\varepsilon\otimes u^\varepsilon)|^2 dx + \int \left(a|\nabla u^\varepsilon|^2 - \varepsilon|\nabla\cdot(u^\varepsilon\otimes u^\varepsilon)|^2\right)dx = 0.\end{aligned} \tag{6.14}$$

Then, since $||u^\varepsilon||_{L^\infty} \leq \sqrt{\frac{a}{\varepsilon}}$, we obtain (6.12).

For the second estimate, we consider the equation satisfied by $w^\varepsilon$. Since in two space dimensions we have $w = \partial_2 u_1 - \partial_1 u_2$, then

$$\partial_t w^\varepsilon + u^\varepsilon\cdot\nabla w^\varepsilon - a\Delta w^\varepsilon + \varepsilon\partial_{tt} w^\varepsilon = 0. \tag{6.15}$$

If we multiply this equation by $(w^\varepsilon + 2\varepsilon\partial_t w^\varepsilon)$, we obtain

$$\begin{aligned}&\frac{d}{dt}\int (\frac{1}{2}|w^\varepsilon + \varepsilon\partial_t w^\varepsilon|^2 + \varepsilon^2|\partial_t w^\varepsilon|^2 + \varepsilon a|\nabla w^\varepsilon|^2)dx \\ &+\varepsilon\int |\partial_t w^\varepsilon + u^\varepsilon\cdot\nabla w^\varepsilon|^2 + \int (a|\nabla w^\varepsilon|^2 - \varepsilon|u^\varepsilon\cdot\nabla w^\varepsilon|^2) = 0.\end{aligned} \tag{6.16}$$

The conclusion follows as previously.□

**$L^\infty$ bounds**

Let us prove a uniform $L^\infty$ bound for $u^\varepsilon$ , which implies the assumption made in in the previous statement.

*Proposition.* Under the assumptions of Theorem 6.2.2, if

$$|u_0^\varepsilon|_{H^2(\mathbf{T}^2)} < \frac{C_0}{K_s\sqrt{\varepsilon}},$$

where $C_0$ is a given positive constant such that $C_0 < \sqrt{a}$, then the solution $u^\varepsilon$ verifies the following estimate

$$||u^\varepsilon||_{L^\infty} \leq \frac{C_0}{\sqrt{\varepsilon}}, \tag{6.17}$$

for all positive $t$ and, therefore is global.

*Démonstration.* Take a positive constant $\delta$ such that $\delta < \sqrt{a} - C_0$ and set

$$T^\delta = \sup\{0 \leq t \leq T;\ \sup_{0\leq\tau\leq t} ||u^\varepsilon(\tau)||_{L^\infty} \leq \frac{C_0+\delta}{\sqrt{\varepsilon}}\}.$$

Since $|u_0^\varepsilon|_{H^2(\mathbf{T}^2)} < \frac{C_0}{K_s\sqrt{\varepsilon}}$, we have that $||u_0^\varepsilon||_{L^\infty} < \frac{C_0+\delta}{\sqrt{\varepsilon}}$, thanks to the the Sobolev inequalities. Since $u^\varepsilon \in C^0([0,T], L^\infty(\mathbf{T}^2))$, we have that $T^\delta > 0$. Let us prove now that $T^\delta = T$. If $T^\delta < T$, we have

$$||u^\varepsilon(T^\delta)||_{L^\infty} = \frac{C_0+\delta}{\sqrt{\varepsilon}} < \sqrt{\frac{a}{\varepsilon}}.$$

Then, there exists $\mu > 0$ such that for all $t \leq T^\delta + \mu$, $||u^\varepsilon(t)||_{L^\infty} \leq \sqrt{\frac{a}{\varepsilon}}$. On the other hand, for all $t \leq T^\delta + \mu$, the estimates (6.12) and (6.13) hold true. This implies, again by the Sobolev inequalities, that

$$||u^\varepsilon(T^\delta)||_{L^\infty} \leq \frac{C_0}{\sqrt{\varepsilon}},$$

which yields a contradiction.□

### 6.3.2 Convergence

Let $U$ be a smooth solution of the Navier Stokes equations with $U^0$ as initial data. We shall prove here that $\frac{1}{2}\int |u^\varepsilon - U|^2 dx \leq C_T\sqrt{\varepsilon}$. To prove that, we shall define a specific modulated energy which control this quantity.

#### Definition and properties of the modulated energy

Let us define the energy in the following way

$$E^\varepsilon(t) = \int (\frac{1}{2}|u^\varepsilon + \varepsilon\partial_t u^\varepsilon|^2 + \varepsilon^2|\partial_t u^\varepsilon|^2 + \varepsilon a|\nabla u^\varepsilon|^2)dx. \tag{6.18}$$

For all smooth divergence free $v$, we introduce the modulated energy

$$E_v^\varepsilon(t) = \int (\frac{1}{2}|u^\varepsilon - v(t,x) + \varepsilon\partial_t u^\varepsilon|^2 + \varepsilon^2|\partial_t u^\varepsilon|^2 + \varepsilon a|\nabla u^\varepsilon|^2)dx. \tag{6.19}$$

Let us prove now a useful identity.

*Proposition.* The modulated energy satisfies the identity

$$\begin{aligned}\frac{d}{dt}E_v^\varepsilon(t) &= \int v\cdot\nabla : (u^\varepsilon - v)\otimes(u^\varepsilon - v) + \int (\partial_t v + v\cdot\nabla v - a\Delta v)(v - u^\varepsilon) \\ &\quad -\varepsilon\int |\partial_t u^\varepsilon + \nabla\cdot(u^\varepsilon\otimes u^\varepsilon)|^2 - \varepsilon\int \partial_t v\cdot\partial_t u^\varepsilon \\ &\quad -a\int |\nabla(u^\varepsilon - v)|^2 + \varepsilon\int |\nabla(u^\varepsilon\otimes u^\varepsilon)|^2.\end{aligned} \tag{6.20}$$

*Démonstration.* We have

$$\frac{d}{dt}E_v^\varepsilon(t) = \frac{d}{dt}E(t) - \int v\cdot\partial_t u^\varepsilon - \int \partial_t v\cdot u^\varepsilon - \varepsilon\int \partial_t v\cdot\partial_t u^\varepsilon - \varepsilon\int v\cdot\partial_{tt}u^\varepsilon + \int v\partial_t v.$$

Then, using (6.6), (6.14) and the equality

$$\int v\cdot\nabla : (u^\varepsilon\otimes u^\varepsilon) \;= \int v\cdot\nabla : (u^\varepsilon - v)\otimes(u^\varepsilon - v) + \int v\cdot\nabla : (u^\varepsilon\otimes v)$$

$$+\int v\cdot\nabla : (v\otimes u^\varepsilon) - \int v\cdot\nabla : (v\otimes v)$$

we obtain

$$\begin{aligned}\frac{d}{dt}E_v^\varepsilon(t) \;=\; & \int v\cdot\nabla : (u^\varepsilon - v)\otimes(u^\varepsilon - v) + \int(\partial_t v + v\cdot\nabla v)(v - u^\varepsilon)\\ & -\varepsilon\int|\partial_t u^\varepsilon + \nabla\cdot(u^\varepsilon\otimes u^\varepsilon)|^2 - \varepsilon\int\partial_t v\cdot\partial_t u^\varepsilon \qquad (6.21)\\ & +a\int\Delta u^\varepsilon(u^\varepsilon - v) + \varepsilon\int|\nabla(u^\varepsilon\otimes u^\varepsilon)|^2.\end{aligned}$$

Therefore, since

$$a\int\Delta u^\varepsilon(u^\varepsilon - v) = a\int\Delta(u^\varepsilon - v)(u^\varepsilon - v) + a\int\Delta v(u^\varepsilon - v),$$

we have (6.20). □

**Proof of Theorem 6.2.2**

Thanks to the assumptions on the initial data, we have that

$$\int|u^\varepsilon|^2 \le CE^\varepsilon(t) \le CE^\varepsilon(0) \le C.$$

Moreover, we have the inequality

$$\int|u^\varepsilon - v|^2dx \le CE_v^\varepsilon(t).$$

Now, we assume $v = U$, where $U$ is a smooth solution to the incompressible Navier-Stokes equations (6.5) with $U^0$ as initial data. From (6.20), we obtain

$$\tfrac{d}{dt}E_v^\varepsilon(t) \le CE_v^\varepsilon(t) - \varepsilon\int\partial_t v\cdot\partial_t u^\varepsilon - a\int|\nabla(u^\varepsilon - v)|^2 + \varepsilon|u^\varepsilon\cdot\nabla u^\varepsilon|^2.$$

We have used, in the right-hand side of (6.20), i) that $v$ is smooth in order to bound the first term by $CE_v^\varepsilon$, ii) that $v$ is a solution to the Navier-Stokes equations to cancel the second term. We see that

$$-\varepsilon\int\partial_t v\cdot\partial_t u^\varepsilon = -\varepsilon\frac{d}{dt}\int\partial_t v\cdot u^\varepsilon + \varepsilon\int\partial_{tt}v\cdot u^\varepsilon,$$

which is of order $\varepsilon$. We want to prove now that the term

$$A^\varepsilon = -a\int|\nabla(u^\varepsilon - v)|^2 + \varepsilon\int|u^\varepsilon\cdot\nabla u^\varepsilon|^2$$

goes to zero, as $\varepsilon \to 0$. In this regard, let us write

$$\varepsilon \int |u^\varepsilon \cdot \nabla u^\varepsilon|^2 \leq \varepsilon(1+\theta) \int |u^\varepsilon \cdot \nabla (u^\varepsilon - v)|^2 + \varepsilon(1+\frac{1}{\theta}) \int |u^\varepsilon \cdot \nabla v|^2.$$

Then, since $||u^\varepsilon||_{L^\infty} \leq \frac{\sqrt{a}}{\sqrt{\varepsilon}}$, we have the inequality

$$A^\varepsilon \leq \theta a \int |\nabla (u^\varepsilon - v)|^2 + \varepsilon(1+\frac{1}{\theta}) \int |u^\varepsilon \cdot \nabla v|^2.$$

This yields

$$A^\varepsilon \leq \theta a ||\nabla u^\varepsilon||^2_{L^2(\mathbf{T}^2)} + \varepsilon(1+\frac{1}{\theta}) c ||u^\varepsilon||^2_{L^2(\mathbf{T}^2)}.$$

Since, thanks to the estimates (6.12) and (6.13), we have

$$\int |u^\varepsilon|^2 + \int |\nabla (u^\varepsilon - v)|^2 \leq C.$$

If we take $\theta = \sqrt{\varepsilon}$, we obtain that $A^\varepsilon = O(\sqrt{\varepsilon})$, when $\varepsilon$ goes to zero. Thus, we have obtained

$$\frac{d}{dt}(E_v(t) + O(\varepsilon)) \leq C E_v(t) + O(\sqrt{\varepsilon}).$$

The assumptions that we have made on the initial data imply that

$$E_v(0) = O(\sqrt{\varepsilon}).$$

We conclude that

$$\sup_{t \in [0,T]} \int |u^\varepsilon - v|^2 dx \leq C E_v^\varepsilon \leq C_T \sqrt{\varepsilon},$$

where $C_T$ depends only on $T$, $v$ and the initial conditions. $\square$

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# Chapitre 7

# Diffusion and homogenization limits with separate scales

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We consider the simultaneous diffusion and homogenization limit of the linear Boltzmann equation in a periodic medium in the regime where the mean free path is much smaller than the lattice constant. The resulting equation is a diffusion equation, with an averaged diffusion matrix that is formally obtained by first performing the diffusion limit and then the homogenization one. The rigorous proof relies on the use of two-scale limits, in combination with an asymptotic expansion of the equilibrium profile in powers of the ratio between the mean free path and the lattice constant.

## 7.1 Introduction

### 7.1.1 Setting of the problem

The derivation of macroscopic equations like diffusion equations or hydrodynamic equations from kinetic theory is a topic which has received a lot of attention in the last decades (see for instance [13], [6], [14], [21], [22], [23],[24], [17], [18]). The starting point is the Boltzmann equation, the unknown of which is the particle distribution function $f$, a function of time, positition and velocity ($f = f(t, \mathbf{x}, \mathbf{v})$). A typical Boltzmann equation asserts that

$$\partial_t f + \mathbf{v} \cdot \nabla_{\mathbf{x}} f = Q(f) \ ,$$

where the left hand side is the total derivative taking into account the free streaming of particles, while the right hand side is the so-called collision operator, which describes in a statistical way the scattering of the particles by an exterior medium and/or other particles. A Boltzmann equation may contain other terms, describing the effects of driving forces, sources or absorption ; it may also include other degrees of freedom than time, space and velocity.

Macroscopic effects prevail when the time scale between collisions is much smaller than the observation time scale, or equivalently when the mean free path (which is the average distance between two successive collisions) becomes much smaller than the specimen length scale. The rescaled Boltzmann equation reads

$$\partial_t f + \frac{1}{\varepsilon} \mathbf{v} \cdot \nabla_{\mathbf{x}} f = \frac{Q(f)}{\varepsilon^2} \ ,$$

where $\epsilon$ is the small mean free path. As $\varepsilon \to 0$, the particles are driven towards a local equilibrium characterized by the equation $Q(f) = 0$.

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One of the simplest collision operators is the linear BGK operator

$$Q(f)(\mathbf{v}) = \int (\sigma(\mathbf{v}' \to \mathbf{v}) f(\mathbf{v}') - \sigma(\mathbf{v} \to \mathbf{v}') f(\mathbf{v}))\, d\nu(\mathbf{v}') \ ,$$

whose kernel is generated by a positive function [13] (with an abuse of notation, we shall refer to this function as the Maxwellian). The local equilibrium is then a multiple of the Maxwellian, the multiplier being the limiting particle density, a function of space and time. This particle density can be shown to solve a diffusion equation (see also [12]).

The cross section $\sigma(\mathbf{v}' \to \mathbf{v})$, which represents the probability per unit time that a collision changes the particle velocity from $\mathbf{v}'$ to $\mathbf{v}$, will depend on position in the case of a non homogeneous medium. We shall consider the case when this position dependence is periodic, with a small scale of periodicity (this scale is referred to as the lattice constant). In this situation one might expect some form of homogenization to occur, giving rise to a macroscopic equation with averaged coefficients (see [11], [1], [15], [25], [5], [13], [14]). The linear Boltzmann equation has been widely studied when the mean free path and the lattice constant are small but of the same order of magnitude, see for instance Bal [3] , Allaire - Bal [2], Goudon-Poupaud [19], [20] , Goudon-Mellet [17] and Ben Abdallah-Tayeb [10]. In this case the local equilibrium profile is a solution of the cell problem $\mathbf{v} \cdot \nabla_{\mathbf{y}} f = Q(f)$. The equilibrium is in general a function of both the velocity and the scaled position variable $y$.

The aim of the present paper is to tackle the simultaneous diffusion and homogenization limit of the linear Boltzmann equation in the case when the mean free path is much smaller than the already small lattice constant. Namely, we consider the equation

$$\partial_t f + \frac{1}{\varepsilon} \mathbf{v} \cdot \nabla_{\mathbf{x}} f = \frac{Q_\alpha(f)}{\varepsilon^2} \ , \tag{7.1}$$

where

$$Q_\alpha(f)(\mathbf{v}) = \int \left( \sigma(\frac{\mathbf{x}}{\alpha}, \mathbf{v}, \mathbf{v}') f(\mathbf{v}') - \sigma(\frac{\mathbf{x}}{\alpha}, \mathbf{v}', \mathbf{v}) f(\mathbf{v}) \right) d\nu(\mathbf{v}') \ , \tag{7.2}$$

and the mean free path, $\varepsilon$, and the lattice constant, $\alpha$, both tend to zero, subject to

$$\varepsilon \ll \alpha \ .$$

The cross section $\sigma$ is 1-periodic with respect to the variable $\mathbf{y} = \mathbf{x}/\alpha$. Notice that the operator $Q_\alpha$ satisfies $\int_{\mathbf{v}} Q_\alpha(f)(\mathbf{v})\, d\nu(\mathbf{v}) = 0$. Thus, by integration of the equation (7.1) with respect to $\mathbf{x}$ and $\mathbf{v}$, we obtain that total mass is conserved, in the sense that the solution satisfies $\partial_t \int_{x,\mathbf{v}} f = 0$.

One expects that the diffusion limit is achieved relatively "faster" than the homogenization limit. And indeed, the simultaneous limiting equation we rigorously derive corresponds to first performing the diffusion limit (obtaining a drift-diffusion equation with oscillating parameters) and then averaging this equation by homogenization techniques. Such a convergence result has previously been proven by Sentis, [25] when $\alpha = \varepsilon^p$ for some constant $p < 1$, and provided the local equilibrium (the Maxwellian) does not depend on the "fast" variable $\mathbf{y}$. His proof relies on a formal expansion in powers of $\varepsilon$ and $\alpha$, which yields an estimate of the remainder when a sufficiently large number of terms is included (basically, the closer $p$ is to 1, the more terms are needed in the expansion). The aim of our work is to remove these two hypotheses, thus allowing for instance $\alpha = \varepsilon |\log(\varepsilon)|$ ) and also permitting the local Maxwellian to depend on the periodic position variable. By removal of either of the hypotheses, the method proposed in [25] fails, and we need to find another route. It is convenient to introduce the small parameter $\eta = \dfrac{\varepsilon}{\alpha}$. With this notation our proof relies heavily on the expansion of solutions to the cell problem

$$\eta \mathbf{v} \cdot \nabla_{\mathbf{y}} f = Q(f)$$

with respect to the parameter $\eta$.

### 7.1.2 Notations, Hypotheses and Main Result

The position variable $x$ lies in $\mathbf{R}^d$, where $d$ is an integer, while the velocity variables $\mathbf{v}$ lies in a compact, symmetric set $V$ of $\mathbf{R}^d$ equiped with a symmetric probability measure $\nu$. We shall rely on the results of [17] ; for that reason we shall throughout this paper also assume that the measure $\nu$ satisfies

– There exist constants $C,\ \gamma > 0$ such that $\nu(\{\mathbf{v} \in V,\ |\mathbf{v}\cdot\xi| \le h\}) \le Ch^\gamma, \quad$ for all $\xi \in S^{d-1},\ h > 0$ .

This in particular implies that $\nu(\{\mathbf{v} \in V,\ |\mathbf{v}\cdot\xi| = 0\}) = 0$ for all $\xi \neq 0$, and so

$$\mathbf{v}\cdot\xi = 0 \quad \text{a.e. in } \mathbf{v} \text{ implies } \xi = 0 \ .$$

The time variable $t$ lies in an arbitrarily large time interval $[0,T]$. The distribution function $f^{\varepsilon,\eta}(t,\mathbf{x},\mathbf{v})$ is the unique weak solution to the equation

$$\partial_t f^{\varepsilon,\eta} + \frac{1}{\varepsilon}\mathbf{v}\cdot\nabla_{\mathbf{x}} f^{\varepsilon,\eta} = \frac{1}{\varepsilon^2} Q_{\varepsilon/\eta}(f^{\varepsilon,\eta}) \quad , \quad f^{\varepsilon,\eta}(0,\mathbf{x},\mathbf{v}) = f_{ini}(\mathbf{x},\mathbf{v}) \ , \tag{7.3}$$

where

$$Q_\alpha(f)(\mathbf{v}) = Q[\frac{\mathbf{x}}{\alpha}](f) \ , \quad \text{with} \quad Q[\mathbf{y}](f) = \int (\sigma(\mathbf{y},\mathbf{v},\mathbf{v}')f(\mathbf{v}') - \sigma(\mathbf{y},\mathbf{v}',\mathbf{v})f(\mathbf{v}))\ d\nu(\mathbf{v}') \ . \tag{7.4}$$

Whenever one of the traditional normed function spaces is given a subscript *per* it signifies that the correponding elements are 1-periodic with respect to the variable $\mathbf{y}$ ; the norm is the one induced from the *unit cell* $Y = [-1/2,1/2]^d$. The cross section $\sigma$ is assumed to satisfy

– **(H0)** $\sigma(\mathbf{y},\mathbf{v},\mathbf{v}')$ is in $L^\infty(V_{\mathbf{v}} \times V_{\mathbf{v}'}; C^\infty_{per}(\mathbf{R}^d_{\mathbf{y}}))$. It is bounded from below and above by positive constants and is 1-periodic with respect to the variable $\mathbf{y}$.

– **(H1)** $\sigma$ has the following symmetries :

$$\sigma(-\mathbf{y},\mathbf{v},\mathbf{v}') = \sigma(\mathbf{y},\mathbf{v},\mathbf{v}'), \text{ and } \sigma(\mathbf{y},-\mathbf{v},-\mathbf{v}') = \sigma(\mathbf{y},\mathbf{v},\mathbf{v}').$$

– **(H2)** $\sigma$ furthermore exhibits the detailed balance

$$\sigma(\mathbf{y},\mathbf{v},\mathbf{v}') = \tilde\sigma(\mathbf{y},\mathbf{v},\mathbf{v}')M(\mathbf{y},\mathbf{v}) \quad \text{with} \quad M > 0 \quad \text{and} \quad \tilde\sigma(\mathbf{y},\mathbf{v},\mathbf{v}') = \tilde\sigma(\mathbf{y},\mathbf{v}',\mathbf{v}) \ .$$

The functions $M$ and $\tilde\sigma$ lie in $L^\infty(V_{\mathbf{v}}; C^\infty(\mathbf{R}^d_{\mathbf{y}}))$ and $L^\infty(V_{\mathbf{v}} \times V_{\mathbf{v}'}; C^\infty(\mathbf{R}^d_{\mathbf{y}}))$, respectively, and they are 1-periodic with respect to the variable $\mathbf{y}$. We may without loss of generality suppose that $\int_V M(\mathbf{y},\mathbf{v})d\nu(\mathbf{v}) = 1$. The function $M$ is referred to as the local Maxwellian; as we shall show later it is also the unique solution to

$$Q[\mathbf{y}](M) = 0, \quad \text{with} \quad \int_V M(\mathbf{y},\mathbf{v})d\nu(\mathbf{v}) = 1 \ .$$

*Theorem* 7.1.1. [31][13]. Let $f_{ini}(\mathbf{x},\mathbf{v})$ be a nonnegative function in $L^1(\mathbf{R}^d \times V)$. The equation (7.3) has a unique weak solution in $C^0(\mathbf{R}^+; L^1(\mathbf{R}^d \times V))$ satisfying :

$$\begin{aligned} &\int_{\mathbf{R}^d\times V} f_{ini}(\mathbf{x},\mathbf{v})\phi(0,\mathbf{x},\mathbf{v})\,d\mathbf{x}\,d\nu(\mathbf{v}) \\ &\quad + \int_{\mathbf{R}^+}\int_{\mathbf{R}^d\times V} f^{\varepsilon,\eta}(\partial_t\phi + \frac{\mathbf{v}\cdot\nabla_{\mathbf{x}}\phi}{\varepsilon} + \frac{Q^*_{\varepsilon/\eta}(\phi)}{\varepsilon^2})\,d\mathbf{x}d\nu(\mathbf{v})\,dt = 0, \qquad \forall\phi \in \mathcal{T} \ . \end{aligned} \tag{7.5}$$

The test functions (the elements of $\mathcal{T}$) are $C^1$ in the variables $(t,\mathbf{x})$, continuous in $\mathbf{v}$, and have compact support. The solution $f^{\varepsilon,\eta}$ is nonnegative and $\|f^{\varepsilon,\eta}(t,\cdot,\cdot)\|_{L^1} = \|f_{ini}\|_{L^1}$.

In our analysis we shall assume that the three parameters $\varepsilon$, $\eta$ and $\varepsilon/\eta$ (also referred to as $\alpha$) all tend to zero. The main result established in this paper is the following convergence theorem.

*Theorem* 7.1.2. Assume $\sigma(\mathbf{y},\mathbf{v},\mathbf{v}')$ satisfies **(H0)**,**(H1)** and **(H2)** above. Let $f^{\varepsilon,\eta}$ be the solution to (7.3) with nonnegative initial datum $f_{ini} \in L^1(\mathbf{R}^d, V)$. Let $\mathbf{X}^* = \mathbf{X}^*(\mathbf{y},\mathbf{v})$, be the unique solution to the cell problem

$$Q^*[\mathbf{y}](\mathbf{X}^*) = -\mathbf{v} \ , \quad \int_V \mathbf{X}^* d\nu(\mathbf{v}) = 0 \ ,$$

where $Q^*[\mathbf{y}]$ is the $L^2(V)$ adjoint of the operator $Q[\mathbf{y}]$. Let $M$ be the local Maxwellian, and let $\lambda = \lambda(\mathbf{y},\mathbf{v})$ be the unique solution to

$$Q[\mathbf{y}](\lambda) = \mathbf{v}\cdot\nabla_{\mathbf{y}} M \ , \quad \int_V \lambda(\mathbf{y},\mathbf{v})d\nu(\mathbf{v}) = 0 \ .$$

There exists a positive, regular Borel measure $\mathcal{F}(dt, d\mathbf{x}, d\mathbf{y}, d\mathbf{v})$ such that $f^{\varepsilon,\eta}$ two scale converges (at the scale 1 and $\frac{\varepsilon}{\eta}$) towards $\mathcal{F}(dt, d\mathbf{x}, d\mathbf{y}, d\mathbf{v})$ for any sequence $\varepsilon \to 0$, $\eta \to 0$, with $\varepsilon/\eta \to 0$.[3] The limiting measure has the form $\mathcal{F}(dt, d\mathbf{x}, d\mathbf{y}, d\mathbf{v}) = N(dt, d\mathbf{x}, d\mathbf{y})M(\mathbf{y}, \mathbf{v})d\nu(\mathbf{v})$, where the two-scale density $N(t, \mathbf{x}, \mathbf{y})$ decomposes as $N(t, \mathbf{x}, \mathbf{y}) = \rho(\mathbf{y})d\mathbf{y}\, n(dt, d\mathbf{x})$. Here $\rho(\mathbf{y})$ is the positive 1-periodic function satisfying

$$L(\rho(\mathbf{y})) = 0 \ , \quad \int_Y \rho(\mathbf{y}) = 1 \ ,$$

with

$$L(\rho(\mathbf{y})) = -\mathrm{div}_{\mathbf{y}}(D(\mathbf{y})\nabla_{\mathbf{y}}\rho(\mathbf{y})) + \mathrm{div}_{\mathbf{y}}(U(\mathbf{y})\rho(\mathbf{y})) \ ,$$

and

$$D(\mathbf{y}) = \int_V \mathbf{X}^*(\mathbf{y}, \mathbf{v}) \otimes \mathbf{v} M(\mathbf{y}, \mathbf{v}) d\nu(\mathbf{v}) \ , \quad U(\mathbf{y}) = \int_V \mathbf{v}\lambda(\mathbf{y}, \mathbf{v}) d\nu(\mathbf{v}) \ ,$$

while the macroscopic density $n(dt, d\mathbf{x})$ is given by $n(dt, d\mathbf{x}) = n(t, \mathbf{x})dt\, d\mathbf{x}$, where the function $n(t, \mathbf{x})$ is the solution to the diffusion initial value problem

$$\partial_t n - \mathrm{div}_{\mathbf{x}}(\mathbf{D}\nabla_{\mathbf{x}} n) = 0 \quad , \quad n(t = 0, \mathbf{x}) = n_{ini}(\mathbf{x}) = \int_V f_{ini}(\mathbf{x}, \mathbf{v})\, d\nu(\mathbf{v}) \ . \tag{7.6}$$

The homogenized diffusion matrix is given by the formula

$$\mathbf{D} = \int_Y \rho(\mathbf{y}) D(\mathbf{y})\, d\mathbf{y} - \int_Y \mathbf{\Theta}^{[-1]}(\mathbf{y}) \otimes H(\mathbf{y})\, d\mathbf{y} \ , \tag{7.7}$$

where

$$H(\mathbf{y}) = \mathrm{div}_{\mathbf{y}}(\rho(\mathbf{y})D^\top(\mathbf{y})) + D(\mathbf{y})\nabla_{\mathbf{y}}\rho - \rho(\mathbf{y})U(\mathbf{y}) \ , \quad \mathbf{\Theta}^{[-1]}(\mathbf{y}) = L^{*-1}(\int M(\mathbf{y}, \mathbf{v})(\mathbf{v} \cdot \nabla_{\mathbf{y}})\, \mathbf{X}^*(\mathbf{y}, \mathbf{v}) d\mathbf{v})$$

and $L^*$ is the $(L^2-)$ adjoint operator of $L$ (acting on vectors).

## 7.2 Heuristics and strategy of proof

In this section, we explain why the rigorous convergence result stated in the main theorem corresponds to performing the diffusion limit of the Boltzmann equation (while keeping the spatial length scale $\alpha$ fixed) and then homogenizing the so obtained drift-diffusion equation by letting $\alpha$ go to zero. We then explain the strategy of the proof which deals simultaneously with both limits.

### 7.2.1 Heuristics

We base our formal argument on a Hilbert expansion, first expanding $f$ in powers of $\varepsilon$ (for fixed $\alpha$). In this context we use the notation $f^{\epsilon,\alpha}$ in place of $f^{\varepsilon,\eta}$ (remember, for fixed $\varepsilon$, there is a one-to-one correspondance between $\eta$ and $\alpha$, given by $\alpha = \varepsilon/\eta$).

$$f^{\varepsilon,\alpha}(t, \mathbf{x}, \mathbf{v}) = f_0^\alpha(t, \mathbf{x}, \mathbf{v}) + \varepsilon f_1^\alpha(t, \mathbf{x}, \mathbf{v}) + \varepsilon^2 f_2^\alpha(t, \mathbf{x}, \mathbf{v}) + \cdots \ .$$

Inserting this expansion in the Boltzmann equation and equating terms of same powers of $\varepsilon$ leads to the three equations

$$Q_\alpha(f_0^\alpha) = 0 \ , \qquad Q_\alpha(f_1^\alpha) = \mathbf{v} \cdot \nabla f_0^\alpha \ , \qquad Q_\alpha(f_2^\alpha) = \mathbf{v} \cdot \nabla f_1^\alpha + \partial_t f_0^\alpha \ .$$

The first equation implies that $f_0^\alpha(t, \mathbf{x}, \mathbf{v}) = N^\alpha(t, \mathbf{x})M(\frac{\mathbf{x}}{\alpha}, \mathbf{v})$ where $M(\mathbf{y}, \cdot)$ is the Maxwellian profile, namely the unique solution of $Q[\mathbf{y}](M) = 0$ , $\int_V M(\mathbf{y}, \mathbf{v})d\nu(\mathbf{v}) = 1 \quad \forall \mathbf{y} \in \mathbf{R}^d$. The formula for $Q$ (ensuring mass conservation) implies that 1 belongs to the kernel of the adjoint $Q^*$. Furthermore, in Proposition 7.3.1, we show that 1 generates the kernel of the adjoint $Q^*$. The Fredholm alternative

3. For the precise definition of this notion of convergence see section 7.4

therefore implies that the equation for $f_1^\alpha$ has a solution if the Maxwellian profile satisfies the non drift-condition

$$\int_V \mathbf{v} M(\mathbf{y},\mathbf{v}) d\nu(\mathbf{v}) = 0 \quad \forall \mathbf{y} \in \mathbf{R}^d \ .$$

This condition is satisfied thanks to Hypothesis **(H1)**, which implies that $M$ is even with respect to the velocity variable $\mathbf{v}$. Therefore

$$f_1^\alpha = \frac{1}{\alpha} N^\alpha(t,\mathbf{x}) \lambda(\frac{\mathbf{x}}{\alpha},\mathbf{v}) - \mathbf{X}(\frac{\mathbf{x}}{\alpha},\mathbf{v}) \cdot \nabla N^\alpha(t,\mathbf{x})$$

where $\lambda$ is defined by $Q[\mathbf{y}](\lambda) = \mathbf{v} \cdot \nabla_{\mathbf{y}} M$, and $\mathbf{X}$ is defined by $Q[\mathbf{y}](\mathbf{X}) = -\mathbf{v}M$.

Finally, the Fredholm alternative asserts that the equation defining $f_2^\alpha$ can be solved if and only if

$$\int_V \left(\partial_t f_0^\alpha + \mathbf{v} \cdot \nabla_{\mathbf{x}} f_1^\alpha\right) d\nu(\mathbf{v}) = 0 \ .$$

This last compatibility condition gives the equation that has to be satisfied by $N^\alpha$, namely

$$\partial_t N^\alpha - \mathrm{div}(D(\frac{\mathbf{x}}{\alpha}) \cdot \nabla N^\alpha) + \frac{1}{\alpha} \mathrm{div}(U(\frac{\mathbf{x}}{\alpha}) N^\alpha) = 0 \ , \tag{7.8}$$

where

$$D(\mathbf{y}) = \int_V \mathbf{v} \otimes \mathbf{X}(\mathbf{y},\mathbf{v}) d\nu(\mathbf{v}) = \int_V \mathbf{X}^*(\mathbf{y},\mathbf{v}) \otimes \mathbf{v} M(\mathbf{y},\mathbf{v}) d\nu(\mathbf{v}) \ , \quad \text{and} \quad U(\mathbf{y}) = \int_V \mathbf{v} \lambda(\mathbf{y},\mathbf{v}) d\nu(\mathbf{v}) \ . \tag{7.9}$$

The conclusion of this first step of the heuristics is that $f^\varepsilon(t,\mathbf{x},\mathbf{v}) \sim N^\alpha(t,\mathbf{x}) M(\frac{\mathbf{x}}{\alpha},\mathbf{v}) + O(\varepsilon)$. But one has to be aware that $O(\varepsilon)$ is not necessarily uniform in $\alpha$.

Now we continue the heuristics by passing to the limit $\alpha \to 0$ in the above drift-diffusion equation. Here we do not expect a simple Hilbert expansion as in the previous step, since the fast variable $\mathbf{y} = \frac{\mathbf{x}}{\alpha}$ must be taken into account. The usual approach consists in looking for a two-scale expansion involving both the space variable and the fast space variable

$$N^\alpha(t,\mathbf{x}) \sim N_0(t,\mathbf{x},\mathbf{y}) + \alpha N_1(t,\mathbf{x},\mathbf{y}) + \alpha^2 N_2(t,\mathbf{x},\mathbf{y}) \ , \quad \mathbf{y} = \mathbf{x}/\alpha \ ,$$

$N_i$ being 1-periodic in $\mathbf{y}$. Consequently, one can write $\nabla N^\alpha \sim \sum_{i=0}^{2} \alpha^i (\nabla_{\mathbf{x}} N_i + \frac{1}{\alpha} \nabla_{\mathbf{y}} N_i)$. Equating terms of the same power of $\alpha$ one is led to the following set of equations

$$L(N_0) = 0 \ , \tag{7.10}$$

$$L(N_1) = \mathrm{div}_{\mathbf{y}}(D(\mathbf{y}) \nabla_{\mathbf{x}} N_0) + \mathrm{div}_{\mathbf{x}}(D(\mathbf{y}) \nabla_{\mathbf{y}} N_0) - \mathrm{div}_{\mathbf{x}}(U(\mathbf{y}) N_0) \ , \tag{7.11}$$

$$L(N_2) = -\partial_t N_0 + \mathrm{div}_{\mathbf{x}}(D(\mathbf{y}) \nabla_{\mathbf{x}} N_0) + \mathrm{div}_{\mathbf{y}}(D(\mathbf{y}) \nabla_{\mathbf{x}} N_1) + \mathrm{div}_{\mathbf{x}}(D(\mathbf{y}) \nabla_{\mathbf{y}} N_1) - \mathrm{div}_{\mathbf{x}}(U(\mathbf{y}) N_1) \ , \tag{7.12}$$

where the operator $L$ is defined by

$$L(n) = -\mathrm{div}_{\mathbf{y}}(D(\mathbf{y}) \nabla_{\mathbf{y}} n) + \mathrm{div}_{\mathbf{y}}(U(\mathbf{y}) n) \quad \text{with periodic boundary conditions} \ . \tag{7.13}$$

As shall be proven in Proposition 7.3.1 (see the following section) the operator $L$ has a one dimensional kernel spanned by a positive function $\rho$ (that we normalize such that its integral over $Y$ is equal to one). Equation (7.10) now implies that $N_0(t,\mathbf{x},\mathbf{y}) = n(t,\mathbf{x}) \rho(\mathbf{y})$. Since $L$ has divergence form (and a one dimensional kernel) its range is exactly the set of functions with zero average in $\mathbf{y}$. Therefore, (7.11) has a solution if and only if its right hand side has zero $\mathbf{y}$ average, *i.e.*,

$$\begin{aligned} 0 &= \int_Y \mathrm{div}_{\mathbf{x}}(D(\mathbf{y}) \nabla_{\mathbf{y}} N_0) - \mathrm{div}_{\mathbf{x}}(U(\mathbf{y}) N_0) \\ &= \left[ \int_Y \int_V \left[\mathbf{X}^*(\mathbf{y},\mathbf{v}) \otimes \mathbf{v}\right] \nabla_{\mathbf{y}} \left[\rho(\mathbf{y}) M(\mathbf{y},\mathbf{v})\right] d\mathbf{y} d\nu(\mathbf{v}) \right] \cdot \nabla_{\mathbf{x}} n(t,\mathbf{x}) \ . \end{aligned} \tag{7.14}$$

As will be shown later on, this non drift condition is true due to the symmetry hypothesis **(H1)**. Proceeding analogously we deduce that (7.12) has a solution if and only if

$$\int_Y [\partial_t N_0 - \mathrm{div}_{\mathbf{x}}(D(\mathbf{y})\nabla_{\mathbf{x}} N_0) - \mathrm{div}_{\mathbf{x}}(D(\mathbf{y})\nabla_{\mathbf{y}} N_1) + \mathrm{div}_{\mathbf{x}}(U(\mathbf{y})N_1)] = 0 \ ,$$

which can be rewritten

$$\partial_t n - \mathrm{div}_{\mathbf{x}}(\int_Y \rho(\mathbf{y})D(\mathbf{y})\, d\mathbf{y}\nabla_{\mathbf{x}} n) + \mathrm{div}_{\mathbf{x}} \int_Y (\mathrm{div}_{\mathbf{y}} D(\mathbf{y}) + U(\mathbf{y}))N_1\, d\mathbf{y} = 0 \ .$$

Here the divergence of a matrix is taken by row. Furthermore, since

$$L(N_1) = H(\mathbf{y}) \cdot \nabla_{\mathbf{x}} n, \quad \text{with} \quad H(\mathbf{y}) = \mathrm{div}_{\mathbf{y}}(\rho(\mathbf{y})D^{\top}(\mathbf{y})) + D(\mathbf{y})\nabla_{\mathbf{y}}\rho(\mathbf{y}) - \rho(\mathbf{y})U(\mathbf{y}) \ ,$$

we can write $N_1 = L^{-1}(H(\mathbf{y})) \cdot \nabla_{\mathbf{x}} n$ (modulo the kernel of $L$), and so the solvability condition for $N_2$ can be rewritten

$$\partial_t n - \mathrm{div}_{\mathbf{x}}(\int_Y \rho(\mathbf{y})D(\mathbf{y}) d\mathbf{y}\nabla_{\mathbf{x}} n) + \mathrm{div}_{\mathbf{x}}(\int_Y (\mathrm{div}_{\mathbf{y}} D(\mathbf{y}) + U(\mathbf{y}))(L^{-1}H(\mathbf{y}) \cdot \nabla_{\mathbf{x}} n)\, d\nu(\mathbf{v})\, d\mathbf{y}) = 0 \ .$$

Thanks to (7.14), this equation is independent of the (unkown) additive kernel element in the definition of $L^{-1}$. Now

$$\begin{aligned}
\int_Y (\mathrm{div}_{\mathbf{y}} D(\mathbf{y}) + U(\mathbf{y}))(L^{-1}H(\mathbf{y}) \cdot \nabla_{\mathbf{x}} n)\, d\mathbf{y} &= \left[\int_Y (\mathrm{div}_{\mathbf{y}} D(\mathbf{y}) + U(\mathbf{y})) \otimes L^{-1}H(\mathbf{y})\, d\mathbf{y}\right] \nabla_{\mathbf{x}} n \\
&= \left[\int_Y L^{*-1}(\mathrm{div}_{\mathbf{y}} D(\mathbf{y}) + U(\mathbf{y})) \otimes H(\mathbf{y})\, d\mathbf{y}\right] \nabla_{\mathbf{x}} n \\
&= \left[\int_Y \mathbf{\Theta}^{[-1]}(\mathbf{y}) \otimes H(\mathbf{y})\, d\mathbf{y}\right] \nabla_{\mathbf{x}} n \ ,
\end{aligned}$$

where

$$\begin{aligned}
\mathbf{\Theta}^{[-1]}(\mathbf{y}) = L^{*-1}(\mathrm{div}_{\mathbf{y}} D(\mathbf{y}) + U(\mathbf{y})) &= L^{*-1} \int_V [(\mathbf{v} \cdot \nabla_{\mathbf{y}})\mathbf{X}^* M + (\mathbf{v} \cdot \nabla_{\mathbf{y}} M)\mathbf{X}^* + \mathbf{v}\lambda(\mathbf{y}, \mathbf{v})\,]\, d\nu(\mathbf{v}) \\
&= L^{*-1} \int_V (\mathbf{v} \cdot \nabla_{\mathbf{y}})\mathbf{X}^* M d\nu(\mathbf{v}) \ .
\end{aligned}$$

The diffusion equation for $n$ can finally be written

$$\partial_t n - \mathrm{div}_{\mathbf{x}}(\mathbf{D}\nabla n) = 0 \ ,$$

where

$$\mathbf{D} = \int_Y \left(\rho(\mathbf{y})D(\mathbf{y}) - \mathbf{\Theta}^{[-1]}(\mathbf{y}) \otimes H(\mathbf{y})\right)\, d\mathbf{y} \ .$$

### 7.2.2 Strategy of Proof

The above calculations provide reasonable heuristics for the result contained in the main theorem of this paper. For a rigorous proof, however, decoupling the diffusion step from the homogenization step does not lead to satisfactory bounds, and we shall proceed in a different way. The heuristics for this different approach is based on the use of the parameter $\eta$ : first one performs the limit as $\epsilon$ and $\alpha$ simultaneously tend to zero (for a fixed $\eta$) and then one takes the limit as $\eta$ tends to zero. The first limit leads to a decomposition of the form

$$f^{\varepsilon,\eta} \sim n^{\eta}(t, x)F^{\eta}(\mathbf{y}, \mathbf{v}) \ ,$$

where $F^{\eta}$ is the solution to $\eta\mathbf{v} \cdot \nabla_{\mathbf{y}} F^{\eta} = Q(F^{\eta})$, $\int_{Y\times V} F^{\eta} = 1$, and $n^{\eta}$ satisfies

$$\partial_t n^{\eta} - \mathrm{div}_{\mathbf{x}}\mathbf{D}^{\eta}\nabla_{\mathbf{x}} n^{\eta} = 0 \quad \text{with} \quad \mathbf{D}^{\eta} = \int_V \int_Y \mathbf{X}^{\eta *}(\mathbf{y}, \mathbf{v}) \otimes \mathbf{v}F^{\eta}(\mathbf{y}, \mathbf{v}) d\mathbf{y} d\nu(\mathbf{v}) \ .$$

Here $\mathbf{X}^{\eta *}$ solves an equation similar to $F^\eta$, namely $-\eta \mathbf{v}\cdot\nabla_{\mathbf{y}} X^{\eta *} = Q^*(X^{\eta *}) + \mathbf{v}$. To arrive at the second limit one may use expansions of $F^\eta$ and $X^{\eta *}$ in powers of $\eta$. A similar strategy was employed in [10] to analyze the diffusion limit in the presence of a very high magnetic field.

Our rigorous proof leading directly from the Boltzmann equation to the homogenized diffusion equation combines these two steps into one. First we perform a careful study of the cell problems for the equilibrium profile $F^\eta$ and the auxiliary function $X^{\eta *}$, and derive their limiting expansions (in Section 7.3). These expansions are then used directly to find the two-scale limit (in Section 7.4).

## 7.3 Study of the cell equations

As in [3] and [17], two cell problems are involved in the proof. However, in our context these cell problems depend on the parameter $\eta = \frac{\varepsilon}{\alpha}$. In this section we give expansions of the corresponding solutions with respect to this parameter. More precisely, we define $F^\eta$ and $\mathbf{X}^{\eta *}$ as the solutions to

$$T^\eta(F^\eta) = 0\ , \qquad \int F^\eta\, d\mathbf{y}\, d\nu(\mathbf{v}) = 1\ , \tag{7.15}$$

and

$$T^{\eta *}(\mathbf{X}^{\eta *}) = \mathbf{v}\ , \qquad \int \mathbf{X}^{\eta *}\, d\mathbf{y}\, d\nu(\mathbf{v}) = 0\ , \tag{7.16}$$

where

$$T^\eta = \eta \mathbf{v}\cdot\nabla_{\mathbf{y}} - Q, \quad \text{on } \mathcal{D} \subset L^2_{per}(\mathbf{R}^d_{\mathbf{y}} \times V)\ , \tag{7.17}$$

and

$$T^{\eta *} = -\eta \mathbf{v}\cdot\nabla_{\mathbf{y}} - Q^* \tag{7.18}$$

is the formal adjoint with respect to the scalar product of $L^2_{per}(\mathbf{R}^d_{\mathbf{y}} \times V)$. For the definition of the domain of definition $\mathcal{D}$, see Proposition 7.3.1. $Q^*$ is given by the formula

$$Q^*[\mathbf{y}](f)(\mathbf{v}) = \int_V \sigma(\mathbf{y}, \mathbf{v}', \mathbf{v})[f(\mathbf{v}') - f(\mathbf{v})]\, d\nu(\mathbf{v}')\ . \tag{7.19}$$

### 7.3.1 Preliminary results

We now list three propositions that will be useful for the proof of the main theorem.

*Proposition.* Let $L$ be the unbounded operator on $L^2_{per}(\mathbf{R}^d_{\mathbf{y}})$ with domain $H^2_{per}(\mathbf{R}^d_{\mathbf{y}})$, defined by (7.13), and let $L^*$ be its adjoint, defined by

$$L^*(n) = -\mathrm{div}_{\mathbf{y}}(D^\top(\mathbf{y})\nabla_{\mathbf{y}} n) - U(\mathbf{y})\cdot\nabla_{\mathbf{y}} n \quad \text{with periodic boundary conditions.} \tag{7.20}$$

The matrix-valued function $D$ and the vector field $U$ are given by (7.9). The following statements are true

(*i*) $Im(L) = \{u \in L^2_{per}(\mathbf{R}^d_{\mathbf{y}}), \quad s.t. \quad \int_Y u\, d\mathbf{y} = 0\}$ .

(*ii*) There exists a unique, positive function $\rho(\mathbf{y}) \in H^2_{per}(\mathbf{R}^d_{\mathbf{y}})$ such that $L(\rho) = 0$ and $\int_Y \rho\, d\mathbf{y} = 1$ .

(*iii*) For any function $v \in Im(L)$, there exists a unique solution to $L(u) = v$, $\int_Y u\, d\mathbf{y} = 0$ with $u \in H^2_{per}(\mathbf{R}^d_{\mathbf{y}})$. This solution will be denoted $u = L^{-1}(v)$, and $L^{-1}$ will be referred to as the pseudo inverse of $L$.

(*iv*) $Im(L^*) = \{u \in L^2_{per}(\mathbf{R}^d_{\mathbf{y}}), \quad s.t. \quad \int_Y u(\mathbf{y})\rho(\mathbf{y})\, d\mathbf{y} = 0\}$ .

(*v*) The kernel of $L^*$ is the set of constant functions (in the variable $\mathbf{y}$).

(*vi*) For any function $v^* \in Im(L^*)$, there exists a unique solution to $L^*(u^*) = v^*$, $\int_Y u^*(\mathbf{y})\, d\mathbf{y} = 0$ with $u^* \in H^2_{per}(\mathbf{R}^d_{\mathbf{y}})$. This solution will be denoted $u^* = L^{*-1}(v^*)$, and $L^{*-1}$ will be referred to as the pseudo inverse of $L^*$.

(*vii*) $D(\mathbf{y})$, $U(\mathbf{y})$ and $\rho(\mathbf{y})$ have the following symmetry properties
- The diffusion matrix $D(\mathbf{y})$ is even with respect to $\mathbf{y}$. The Flux $U(\mathbf{y})$ is odd with respect to $\mathbf{y}$. The equilibrium function $\rho(\mathbf{y})$ is even.
- $L^{-1}$ and $L^{*-1}$ leave invariant the set of even as well as the set of odd functions in $\mathbf{y}$.

(*viii*) The diffusion matrix $D(\mathbf{y})$ is in $C^\infty_{per}(\mathbf{R}^d_\mathbf{y})$ and satisfies

$$D(y)\xi\cdot\xi = 0 \iff \xi = 0 \quad \text{and furthermore} \quad D(y)\xi\cdot\xi \geq \beta|\xi|^2 \ , \tag{7.21}$$

for some constant $\beta > 0$.

(*ix*) There exists a constant $C$ such that

$$\begin{cases} \|L^{*-1}v^*\|_{H^2_{per}} \leq C\|v^*\|_{L^2_{per}}, \qquad \|L^{-1}v\|_{H^2_{per}} \leq C\|v\|_{L^2_{per}}, \\ \|L^{*-1}v^*\|_{L^2_{per}} \leq C\|v^*\|_{H^{-2}_{per}}, \qquad \|L^{-1}v\|_{L^2_{per}} \leq C\|v\|_{H^{-2}_{per}} \ , \end{cases} \tag{7.22}$$

for all $v \in Im(L)$ and $v^* \in Im(L^*)$.

*Démonstration.* We only give the proof of item 5 (thus also item 2) and item 8, leaving the others to the reader. Let $c$ be positive and sufficiently large. The function 1 is a positive eigenvector for $(L^* + c)^{-1}$ (with corresponding eigenvalue $c^{-1}$). Furthermore, since $(L^* + c)^{-1}$ is a strongly positive operator (in the sense that it maps positive functions to positive functions) the Krein-Rutmann theorem [26] [13] asserts that a positive eigenvector can only be associated with the principal eigenvalue (the eigenvalue that equals the spectral radius). This theorem also asserts that the principal eigenvalue is simple, that it is at the same time the simple, principal eigenvalue for the adjoint ( $[(L^* + c)^{-1}]^* = (L + c)^{-1}$), and that the corresponding eigenspace is generated by a positive eigenvector. It follows that $c$ is the lowest eigenvalue for $L + c$ and $L^* + c$, and that the corresponding eigenspaces are simple and generated by positive eigenfunctions. Consequently 0 is the lowest eigenvalue for $L$ and $L^*$ and the eigenspaces are simple and generated by positive eigenfunctions.

The dissipation relation gives

$$\begin{aligned} D(\mathbf{y})\xi\cdot\xi &= \int_V \mathbf{X}^*(\mathbf{y},\mathbf{v})\cdot\xi\mathbf{v}\cdot\xi M(\mathbf{y},\mathbf{v})d\nu(\mathbf{v}) \\ &= \frac{1}{2}\int_V\int_V \sigma(\mathbf{y},\mathbf{v},\mathbf{v}')M(\mathbf{y},\mathbf{v})|\mathbf{X}^*(\mathbf{y},\mathbf{v}')\cdot\xi - \mathbf{X}^*(\mathbf{y},\mathbf{v})\cdot\xi|^2 d\nu(\mathbf{v})d\nu(\mathbf{v}') \geq 0 \ , \end{aligned}$$

see [17]. Equality implies that $\mathbf{X}^*(\mathbf{y},\cdot)\cdot\xi$ does not depend on $\mathbf{v}$, and therefore it belongs to the kernel of $Q^*$. Since $Q^*[\mathbf{y}](\mathbf{X}^*\cdot\xi)(\mathbf{v}) = -\mathbf{v}\cdot\xi$, we get $\mathbf{v}\cdot\xi = 0$ for all $\mathbf{v}$, and so $\xi = 0$. The coercivity estimate of (7.21) (with $\beta > 0$, independent of $\mathbf{y}$) follows by continuity.

*Proposition.* Assume hypotheses **(H0)**, **(H1)** and **(H2)** are satisfied, then

(*i*) The operator $Q[\mathbf{y}]$ is a bounded operator on $L^2(V)$ with a uniformly bounded norm (when $\mathbf{y}$ varies in $Y$).

(*ii*) The kernel of $Q[\mathbf{y}]$ is a one dimensional space spanned by the positive function $M(\mathbf{y},\mathbf{v})$ (the local Maxwellian).

(*iii*) The range of $Q[\mathbf{y}]$ is the set of functions $f \in L^2(V)$ such that $\int f(\mathbf{v})\, d\nu(\mathbf{v}) = 0$.

(*iv*) Suppose $f = f(\mathbf{y},\mathbf{v})$ is a function in $L^p(V; H^k_{per}(\mathbf{R}^d_\mathbf{y}))$, with $p = 2$ or $+\infty$ and $k$ an integer, and suppose $f$ has zero velocity average (pointwise in $y$). Then there exists a unique function $u$, denoted $Q[\mathbf{y}]^{-1}(f)$, satisfying

$$Q[\mathbf{y}](u) = f, \qquad \int_V u(\mathbf{y},\mathbf{v})\, d\nu(\mathbf{v}) = 0 \ , \quad a.e. \text{ in } Y \ .$$

Moreover $u$ lies in $L^p(V; H^k_{per}(\mathbf{R}^d_{\mathbf{y}}))$, and we have the estimate

$$\|u\|_{L^p(V;H^k_{per})} \leq C_k \|f\|_{L^p(V;H^k_{per})} \ . \tag{7.23}$$

Analogously, the adjoint operator $Q^*[\mathbf{y}]$ (in $L^2(V)$) has the following properties

(*v*) The operator $Q^*[\mathbf{y}]$ is a bounded operator on $L^2(V)$ with a uniformly bounded norm (when $\mathbf{y}$ varies in $Y$).

(*vi*) The kernel of $Q^*[\mathbf{y}]$ is the set of constant functions (in $\mathbf{v}$).

(*vii*) The range of $Q^*[\mathbf{y}]$ is the set of functions $f \in L^2(V)$ such that $\int f(\mathbf{v}) M(\mathbf{y}, \mathbf{v})\, d\nu(\mathbf{v}) = 0$.

(*viii*) Suppose $f = f(\mathbf{y}, \mathbf{v})$ is a function in $L^p(V; H^k_{per}(\mathbf{R}^d_{\mathbf{y}}))$, with $p = 2$ or $+\infty$ and $k$ an integer, and suppose $\int f(\mathbf{y}, \mathbf{v}) M(\mathbf{y}, \mathbf{v})\, d\nu(\mathbf{v}) = 0$ a.e. in $Y$. Then there exists a unique function $u^*$, denoted $Q[\mathbf{y}]^{*-1}(f)$, satisfying

$$Q^*[\mathbf{y}](u^*) = f \ , \qquad \int_V u^*(\mathbf{y}, \mathbf{v})\, d\nu(\mathbf{v}) = 0, \quad a.e. \text{ in } Y \ .$$

Moreover $u^*$ lies in $L^p(V; H^k_{per}(\mathbf{R}^d_{\mathbf{y}}))$, and we have the estimate

$$\|u^*\|_{L^p(V;H^k_{per})} \leq C_k \|f\|_{L^p(V;H^k_{per})} \ . \tag{7.24}$$

Finally, we have the following symmetry invariants

(*ix*) The operators $Q[\mathbf{y}]^{-1}$ and $Q[\mathbf{y}]^{*-1}$ leave invariant the set of even functions, as well as the set of odd functions in $\mathbf{v}$.

*Démonstration.* We start by proving items 2 and 6. The fact that $M$ satisfies $Q[\mathbf{y}](M) = 0$ is a simple consequence of the assumption **(H2)**. The well known dissipation relation (with $Q[\mathbf{y}]$ abbreviated $Q$) states that

$$2Q(g)\frac{g}{f} = Q(\frac{g^2}{f}) + Q(f)(\frac{g}{f})^2 - \int_V \sigma(\mathbf{y}, \mathbf{v}, \mathbf{v}') f(\mathbf{v}') |\frac{g}{f}(\mathbf{v}') - \frac{g}{f}(\mathbf{v})|^2 d\nu(\mathbf{v}') \ ,$$

which, by insertion of $f = M$ and integration, leads to

$$\frac{1}{2}\int_{V\times V} \sigma(\mathbf{y}, \mathbf{v}, \mathbf{v}') M(\mathbf{y}, \mathbf{v}') |\frac{g}{M}(\mathbf{y}, \mathbf{v}') - \frac{g}{M}(\mathbf{y}, \mathbf{v})|^2 d\nu(\mathbf{v}) d\nu(\mathbf{v}') = -\int_V Q[\mathbf{y}](g) \frac{g}{M(\mathbf{y}, \mathbf{v})} d\nu(\mathbf{v}) \ . \tag{7.25}$$

For any $g$ with $Q[\mathbf{y}](g) = 0$ this implies that $g/M$ does not depend on $\mathbf{v}$ and so $g = CM(\mathbf{y}, \mathbf{v})$ a.e. in $\mathbf{v}$. This verifies 2. To prove 6, we already know that 1 belongs to the kernel of $Q^*[\mathbf{y}]$. If we insert $g = M\tilde{g}$ into (7.25) then we arrive at

$$\frac{1}{2}\int_{V\times V} \sigma(\mathbf{y}, \mathbf{v}, \mathbf{v}') M(\mathbf{y}, \mathbf{v}') |\tilde{g}(\mathbf{v}') - \tilde{g}(\mathbf{v})|^2 d\nu(\mathbf{v}) d\nu(\mathbf{v}') = -\int_V M(\mathbf{y}, \mathbf{v}) \tilde{g} Q^*[\mathbf{y}](\tilde{g}) d\nu(\mathbf{v}) \ .$$

For any $\tilde{g}$ with $Q^*[\mathbf{y}](\tilde{g}) = 0$, this implies that $\tilde{g}$ does not depend on $\mathbf{v}$, and so item 6 is verified. The items 1,3, 5, 7 and 9 are quite elementary, and their proof is left to the reader. It thus remains to verify item 4 and 8. We shall only prove the estimate (7.23) and note that (7.24) follows by a similar argument. We first consider the case $p = 2$. The starting point is the bound $\|u(\mathbf{y}, \cdot)\|_{L^2(V)} \leq \|Q[\mathbf{y}]^{-1}\| \|f(\mathbf{y}, \cdot)\|_{L^2(V)}$. Since $\|Q[\mathbf{y}]^{-1}\|$ is bounded uniformly in $Y$, we obtain (after taking the $L^2$ norm in $\mathbf{y}$) that

$$\|u\|_{L^2(V\times Y)} \leq C \|f\|_{L^2(V\times Y)} \ .$$

To obtain estimates on the derivatives, we differentiate the equation with respect to an arbitrary coordinate $y_i$ which yields $Q(\partial_{y_i} u) = -\partial_{y_i} Q(u) + \partial_{y_i} f$, where we denote by $\partial_{y_i} Q$ the operator in which $\sigma$ is replaced by $\partial_{y_i}\sigma$. An application of the zeroth order inequality leads to

$$\begin{aligned} \|\partial_{y_i} u\|_{L^2(V\times Y)} \leq C(\|\partial_{y_i} Q(u)\|_{L^2(V\times Y)} + \|\partial_{y_i} f\|_{L^2(V\times Y)}) &\leq C'\|u\|_{L^2(V\times Y)} + C\|\partial_{y_i} f\|_{L^2(V\times Y)} \\ &\leq C_1 \|f\|_{L^2(V;H^1(Y))} \ . \end{aligned}$$

Differentiating successively with respect to $\mathbf{y}$ and proceeding by induction, we arrive at the inequality

$$\|u\|_{L^2(V;H^k(Y))} \leq C_k \|f\|_{L^2(V;H^k(Y))} \ .$$

In order to prove the result in $L^\infty$, we write

$$u(\mathbf{y},\mathbf{v}) = \frac{1}{\Sigma(\mathbf{y},\mathbf{v})}\left(-f(\mathbf{y},\mathbf{v}) + \int_V \sigma(\mathbf{y},\mathbf{v},\mathbf{v}')u(y,\mathbf{v}')d\nu(\mathbf{v}')\right) \ ,$$

where $\Sigma(\mathbf{y},\mathbf{v}) = \int \sigma(\mathbf{y},\mathbf{v}',\mathbf{v})\, d\nu(\mathbf{v}')$. As $\Sigma$ is also bounded from below by a positive constant, we have the pointwise estimate

$$|u(\mathbf{y},\mathbf{v})| \leq C|f(\mathbf{y},\mathbf{v})| + C\|u(\mathbf{y},\cdot)\|_{L^2(V)} \leq C|f(\mathbf{y},\mathbf{v})| + C\|f(\mathbf{y},\cdot)\|_{L^2(V)} \ .$$

Hence

$$\|u(\cdot,\mathbf{v})\|_{L^2(Y)} \leq C\|f(\cdot,\mathbf{v})\|_{L^2(Y)} + C\|f\|_{L^2(Y\times V)} \leq C\|f(\cdot,\mathbf{v})\|_{L^2(Y)} + C\|f\|_{L^\infty(V;L^2(Y))} \ .$$

Taking supremum over all $\mathbf{v}$, we finally get

$$C\|u\|_{L^\infty(V;L^2(Y))} \leq C_0\|f\|_{L^\infty(V;L^2(Y))} \ .$$

Applying this inequality to the equation $Q(\partial_{y_i}u) = -\partial_{y_i}Q(u) + \partial_{y_i}f$ we are led to the estimate

$$\|u\|_{L^\infty(V;H^1(Y))} \leq C_1\|f\|_{L^\infty(V;H^1(Y))} \ .$$

As was the case in the $L^2$ case, the estimate for arbitrary Sobolev index $k$ follows by induction. □

*Remarque* 7.3.1. We note that $M$ is bounded from below since

$$M(\mathbf{y},\mathbf{v}) = \frac{1}{\Sigma(\mathbf{y},\mathbf{v})}\int \sigma(\mathbf{y},\mathbf{v},\mathbf{v}')M(\mathbf{y},\mathbf{v}')d\nu(\mathbf{v}') \geq c\int_V M(\mathbf{y},\mathbf{v}')d\nu(\mathbf{v}') = c \ .$$

Therefore $\dfrac{1}{M}$ belongs to $L^\infty(V;C^\infty_{per}(Y))$.

We now proceed to the analysis of the operators $T^\eta$ and $T^{\eta *}$ defined by (7.17) and (7.18). Parts of the following proposition are taken from [17].

*Proposition.* The operator $T^\eta = \eta\mathbf{v}\cdot\nabla_{\mathbf{y}} - Q[\mathbf{y}](\cdot)$ is un unbounded operator on $L^2_{per}(\mathbf{R}^d_{\mathbf{y}}\times V)$ with domain

$$\mathcal{D} = \{u \in L^2_{per}(\mathbf{R}^d_{\mathbf{y}}\times V), \quad \text{such that } \mathbf{v}\cdot\nabla_{\mathbf{y}}u \in L^2_{per}(\mathbf{R}^d_{\mathbf{y}}\times V)\ \} \ .$$

Let $T^{\eta^*} = -\eta\mathbf{v}\cdot\nabla_{\mathbf{y}} - Q^*[\mathbf{y}](\cdot)$ be the adjoint of $T^\eta$ . Then

(*i*) The kernel of $T^\eta$ is a one dimensional space spanned by a positive function $F^\eta(\mathbf{y},\mathbf{v})$, that we normalize by $\displaystyle\int_{V\times Y} F^\eta d\nu(\mathbf{v})d\mathbf{y} = 1$. The function $F^\eta$ satisfies

$$F^\eta(-\mathbf{y},-\mathbf{v}) = F^\eta(\mathbf{y},\mathbf{v}), \quad a.e. \text{ in } \mathbf{y} \text{ and } \mathbf{v} \ . \tag{7.26}$$

(*ii*) The range of $T^\eta$ is the set of functions $g \in L^2_{per}(\mathbf{R}^d_{\mathbf{y}}\times V)$ such that $\displaystyle\int_{V\times Y} g(\mathbf{y},\mathbf{v})\, d\nu(\mathbf{v})d\mathbf{y} = 0$ .

(*iii*) The adjoint $T^{\eta *}$ has the same domain $\mathcal{D}$. Its range is the set of functions $g$ such that

$$\int_{V\times Y} F^\eta(\mathbf{y},\mathbf{v})\, g(\mathbf{y},\mathbf{v})d\nu(\mathbf{v})d\mathbf{y} = 0 \ .$$

Its kernel is the set of constant functions (in $\mathbf{y}$ and $\mathbf{v}$).

(*iv*) For $g \in L^2_{per}(\mathbf{R}^d_{\mathbf{y}} \times V)$ satisfying $\displaystyle\int_{V\times Y} g(\mathbf{y},\mathbf{v})d\nu(\mathbf{v})d\mathbf{y} = 0$ there exists a unique function $R^\eta \in \mathcal{D}$ such that

$$T^\eta(R^\eta) = g \quad \text{and} \quad \int_{V\times Y} R^\eta(\mathbf{y},\mathbf{v})d\nu(\mathbf{v})d\mathbf{y} = 0\ . \tag{7.27}$$

We shall denote $R^\eta = (T^\eta)^{-1}(g)$ . There exists $\eta_0 > 0$ such that $R^\eta$ satisfies the following bound

$$||R^\eta||_{L^2(Y\times V)} \leq \frac{C}{\eta^2}||g||_{L^2(Y\times V)}\ , \quad 0 < \eta < \eta_0\ . \tag{7.28}$$

Moreover, the following symmetry implications hold true

$$\begin{cases} \text{If } g(-\mathbf{y},-\mathbf{v}) = \ g(\mathbf{y},\mathbf{v}) \quad a.e. \quad \text{then} \quad R^\eta(-\mathbf{y},-\mathbf{v}) = \ R^\eta(\mathbf{y},\mathbf{v}) \quad a.e. \\ \text{If } g(-\mathbf{y},-\mathbf{v}) = -g(\mathbf{y},\mathbf{v}) \quad a.e. \quad \text{then} \quad R^\eta(-\mathbf{y},-\mathbf{v}) = -R^\eta(\mathbf{y},\mathbf{v}) \quad a.e. \end{cases} \tag{7.29}$$

(*v*) For $g^* \in L^2_{per}(\mathbf{R}^d_{\mathbf{y}} \times V)$ satisfying $\displaystyle\int_{V\times Y} F^\eta(\mathbf{y},\mathbf{v})g^*(\mathbf{y},\mathbf{v})d\nu(\mathbf{v})d\mathbf{y} = 0$ there exists a unique function $R^{\eta *} \in \mathcal{D}$ such that

$$T^{\eta *}(R^{\eta *}) = g^* \quad \text{and} \quad \int_{V\times Y} R^{\eta *}(\mathbf{y},\mathbf{v})d\nu(\mathbf{v})d\mathbf{y} = 0\ . \tag{7.30}$$

We shall denote $R^{\eta *} = (T^{\eta *})^{-1}(g^*)$ . There exist $\eta_0 > 0$ such that $R^{\eta *}$ satisfies the following bound

$$||R^{\eta *}||_{L^2(Y\times V)} \leq \frac{C}{\eta^2}||g^*||_{L^2(Y\times V)}\ , \quad 0 < \eta < \eta_0\ . \tag{7.31}$$

Moreover, the following symmetry implications hold true

$$\begin{cases} \text{If } g^*(-\mathbf{y},-\mathbf{v}) = \ g^*(\mathbf{y},\mathbf{v}) \quad a.e. \quad \text{then} \quad R^{\eta *}(-\mathbf{y},-\mathbf{v}) = \ R^{\eta *}(\mathbf{y},\mathbf{v}) \quad a.e. \\ \text{If } g^*(-\mathbf{y},-\mathbf{v}) = -g^*(\mathbf{y},\mathbf{v}) \quad a.e. \quad \text{then} \quad R^{\eta *}(-\mathbf{y},-\mathbf{v}) = -R^{\eta *}(\mathbf{y},\mathbf{v}) \quad a.e. \end{cases} \tag{7.32}$$

*Démonstration.* Parts 1 thru 3 of this proposition are proven in [17] and require assumption **(A1)** on the probability measure $d\nu(v)$. Of the remaining two items we shall only prove 4. Suppose $\displaystyle\int_{V\times Y} g(\mathbf{y},\mathbf{v})d\nu(\mathbf{v})d\mathbf{y} = 0$. Then (according to item 2) there exists a unique solution, $R^\eta$, to (7.27). We introduce $\tilde{R}^\eta = \eta^2 R^\eta$ , $\gamma^\eta(\mathbf{y}) = \displaystyle\int_V \frac{\tilde{R}^\eta(\mathbf{y},\mathbf{v})}{M(\mathbf{y},\mathbf{v})}d\nu(\mathbf{v})$ and perform a first order Chapman-Enskog expansion

$$\tilde{R}^\eta(\mathbf{y},\mathbf{v}) = \gamma^\eta(\mathbf{y})M(\mathbf{y},\mathbf{v}) + r^\eta(\mathbf{y},\mathbf{v}) \quad \text{with} \quad \int_V \frac{r^\eta(\mathbf{y},\mathbf{v})}{M(\mathbf{y},\mathbf{v})}d\nu(\mathbf{v}) = 0\ .$$

Following the same argument as in [19], we prove by contradiction that $||\tilde{R}^\eta||_{L^2(Y\times V)} \leq C||g||_{L^2(Y\times V)}$ for $\eta$ sufficiently close to zero. This verifies the bound 7.28 on $R^\eta = \eta^{-2}\tilde{R}^\eta$ . To arrive at a contradiction suppose

$$||\tilde{R}^\eta||_{L^2(Y\times V)} = 1 \quad \text{and} \quad ||g^\eta||_{L^2(Y\times V)} \to 0\ ,$$

for some sequence $\eta \to 0$. By equation (7.25), we obtain

$$||\frac{g}{M} - \int_V \frac{g}{M}d\nu(\mathbf{v})||^2_{L^2(V)} \leq -\int_V Q(g)\frac{g}{M}d\nu(\mathbf{v}).$$

Therefore

$$\begin{aligned}
||\tilde{R}^\eta - \gamma^\eta M||^2_{L^2(Y\times V)} &\leq ||\tfrac{\tilde{R}^\eta}{M} - \gamma^\eta||^2_{L^2(Y\times V)} \\
&\leq -\int_Y\int_V Q(\tilde{R}^\eta)\frac{\tilde{R}^\eta}{M}dvdy \\
&= \int_Y\int_V(\eta^2 g^\eta - \eta v\cdot\nabla_y\tilde{R}^\eta)\frac{\tilde{R}^\eta}{M}dvdy \\
&\leq C(\eta^2||\tilde{R}^\eta||_{L^2(Y\times V)}||g^\eta||_{L^2(Y\times V)} + \eta||\tilde{R}^\eta||^2_{L^2(Y\times V)}) \ .
\end{aligned}$$

This shows that $||r^\eta||_{L^2(Y\times V)} \to 0$ . Since $\tilde{R}^\eta = \gamma^\eta M(\mathbf{y},\mathbf{v}) + r^\eta$, we have

$$\eta\mathbf{v}\cdot\nabla_\mathbf{y}(\gamma^\eta M(\mathbf{y},\mathbf{v})) + \eta\mathbf{v}\cdot\nabla_\mathbf{y}(r^\eta) = Q(\gamma^\eta M(\mathbf{y},\mathbf{v})) + Q(r^\eta) + \eta^2 g^\eta \ ,$$

and since $Q(M) = 0$ , the equation satisfied by the remainder is

$$\eta\mathbf{v}\cdot\nabla_\mathbf{y}(r^\eta) - Q(r^\eta) = \eta^2 g^\eta - \eta\mathbf{v}\cdot\nabla_\mathbf{y}(\gamma^\eta(\mathbf{y})M(\mathbf{y},\mathbf{v})) \ .$$

Integration of this equation with respect to $\mathbf{v}$ leads to

$$\mathrm{div}_\mathbf{y}(\int_V(\mathbf{v}r^\eta)d\nu(\mathbf{v})) = \eta\int_V g^\eta d\nu(\mathbf{v}) \ . \tag{7.33}$$

For this identity we use the facts that $\int_V Q(r^\eta)\ d\nu(\mathbf{v}) = 0$ and $\int_V \mathbf{v}M(\mathbf{y},\mathbf{v})\,d\nu(\mathbf{v}) = 0$. Since

$$\begin{aligned}
\mathrm{div}_\mathbf{y}(\int_V(\mathbf{v}r^\eta)d\nu(\mathbf{v})) &= -\mathrm{div}_\mathbf{y}(\int_V \mathbf{X}^*Q(r^\eta)d\nu(\mathbf{v})) \\
&= -\eta\mathrm{div}_\mathbf{y}(\int_V \mathbf{X}^*\mathbf{v}\cdot\nabla_\mathbf{y}(\gamma^\eta M)d\nu(\mathbf{v})) - \eta\mathrm{div}_\mathbf{y}(\int_V \mathbf{X}^*\mathbf{v}\cdot\nabla_\mathbf{y}(r^\eta)d\nu(\mathbf{v})) \\
&\quad +\eta^2\mathrm{div}_\mathbf{y}(\textstyle\int_V g^\eta\mathbf{X}^*d\nu(\mathbf{v})) \ ,
\end{aligned}$$

we deduce from (7.33) that $\gamma^\eta$ satisfies the elliptic equation

$$L(\gamma^\eta) = \int_V g^\eta d\nu(\mathbf{v}) + \mathrm{div}_\mathbf{y}(\int_V \mathbf{X}^*\mathbf{v}\cdot\nabla_\mathbf{y}(r^\eta)d\nu(\mathbf{v})) - \eta\mathrm{div}_\mathbf{y}(\int_V g^\eta\mathbf{X}^*d\nu(\mathbf{v})) \ , \tag{7.34}$$

where $L$ is defined in (7.13). By elliptic regularity this yields the convergence of the sequence $(\gamma^\eta)$ in $L^2(Y)$. Indeed, $\gamma^\eta = \gamma_1^\eta + \gamma_2^\eta + C^\eta\rho$ where $\rho$ satisfies

$$L(\rho) = 0 \ , \qquad \int_Y \rho(\mathbf{y})d\mathbf{y} = 1 \ ,$$

and $\gamma_1^\eta$ and $\gamma_2^\eta$ satisfy

$$L(\gamma_1^\eta) = \int_V g^\eta d\nu(\mathbf{v}), \qquad \gamma_1^\eta \in (span\{1\})^\perp \ ,$$

$$L(\gamma_2^\eta) = \mathrm{div}_\mathbf{y}(\int_V \mathbf{X}^*\mathbf{v}\cdot\nabla_\mathbf{y}(r^\eta)d\nu(\mathbf{v})) - \eta\mathrm{div}_\mathbf{y}(\int_V g^\eta\mathbf{X}^*d\nu(\mathbf{v})) \ , \qquad \gamma_2^\eta \in (span\{1\})^\perp \ .$$

Since $||g^\eta||_{L^2(Y\times V)} \to 0$, $||\nabla_\mathbf{y}(\int_V \mathbf{X}^*\mathbf{v}\cdot\nabla_\mathbf{y}(r^\eta)d\nu(\mathbf{v}))||_{H^{-2}_{per}} \to 0$ and $||\eta\mathrm{div}_\mathbf{y}(\int_V g^\eta\mathbf{X}^*d\nu(\mathbf{v}))||_{H^{-1}_{per}} \to 0$ we conclude that $\gamma_1^\eta + \gamma_2^\eta$ converges strongly to 0 in $L^2_{per}(\mathbf{R}^d_\mathbf{y})$, as $\eta \to 0$. Thus

$$C^\eta - \int_Y \gamma^\eta d\mathbf{y} = -\int_Y(\gamma_1^\eta + \gamma_2^\eta)\,d\mathbf{y} \to 0 \ .$$

At the same time

$$\int_Y \gamma^\eta d\mathbf{y} = \int_Y \int_V \gamma^\eta M(\mathbf{y},\mathbf{v}) d\mathbf{y} d\nu(\mathbf{v}) = -\int_V \int_Y r^\eta(\mathbf{y},\mathbf{v}) d\mathbf{y} d\nu(\mathbf{v}) \to 0 \ ,$$

and so $C^\eta \to 0$. In summary : $(\gamma^\eta)$ converges strongly to zero in $L^2_{per}(\mathbf{R}^d_\mathbf{y})$. The formula $\tilde{R}^\eta = \gamma^\eta M(\mathbf{y},\mathbf{v}) + r^\eta$ implies that $\tilde{R}^\eta$ converges strongly to zero in $L^2_{per}(\mathbf{R}^d_\mathbf{y} \times V)$. This is a direct contradiction to the fact that $||\tilde{R}^\eta||_{L^2(Y\times V)} = 1$. □

### 7.3.2 Expansion of the two-scale equilibrium

Note that the equation (7.15) is in fact a steady state Boltzmann equation on the unit periodic cell, with $\eta$ representing a "cell mean free path". It is therefore not surprising that, in the limit $\eta \to 0$, we have a steady state diffusion approximation, as asserted by the following proposition.

*Proposition.* Assume Hypotheses **(H0)**, **(H1)** and **(H2)** hold true. Let $M$ be the local Maxwellian, and let $\mathbf{X}$ and $\lambda$ be uniquely defined by

$$Q[\mathbf{y}](\mathbf{X}) = -\mathbf{v}M \quad \text{and} \quad Q[\mathbf{y}](\lambda) = \mathbf{v}\cdot\nabla_\mathbf{y} M \ , \qquad \int_V \mathbf{X}(\mathbf{y},\mathbf{v})\, d\nu(\mathbf{v}) = \int_V \lambda(\mathbf{y},\mathbf{v})\, d\nu(\mathbf{v}) = 0 \ . \quad (7.35)$$

The solution, $F^\eta$, to (7.15) can be writen

$$F^\eta(\mathbf{y},\mathbf{v}) = \rho(\mathbf{y})M(\mathbf{y},\mathbf{v}) + \eta\left(-\mathbf{X}(\mathbf{y},\mathbf{v})\cdot\nabla_\mathbf{y}\rho(\mathbf{y}) + \lambda(\mathbf{y},\mathbf{v})\rho(\mathbf{y})\right) + \eta^2 R^\eta(\mathbf{y},\mathbf{v}) \ ,$$

where the remainder term satisfies $||R^\eta||_{L^2(Y\times V)} \le C$ , uniformly in $\eta$ , and $\rho$ is the unique 1-periodic solution to

$$-\mathrm{div}_\mathbf{y}\left(D(\mathbf{y})\nabla_\mathbf{y}\rho\right) + \mathrm{div}_\mathbf{y}\left(U(\mathbf{y})\rho\right) = 0 \quad \text{with} \quad \int_Y \rho(\mathbf{y}) d\mathbf{y} = 1 \ .$$

Here

$$\begin{aligned} D(\mathbf{y}) &= \int_V \mathbf{v}\otimes\mathbf{X}(\mathbf{y},\mathbf{v}) d\nu(\mathbf{v}) = \int_V \mathbf{X}^*(\mathbf{y},\mathbf{v})\otimes\mathbf{v}M(\mathbf{y},\mathbf{v}) d\nu(\mathbf{v}) \ , \\ U(\mathbf{y}) &= \int_V \mathbf{v}\lambda(\mathbf{y},\mathbf{v}) d\nu(\mathbf{v}) = -\int_V (\mathbf{v}\cdot\nabla_\mathbf{y} M)\mathbf{X}^*\, d\nu(\mathbf{v}) \ . \end{aligned}$$

*Démonstration.* The first step of the proof consists in writing a finite expansion of $F^\eta$ in powers of $\eta$. Namely, we write

$$\begin{aligned} F^\eta(\mathbf{y},\mathbf{v}) &= F_0(\mathbf{y},\mathbf{v}) + \eta F_1(\mathbf{y},\mathbf{v}) + \eta^2 F_2(\mathbf{y},\mathbf{v}) + \eta^3 F_3(\mathbf{y},\mathbf{v}) + \tilde{R}^\eta(\mathbf{y},\mathbf{v}) \\ &= F_0(\mathbf{y},\mathbf{v}) + \eta F_1(\mathbf{y},\mathbf{v}) + \eta^2 R^\eta \ , \end{aligned} \quad (7.36)$$

with $R^\eta = F_2(\mathbf{y},\mathbf{v}) + \eta F_3(\mathbf{y},\mathbf{v}) + \eta^{-2}\tilde{R}^\eta(\mathbf{y},\mathbf{v})$. Insertion of this expansion in (7.15), and a match of powers of $\eta$ leads to the following equations

$$\begin{aligned} Q[\mathbf{y}](F_0) &= 0 \ , \ \textstyle\int_{V\times Y} F_0 d\nu(\mathbf{v}) d\mathbf{y} = 1 \ , \\ Q[\mathbf{y}](F_1) &= \mathbf{v}\cdot\nabla_\mathbf{y} F_0 \ , \ \textstyle\int_{V\times Y} F_1 d\nu(\mathbf{v}) d\mathbf{y} = 0 \ , \\ Q[\mathbf{y}](F_2) &= \mathbf{v}\cdot\nabla_\mathbf{y} F_1 \ , \ \textstyle\int_{V\times Y} F_2 d\nu(\mathbf{v}) d\mathbf{y} = 0 \ , \\ Q[\mathbf{y}](F_3) &= \mathbf{v}\cdot\nabla_\mathbf{y} F_2 \ , \ \textstyle\int_{V\times Y} F_3 d\nu(\mathbf{v}) d\mathbf{y} = 0 \ . \end{aligned}$$

We may satisfy the last three integral constraints by requiring that $\int_V F_i d\nu(\mathbf{v}) = 0$ a.e. in $\mathbf{y}$, for $1 \le i \le 3$. The first set of equations leads to $F_0 = \rho(\mathbf{y})M(\mathbf{y},\mathbf{v})$, where $M(\mathbf{y},\mathbf{v})$ is the Maxwellian. For the second set of equations to be solvable, the right hand side needs to be in the range of the operator $Q$, which means

$$\int_V \mathbf{v}\cdot\nabla_\mathbf{y} F_0 d\nu(\mathbf{v}) = 0 \ .$$

This is satisfied since $M$ is even with repect to $\mathbf{v}$. For the third set of equations to have a solution, we need

$$\int_V \mathbf{v} \cdot \nabla_{\mathbf{y}} F_1 = 0 \ .$$

This will provide us (as is usual for diffusion limits) a diffusion equation for the function $\rho(\mathbf{y})$ . Indeed, if $\mathbf{X}(\mathbf{y}, \mathbf{v})$ and $\lambda(\mathbf{y}, \mathbf{v})$ are as defined by (7.35), then we calculate $F_1(\mathbf{y}, \mathbf{v}) = -\mathbf{X}(\mathbf{y}, \mathbf{v}) \cdot \nabla_{\mathbf{y}} \rho(\mathbf{y}) + \lambda(\mathbf{y}, \mathbf{v})\rho(\mathbf{y})$. The equation $\int_V \mathbf{v} \cdot \nabla_{\mathbf{y}} F_1 = 0$ (together with $\int_{V \times Y} F_0 d\nu(\mathbf{v}) d\mathbf{y} = 1$) now becomes

$$-\mathrm{div}_{\mathbf{y}} \left( D(\mathbf{y}) \nabla_{\mathbf{y}} \rho \right) + \mathrm{div}_{\mathbf{y}} \left( U(\mathbf{y}) \rho \right) = 0 \quad \text{with} \quad \int_Y \rho(\mathbf{y}) d\mathbf{y} = 1 \ , \tag{7.37}$$

a steady state diffusion equation for $\rho$. By Proposition 7.3.1, $Q[\mathbf{y}]^{-1}$ leaves invarient the set of even, as well as the set of odd functions in $\mathbf{v}$. Since we already know that $F_1$ is odd it follows that $F_2$ is even. Therefore the compatiblity condition $\int_v \mathbf{v} \cdot \nabla_{\mathbf{y}} F_2 d\nu(\mathbf{v}) = 0$ holds and this leads to the existence of $F_3$.

The term remainder term $\tilde{R}^\eta$ will now satisfy

$$\eta \mathbf{v} \cdot \nabla_{\mathbf{y}} \tilde{R}^\eta - Q[\mathbf{y}](\tilde{R}^\eta) = -\eta^4 \mathbf{v} \cdot \nabla_{\mathbf{y}} F_3 \ , \quad \text{with} \int_{V \times Y} \tilde{R}^\eta d\nu(\mathbf{v}) d\mathbf{y} = 0 \ . \tag{7.38}$$

Note that this equation can be solved since $\displaystyle\int_Y \int_V \mathbf{v} \cdot \nabla_{\mathbf{y}} F_3 d\mathbf{y} d\nu(\mathbf{v}) = 0$ .

The final step of the proof consists in providing an estimate for $||R^\eta||_{L^2(Y \times V)}$ as $\eta$ goes to zero. To this end, we observe that $||\mathbf{v} \cdot \nabla_{\mathbf{y}} F_3||_{L^2(Y \times V)} \leq C$ due to the smoothness of the coefficients of $Q[\mathbf{y}]$ (see Proposition 7.3.1). By Proposition 7.3.1 $||\tilde{R}^\eta||_{L^2(Y \times V)} \leq C\eta^2 ||\mathbf{v} \cdot \nabla_{\mathbf{y}} F_3||_{L^2(Y \times V)}$, and so

$$||R^\eta||_{L^2(Y \times V)} \leq C||F_2||_{L^2(Y \times V)} + \eta ||F_3||_{L^2(Y \times V)} + \frac{1}{\eta^2} ||\tilde{R}^\eta||_{L^2(Y \times V)} \leq C \ .$$

This concludes the proof of Proposition 7.3.2. □

### 7.3.3 Expansion of the auxiliary function $\mathbf{X}^{\eta *}$

As with $F^\eta$, we need to expand $\mathbf{X}^{\eta *}$ (the solution to (7.16)) with respect to $\eta$. This will be substantially more technical and will involve more compatibility conditions. The largest term in the expansion turns out to be of the order $1/\eta$. Therefore, we write

$$\mathbf{X}^{\eta *} = \sum_{i=-1}^{+\infty} \eta^i \mathbf{X}^*_{[i]}(\mathbf{y}, \mathbf{v}) \ .$$

The above series does not converge, but truncating it at an arbitrary order (say $I$) we obtain a remainder of the subsequent order ($\eta^{I+1}$). The final choice of the index $I$ will be guided by a Sobolev Imbedding Lemma, and chosen larger for larger dimensions. The equations that need to be satisfied by the successive terms are

$$\left\{ \begin{array}{lcl} Q^*(\mathbf{X}^*_{[-1]}) & = & 0, \\ Q^*(\mathbf{X}^*_{[0]}) & = & -\mathbf{v} - (\mathbf{v} \cdot \nabla_{\mathbf{y}}) \mathbf{X}^*_{[-1]}, \\ Q^*(\mathbf{X}^*_{[i]}) & = & -(\mathbf{v} \cdot \nabla_{\mathbf{y}}) \mathbf{X}^*_{[i-1]}, \quad i = 1, 2, \cdots \ , \end{array} \right. \tag{7.39}$$

with $Q^*$ being shorthand for $Q^*[\mathbf{y}]$. Furthermore $\int X^*_{[i]} d\mathbf{y} d\nu(\mathbf{v}) = 0$ , $i = -1, 0, \ldots$ .

*1. Step* $[-1]$ *Determination of* $X^*_{[-1]}$.

Since the kernel of $Q^*$ is the set of constant functions in $\mathbf{v}$, the first equation of (7.39) implies that

$$\mathbf{X}^*_{[-1]} = \mathbf{\Theta}^{[-1]}(\mathbf{y}) \ ,$$

where $\mathbf{\Theta}^{[-1]}$ is a vector valued function to be defined. The second equation defines $\mathbf{X}^*_{[0]}$ if the right hand side is in the range of $Q^*$, which means that it should be orthogonal to $M$ in $L^2_V$. In other words

$$-\int_V (\mathbf{v} + (\mathbf{v}\cdot\nabla_{\mathbf{y}})\mathbf{\Theta}^{[-1]}(\mathbf{y}))M(\mathbf{y},\mathbf{v})\,d\mathbf{v} = 0\ .$$

This identity is satisfied since $M$ is even with respect to $\mathbf{v}$. We therefore have

$$\mathbf{X}^*_{[0]}(\mathbf{y},\mathbf{v}) = Q^{*-1}(-\mathbf{v} - (\mathbf{v}\cdot\nabla_{\mathbf{y}})\mathbf{\Theta}^{[-1]}(\mathbf{y})) + \mathbf{\Theta}^{[0]}(\mathbf{y})\ ,$$

where $\mathbf{\Theta}^{[0]}(\mathbf{y})$ belongs to the kernel of $Q^*$, and will be precisely defined later. We introduce $\mathbf{X}^* = -Q^{*-1}(\mathbf{v})$, in other words

$$Q^*(\mathbf{X}^*) = -\mathbf{v}\ , \quad \int_V \mathbf{X}^* d\nu(\mathbf{v}) = 0\ .$$

With this definition

$$\mathbf{X}^*_{[0]}(\mathbf{y},\mathbf{v}) = \mathbf{X}^* + (\mathbf{X}^*\cdot\nabla_{\mathbf{y}})\mathbf{\Theta}^{[-1]}(\mathbf{y}) + \mathbf{\Theta}^{[0]}(\mathbf{y})\ ,$$

where the functions $\mathbf{\Theta}^{[-1]}$ and $\mathbf{\Theta}^{[0]}$ still need to be determined. We also define

$$\overline{\mathbf{X}}^*_{[0]} = \mathbf{X}^* + (\mathbf{X}^*\cdot\nabla_{\mathbf{y}})\mathbf{\Theta}^{[-1]}(\mathbf{y})\ , \tag{7.40}$$

so that

$$\mathbf{X}^*_{[0]}(\mathbf{y},\mathbf{v}) = \overline{\mathbf{X}}^*_{[0]}(\mathbf{y},\mathbf{v}) + \mathbf{\Theta}^{[0]}(\mathbf{y})\ . \tag{7.41}$$

We proceed to consider $\mathbf{X}^*_{[1]}$. Since $Q^*(\mathbf{X}^*_{[1]}) = -(\mathbf{v}\cdot\nabla_{\mathbf{y}})\mathbf{X}^*_{[0]}$, a necessary and sufficient condition for the existence of $\mathbf{X}^*_{[1]}$ is that $\int M(\mathbf{y},\mathbf{v})(\mathbf{v}\cdot\nabla_{\mathbf{y}})\mathbf{X}^*_{[0]}(\mathbf{y},\mathbf{v})\,d\mathbf{v} = 0$, which is equivalent to $\int M(\mathbf{y},\mathbf{v})(\mathbf{v}\cdot\nabla_{\mathbf{y}})\overline{\mathbf{X}}^*_{[0]}(\mathbf{y},\mathbf{v})\,d\mathbf{v} = 0$. Replacing $\overline{\mathbf{X}}^*_{[0]}$ by the right hand side of (7.40), we obtain the following elliptic equation for $\mathbf{\Theta}^{[-1]}(\mathbf{y}))$ :

$$L^*(\mathbf{\Theta}^{[-1]}) = S^*_{[-1]}\ , \quad \int \mathbf{\Theta}^{[-1]}(\mathbf{y})\ d\mathbf{y} = 0\ . \tag{7.42}$$

Here $L^*$ is the operator $-\mathrm{d}iv_{\mathbf{y}}(D^\top(y)\nabla_{\mathbf{y}}) - (U(y)\cdot\nabla_{\mathbf{y}})$, defined in (7.20) (with periodic boundary conditions), and

$$S^*_{[-1]}(\mathbf{y}) = \int_V M(\mathbf{y},\mathbf{v})(\mathbf{v}\cdot\nabla_{\mathbf{y}})\,\mathbf{X}^*(\mathbf{y},\mathbf{v})d\mathbf{v}\ . \tag{7.43}$$

Proposition 7.3.1 item 4 implies the existence and uniqueness of $\mathbf{\Theta}^{[-1]} = L^{*-1}(S^*_{[-1]})$. Indeed, $\mathbf{X}^*$ is even with respect to $\mathbf{y}$ and odd with respect to $\mathbf{v}$. Therefore $S^*_{[-1]}(\mathbf{y})$ is odd, which implies that $\int_Y S^*_{[-1]}(\mathbf{y})\rho(\mathbf{y})\,d\mathbf{y} = 0$ , the function $\rho$ being even. As a matter of fact, the function $\mathbf{\Theta}^{[-1]}$ is odd with respect to $\mathbf{y}$. To summarize the information collected at this stage

$$\begin{cases}
\mathbf{\Theta}^{[-1]}(\mathbf{y}) = L^{*-1}(S^*_{[-1]})\ , & S^*_{[-1]}(\mathbf{y}) = \int M(\mathbf{y},\mathbf{v})(\mathbf{v}\cdot\nabla_{\mathbf{y}})\,\mathbf{X}^*(\mathbf{y},\mathbf{v})d\mathbf{v}\ , \\
\mathbf{X}^*_{[0]} = \overline{\mathbf{X}}^*_{[0]}(\mathbf{y},\mathbf{v})) + \mathbf{\Theta}^{[0]}(\mathbf{y})\ , & \overline{\mathbf{X}}^*_{[0]} = \mathbf{X}^* + (\mathbf{X}^*\cdot\nabla_{\mathbf{y}})\mathbf{\Theta}^{[-1]}(\mathbf{y})\ , \\
\mathbf{X}^*_{[1]} = \overline{\mathbf{X}}^*_{[1]}(\mathbf{y},\mathbf{v})) + \mathbf{\Theta}^{[1]}(\mathbf{y})\ , & \overline{\mathbf{X}}^*_{[1]}(\mathbf{y},\mathbf{v}) = -Q^{*-1}((\mathbf{v}\cdot\nabla_{\mathbf{y}})\mathbf{X}^*_{[0]})\ , \\
S^*_{[-1]}(-\mathbf{y}) = -S^*_{[-1]}(\mathbf{y})\ , & \overline{\mathbf{X}}^*_{[0]}(-\mathbf{y},\mathbf{v}) = \overline{\mathbf{X}}^*_{[0]}(\mathbf{y},\mathbf{v}),\quad \overline{\mathbf{X}}^*_{[0]}(\mathbf{y},-\mathbf{v}) = -\overline{\mathbf{X}}^*_{[0]}(\mathbf{y},\mathbf{v})\ .
\end{cases}$$

The functions $\mathbf{\Theta}^{[0]}$ and $\mathbf{\Theta}^{[1]}$ are not determined yet. The equation that determines $\mathbf{\Theta}^{[-1]}(\mathbf{y})$ is the compatibity condition that ensures the existence of $\overline{\mathbf{X}}^*_{[1]}(\mathbf{y},\mathbf{v})$.

*2. Step $[i]$ : Determination of $\mathbf{X}^*_{[i]}$ , $i \geq 0$.*

We now describe how to iteratively determine the functions $\mathbf{X}^*_{[i]}$. Suppose at the end of step $[i-1]$ we

already know that

$$\left\{\begin{array}{lcll} \mathbf{X}^*_{[i-1]}(\mathbf{y},\mathbf{v}) & = & \overline{\mathbf{X}}^*_{[i-1]}(\mathbf{y},\mathbf{v}) + \mathbf{\Theta}^{[i-1]}(\mathbf{y}) \ , & \mathbf{\Theta}^{[i-1]}(\mathbf{y}) = L^{*-1}(S^*_{[i-1]}) \text{ is odd w.r.t. } \mathbf{y} \ , \\ \mathbf{X}^*_{[i]} & = & \overline{\mathbf{X}}^*_{[i]}(\mathbf{y},\mathbf{v})) + \mathbf{\Theta}^{[i]}(\mathbf{y}) \ , & \overline{\mathbf{X}}^*_{[i]} \text{ is odd w.r.t. } (\mathbf{y},\mathbf{v}) \text{ , i.e. , } \mathbf{X}^*_{[i]}(-\mathbf{y},-\mathbf{v}) = -\mathbf{X}^*_{[i]}(\mathbf{y},\mathbf{v}) \ , \\ \mathbf{X}^*_{[i+1]} & = & \overline{\mathbf{X}}^*_{[i+1]}(\mathbf{y},\mathbf{v})) + \mathbf{\Theta}^{[i+1]}(\mathbf{y}) \ , & \overline{\mathbf{X}}^*_{[i+1]}(\mathbf{y},\mathbf{v}) = -Q^{*-1}((\mathbf{v}\cdot\nabla_{\mathbf{y}})\mathbf{X}^*_{[i]}) \ . \end{array}\right. \tag{7.44}$$

For $i = 0$ this is satisfied with $\overline{\mathbf{X}}^*_{[-1]} = 0$, $S^*_{[-1]}(\mathbf{y}) = \int M(\mathbf{y},\mathbf{v})(\mathbf{v}\cdot\nabla_{\mathbf{y}})\,\mathbf{X}^*(\mathbf{y},\mathbf{v})d\mathbf{v}$ and $\overline{\mathbf{X}}^*_{[0]} = X^* + (X^*\cdot\nabla_{\mathbf{y}})\mathbf{\Theta}^{[-1]}(\mathbf{y})$ according to step [-1]. At this stage $\mathbf{\Theta}^{[i-1]}$, $S^*_{[i-1]}$, $\overline{\mathbf{X}}^*_{[i-1]}$ and $\overline{\mathbf{X}}^*_{[i]}$ are known, but $\mathbf{\Theta}^{[i]}$ and $\mathbf{\Theta}^{[i+1]}$ are not yet determined. We advance one step and look for $\mathbf{X}^*_{[i+2]}$, a solution to

$$Q^*(\mathbf{X}^*_{[i+2]}) = -(\mathbf{v}\cdot\nabla_{\mathbf{y}})\mathbf{X}^*_{[i+1]} \ .$$

This equation has a solution if and only if $\int_V M(\mathbf{y},\mathbf{v})(\mathbf{v}\cdot\nabla_{\mathbf{y}})\mathbf{X}^*_{[i+1]}\,d\nu(\mathbf{v})$ vanishes, which is equivalent to

$$\int_V M(\mathbf{y},\mathbf{v})(\mathbf{v}\cdot\nabla_{\mathbf{y}})\overline{\mathbf{X}}^*_{[i+1]}\,d\nu(\mathbf{v}) = 0 \ . \tag{7.45}$$

As a consequence

$$\mathbf{X}^*_{[i+2]} = \overline{\mathbf{X}}^*_{[i+2]}(\mathbf{y},\mathbf{v})) + \mathbf{\Theta}^{[i+2]}(\mathbf{y}) \ , \qquad \overline{\mathbf{X}}^*_{[i+2]}(\mathbf{y},\mathbf{v}) = -Q^{*-1}((\mathbf{v}\cdot\nabla_{\mathbf{y}})\mathbf{X}^*_{[i+1]}) \ ,$$

where $\mathbf{\Theta}^{[i+2]}(\mathbf{y})$ is to be determined. Using the formula

$$\overline{\mathbf{X}}^*_{[i+1]}(\mathbf{y},\mathbf{v}) = -Q^{*-1}((\mathbf{v}\cdot\nabla_{\mathbf{y}})\mathbf{X}^*_{[i]}) = -Q^{*-1}((\mathbf{v}\cdot\nabla_{\mathbf{y}})(\overline{\mathbf{X}}^*_{[i]}(\mathbf{y},\mathbf{v}) + \mathbf{\Theta}^{[i]}(\mathbf{y}))) \ ,$$

the constraint (7.45) becomes

$$L^*(\mathbf{\Theta}^{[i]}) = S^*_{[i]} \ , \quad S^*_{[i]}(\mathbf{y}) = -\int M(\mathbf{y},\mathbf{v})(\mathbf{v}\cdot\nabla_{\mathbf{y}})Q^{*-1}((\mathbf{v}\cdot\nabla_{\mathbf{y}})(\overline{\mathbf{X}}^*_{[i]}(\mathbf{y},\mathbf{v})))\,d\nu(\mathbf{v}) \ .$$

For $\mathbf{\Theta}^{[i]}$ to exist, one needs that $\int S^*_{[i]}(\mathbf{y})\rho(\mathbf{y})\,d\mathbf{y} = 0$. Since $\overline{\mathbf{X}}^*_{[i]}(\mathbf{y},\mathbf{v})$ is odd with respect to $(\mathbf{y},\mathbf{v})$, and since this property is preserved under the action of the operators $\mathbf{v}\cdot\nabla_{\mathbf{y}}$ and $Q^{*-1}$, the integrated function $S^*_{[i]}(\mathbf{y})$ is odd with respect to $\mathbf{y}$. As $\rho(\mathbf{y})$ is even, we deduce that $\int S^*_{[i]}(\mathbf{y})\rho(\mathbf{y})\,d\mathbf{y} = 0$ . In summary

$$\left\{\begin{array}{lcll} S^*_{[i]}(\mathbf{y}) & = & -\int M(\mathbf{y},\mathbf{v})(\mathbf{v}\cdot\nabla_{\mathbf{y}})Q^{*-1}((\mathbf{v}\cdot\nabla_{\mathbf{y}})(\overline{\mathbf{X}}^*_{[i]}(\mathbf{y},\mathbf{v}))\,d\nu(\mathbf{v}) \ , & \\ \mathbf{X}^*_{[i]}(\mathbf{y},\mathbf{v}) & = & \overline{\mathbf{X}}^*_{[i]}(\mathbf{y},\mathbf{v}) + \mathbf{\Theta}^{[i]}(\mathbf{y}) \ , & \mathbf{\Theta}^{[i]}(\mathbf{y}) = L^{*-1}(S^*_{[i]}) \text{ is odd w.r.t. } \mathbf{y} \ , \\ \mathbf{X}^*_{[i+1]} & = & \overline{\mathbf{X}}^*_{[i+1]}(\mathbf{y},\mathbf{v})) + \mathbf{\Theta}^{[i+1]}(\mathbf{y}) \ , & \overline{\mathbf{X}}^*_{[i+1]} = -Q^{*-1}((\mathbf{v}\cdot\nabla_{\mathbf{y}})\mathbf{X}^*_{[i]}) \ , \\ \mathbf{X}^*_{[i+2]} & = & \overline{\mathbf{X}}^*_{[i+2]}(\mathbf{y},\mathbf{v})) + \mathbf{\Theta}^{[i+2]}(\mathbf{y}) \ , & \overline{\mathbf{X}}^*_{[i+2]}(\mathbf{y},\mathbf{v}) = -Q^{*-1}((\mathbf{v}\cdot\nabla_{\mathbf{y}})\mathbf{X}^*_{[i+1]}) \ . \end{array}\right. \tag{7.46}$$

The function $\overline{\mathbf{X}}^*_{[i+1]}$ is an odd function with respect to $(\mathbf{y},\mathbf{v})$. To see this, we note that

$$\overline{\mathbf{X}}^*_{[i+1]} = -Q^{*-1}((\mathbf{v}\cdot\nabla_{\mathbf{y}})\mathbf{X}^*_{[i]}) = -Q^{*-1}((\mathbf{v}\cdot\nabla_{\mathbf{y}})\overline{\mathbf{X}}^*_{[i]}) - Q^{*-1}((\mathbf{v}\cdot\nabla_{\mathbf{y}})\mathbf{\Theta}^{[i]}(\mathbf{y})).$$

We also note that the arguments of $Q^{*-1}$ in the very right hand side of the above formula are odd functions in the variable $(\mathbf{y},\mathbf{v})$, and that this property is preserved by the operator $Q^{*-1}$. At this point we have derived the equivalent of (7.44) with $i-1$ replaced by $i$, and so the induction step is complete. The formulas (7.46) therefore determine $\mathbf{X}^*_{[i]}$ iteratively for all $i \geq 0$.

We are now in a position to prove the following proposition

*Proposition.* Under the Hypotheses **(H0)**, **(H1)**, and **(H2)** the auxiliary function $\mathbf{X}^{\eta*}$, the solution of (7.16), has the asymptotic expansion

$$\mathbf{X}^{\eta*}(\mathbf{y},\mathbf{v}) = \sum_{i=-1}^{I-1} \eta^i \mathbf{X}^*_{[i]}(\mathbf{y},\mathbf{v}) \ + R_I(\mathbf{y},\mathbf{v}) \ ,$$

for any $I \geq 0$. The remainder $R_I$ satisfies

$$\|R_I\|_{L^2(Y\times V)} \leq C\eta^I \quad .$$

The terms of the expansion are uniquely defined by

$$\mathbf{X}^*_{[i]}(\mathbf{y},\mathbf{v}) = \overline{\mathbf{X}}^*_{[i]}(\mathbf{y},\mathbf{v}) + \mathbf{\Theta}^{[i]}(\mathbf{y}) \ , \qquad \mathbf{\Theta}^{[i]}(\mathbf{y}) = L^{*-1}(S^*_{[i]}) \ ,$$

with

$$\overline{\mathbf{X}}^*_{[-1]}(\mathbf{y},\mathbf{v}) = 0 \ , \qquad S^*_{[-1]}(\mathbf{y}) = \int_V M(\mathbf{y},\mathbf{v})(\mathbf{v}\cdot\nabla_{\mathbf{y}})\,\mathbf{X}^*(\mathbf{y},\mathbf{v})d\mathbf{v} \ ,$$

$$\overline{\mathbf{X}}^*_{[0]} = \mathbf{X}^* + (\mathbf{X}^*\cdot\nabla_{\mathbf{y}})\mathbf{\Theta}^{[-1]}(\mathbf{y}) \ , \qquad S^*_{[0]}(\mathbf{y}) = -\int_V M(\mathbf{y},\mathbf{v})(\mathbf{v}\cdot\nabla_{\mathbf{y}})Q^{*-1}((\mathbf{v}\cdot\nabla_{\mathbf{y}})(\overline{\mathbf{X}}^*_{[0]}(\mathbf{y},\mathbf{v})))\,d\nu(\mathbf{v}) \ ,$$

and

$$\overline{\mathbf{X}}^*_{[i]} = -Q^{*-1}((\mathbf{v}\cdot\nabla_{\mathbf{y}})\mathbf{X}^*_{[i-1]}) \ , \qquad S^*_{[i]}(\mathbf{y}) = -\int_V M(\mathbf{y},\mathbf{v})(\mathbf{v}\cdot\nabla_{\mathbf{y}})Q^{*-1}((\mathbf{v}\cdot\nabla_{\mathbf{y}})(\overline{\mathbf{X}}^*_{[i]}(\mathbf{y},\mathbf{v})))\,d\nu(\mathbf{v}) \ ,$$

for $i = 1, 2, \cdots$. Here $\mathbf{X}^*$ denotes the function $\mathbf{X}^* = -Q^{*-1}(\mathbf{v})$. All the terms $\mathbf{X}^*_{[i]}(\mathbf{y},\mathbf{v})$ of the expansion belong to $L^\infty(V; C^\infty_{per}(\mathbf{R}^d_{\mathbf{y}}))$.

*Démonstration.* First, the regularity of the terms in the expansion is a direct consequence of the cross section regularity (see Hypothesis **(H0)**). Let now $I$ be an arbitrary integer, and write

$$\mathbf{X}^{\eta*} = \sum_{i=-1}^{I+1} \eta^i \mathbf{X}^*_{[i]}(\mathbf{y},\mathbf{v}) + \tilde{R}_I.$$

If we prove that $\|\tilde{R}_I\|_{L^2} = O(\eta^I)$ this would lead us to the estimate

$$\mathbf{X}^{\eta*} = \sum_{i=-1}^{I-1} \eta^i \mathbf{X}^*_{[i]}(\mathbf{y},\mathbf{v}) + (\eta^I \mathbf{X}^*_{[I]} + \eta^{I+1}\mathbf{X}^*_{[I+1]} + \tilde{R}_I) = \sum_{i=-1}^{I-1} \eta^i \mathbf{X}^*_{[i]}(\mathbf{y},\mathbf{v}) + O(\eta^I) \ ,$$

as desired. To this end, the equation satisfied by $\tilde{R}^I$ reads

$$T^{\eta*}(\tilde{R}_I) = \eta^{I+2}\mathbf{v}\cdot\nabla_{\mathbf{y}}\mathbf{X}^*_{[I+1]} \ .$$

By applying Proposition 7.3.1 Item 6. (see (7.31)), we obtain the inequality $\|\tilde{R}_I\|_{L^2} \leq C\eta^I\|(\mathbf{v}\cdot\nabla_{\mathbf{y}})\mathbf{X}^*_{[I+1]}\|_{L^2}$ , and the proof is complete. □

*Corollary* 7.3.1. The asymptotic expansion $\mathbf{X}^{\eta*}(\mathbf{y},\mathbf{v}) = \sum_{i=-1}^{+\infty} \eta^i \mathbf{X}^*_{[i]}(\mathbf{y},\mathbf{v})$ is valid in $L^\infty_{per}(\mathbf{R}^d_{\mathbf{y}}\times V)$, *i.e.*,

$$\|\mathbf{X}^{\eta*} - \sum_{i=-1}^{I-1} \eta^i \mathbf{X}^*_{[i]}\|_{L^\infty(Y\times V)} = \|R_I\|_{L^\infty(Y\times V)} \leq C\eta^I \quad .$$

In particular

$$\mathbf{X}^{\eta*} = \frac{1}{\eta}\mathbf{\Theta}^{[-1]}(\mathbf{y}) + \mathbf{X}^*_{[0]} + O(\eta) = \frac{1}{\eta}\mathbf{\Theta}^{[-1]}(\mathbf{y}) + \mathbf{X}^* + (\mathbf{X}^*\cdot\nabla_{\mathbf{y}})\mathbf{\Theta}^{[-1]}(\mathbf{y}) + \mathbf{\Theta}^{[0]}(\mathbf{y}) + O(\eta) \ ,$$

with $O(\eta)$ bounded by $C\eta$ uniformly in $Y\times V$.

*Démonstration.* As in the proof of Proposition 7.3.3 we consider an arbirary integer $I$, and write

$$\mathbf{X}^{\eta *} = \sum_{i=-1}^{I+1} \eta^i \mathbf{X}^*_{[i]}(\mathbf{y}, \mathbf{v}) + \tilde{R}_I \ , \quad \text{with} \quad T^{\eta *}(\tilde{R}_I) = \eta^{I+2} \mathbf{v} \cdot \nabla_{\mathbf{y}} \mathbf{X}^*_{[I+1]} \ .$$

We already know that $\|\tilde{R}^I\|_{L^2} = O(\eta^I)$, and that $\mathbf{X}^*_{[I+1]}$ is in $L^\infty(V, C^\infty_{per}(Y))$. Differentiating the equation for $\tilde{R}_I$ with respect to $y_i$, we obtain

$$T^{\eta *}(\partial_{y_i} \tilde{R}_I) = \partial_{y_i} Q^*(\tilde{R}_I) + \eta^{I+2} \mathbf{v} \cdot \nabla_{\mathbf{y}} \partial_{y_i} \mathbf{X}^*_{[I+1]},$$

where $\partial_{y_i} Q^*$ has the same expression as $Q^*$, with $\sigma$ replaced by $\partial_{y_i} \sigma$. Differentiating once more, we obtain

$$T^{\eta *}(\partial^2_{y_i y_j} \tilde{R}_I) = \partial^2_{y_i y_j} Q^*(\tilde{R}_I) + \partial_{y_j} Q^*(\partial_{y_i} \tilde{R}_I) + \partial_{y_i} Q^*(\partial_{y_j} \tilde{R}_I) + \eta^{I+2} (\mathbf{v} \cdot \nabla_{\mathbf{y}}) \partial_{y_i y_j} \mathbf{X}^*_{[I+1]} \ ,$$

where $\partial^2_{y_i y_j} Q^*$ are defined analogously to $\partial_{y_i} Q^*$ . Proposition 7.3.1 leads to the following estimates for the remainder and its derivatives up to second order
(i) $||\tilde{R}_I||_{L^2} \leq C \left( \eta^I ||(\mathbf{v} \cdot \nabla_{\mathbf{y}}) \mathbf{X}^*_{[I+1]}||_{L^2} \right)$,

(ii) $||\partial_{y_i} \tilde{R}_I||_{L^2} \leq C \left( \frac{1}{\eta^2} ||\tilde{R}_I||_{L^2(Y \times V)} + \eta^I ||(\mathbf{v} \cdot \nabla_{\mathbf{y}}) \partial_{y_i} \mathbf{X}^*_{[I+1]}||_{L^2} \right) \leq C_1 \eta^{I-2} \|\mathbf{X}^*_{[I+1]}\|_{L^2(V, H^2(Y))}$ ,

(iii) $||\partial_{y_i, y_j} \tilde{R}_I||_{L^2} \leq C \left( \frac{1}{\eta^2} (||\tilde{R}_I||_{L^2} + ||\nabla_{\mathbf{y}} \tilde{R}_I||_{L^2}) + \eta^I ||(\mathbf{v} \cdot \nabla_{\mathbf{y}}) \partial_{y_i y_i} \mathbf{X}^*_{[I+1]}||_{L^2} \right) \leq C_2 \eta^{I-4} \|\mathbf{X}^*_{[I+1]}\|_{L^2(V, H^3(Y))}$ . Proceeding by induction, we obtain for the $\mathbf{k}^{th}$ derivative (with $\mathbf{k} = (k_1, k_2, \cdots, k_d)$)

$$||\partial^{\mathbf{k}}_{\mathbf{y}} \tilde{R}_I||_{L^2} \leq C_{\mathbf{k}} \eta^{I-2|k|} \|\mathbf{X}^*_{[I+1]}\|_{L^2(V, H^{|k|+1}(Y))} \ .$$

Let $m > \frac{d}{2}$, so that $H^m_{per}(\mathbf{R}^d_{\mathbf{y}}) \subset L^\infty_{per}(\mathbf{R}^d_{\mathbf{y}})$, and let $I = 3m$ and $|k| = 0, 1, 2, \cdots, m$. The above estimate gives that

$$\|\tilde{R}_{3m}\|_{L^2(V, L^\infty(Y))} \leq C \|\tilde{R}_{3m}\|_{L^2(V, H^m(Y))} \leq C \eta^m \|\mathbf{X}^*_{[3m+1]}\|_{L^2(V, H^{m+1}(Y))} \leq C_m \eta^m \ .$$

Since $m$ is arbitrary, this implies that the asymptotic expansion holds true in $L^2(V; L^\infty_{per}(\mathbf{R}^d_{\mathbf{y}}))$. In order to see that this expansion is also asymptotic in $L^\infty_{per}(\mathbf{R}^d_{\mathbf{y}} \times V)$, we write

$$-\eta \mathbf{v} \cdot \nabla_{\mathbf{y}} \tilde{R}_{3m} + \Sigma \tilde{R}_{3m} = \int_V \sigma(\mathbf{y}, \mathbf{v}, \mathbf{v}') \tilde{R}_{3m}(\mathbf{y}, \mathbf{v}') d\nu(\mathbf{v}') + \eta^{3m+2} \mathbf{v} \cdot \nabla_{\mathbf{y}} \mathbf{X}^*_{[3m+1]} \ ,$$

where $\Sigma(\mathbf{y}, \mathbf{v}) = \int \sigma(\mathbf{y}, \mathbf{v}, \mathbf{v}') \, d\nu(\mathbf{v}')$ . The right hand side is an $O(\eta^I)$ in $L^\infty_{per}(\mathbf{R}^d_{\mathbf{y}}, V)$ (since $R_{3m}$ is integrated in velocity). As $\Sigma$ is bounded from below by a positive constant, this implies by inversion of the operator $-\eta \mathbf{v} \cdot \nabla_{\mathbf{y}} + \Sigma$ that $\tilde{R}_{3m}$ is bounded by $O(\eta^m)$ in $L^\infty_{per}(\mathbf{R}^d_{\mathbf{y}}, V)$. [4] □

## 7.4 Study of the limiting process

We consider the full equation (7.3) with all three parameters $\varepsilon$, $\eta$ and $\alpha = \varepsilon/\eta$ going to zero. The main result is a convergence result for $f^{\varepsilon, \eta}$ in a sense of two-scale convergence [1],[30], the two scales being 1 and $\alpha = \frac{\varepsilon}{\eta}$.

### 7.4.1 Existence of a two-scale limit

Our notion of two-scale convergence for a sequence of positive measures is as follows.

4. The $L^\infty$ boundedness of the operator $(-\eta \mathbf{v} \cdot \nabla_{\mathbf{y}} + \Sigma)^{-1}$ follows immediately from the formula $(-\eta \mathbf{v} \cdot \nabla_{\mathbf{y}} + \Sigma)^{-1} h = \eta^{-1} \int_0^\infty \exp\left( -\frac{1}{\eta} \int_0^\sigma \Sigma(\mathbf{y} + \tau \mathbf{v}, \mathbf{v}) \, d\tau \right) h(\mathbf{y} + \sigma \mathbf{v}, \mathbf{v}) \, d\sigma$ .

**Définition. Two-scale convergence.** Let $\alpha_k$ be a sequence with the property that $\alpha_k \to 0$ as $k \to \infty$. A sequence of positive, regular, Borel measures $\mu^k$ is said to two-scale converge (at the scales 1 and $\alpha_k$) towards $\mu$ if and only if : for any test function $\phi \in \mathcal{C}^0_{c,per}([0,T], \mathbf{R}^d_{\mathbf{x}}, \mathbf{R}^d_{\mathbf{y}}, V)$ we have

$$\lim_{k\to\infty} \int_{\mathbf{R}^+}\int_{\mathbf{R}^d}\int_V \phi(t,\mathbf{x},\frac{\mathbf{x}}{\alpha_k},\mathbf{v})\mu^k(dt,d\mathbf{x},d\nu(\mathbf{v})) = \int_{\mathbf{R}^+}\int_{\mathbf{R}^d}\int_Y\int_V \phi(t,\mathbf{x},y,\mathbf{v})\mu(dt,dx,d\mathbf{y},d\nu(\mathbf{v})) \ .$$

$\mathcal{C}^0_{c,per}(\mathbf{R}^+, \mathbf{R}^d_{\mathbf{x}}, \mathbf{R}^d_{\mathbf{y}}, V)$ denotes the set of continuous functions on $\mathbf{R}^+ \times \mathbf{R}^d \times \mathbf{R}^d \times V$ that are compactly supported in $t$ and $\mathbf{x}$, and periodic in $\mathbf{y}$.

*Remarque* 7.4.1. By inserting test functions of the form $\phi(t,\mathbf{x},\frac{\eta\mathbf{x}}{\varepsilon},\mathbf{v})$ in (7.5), we obtain

$$\int_{\mathbf{R}^d\times V} f_{ini}(\mathbf{x},\mathbf{v})\phi(0,\mathbf{x},\frac{\eta\mathbf{x}}{\varepsilon},\mathbf{v})\,d\mathbf{x}\,d\nu(\mathbf{v}) + \int_{\mathbf{R}^+}\int_{\mathbf{R}^d\times V} f^{\varepsilon,\eta}(\partial_t\phi + \frac{\mathbf{v}\cdot\nabla_{\mathbf{x}}\phi}{\varepsilon} + \frac{\eta\mathbf{v}\cdot\nabla_{\mathbf{y}}\phi}{\varepsilon^2} + \frac{Q^*[\mathbf{y}](\phi)}{\varepsilon^2})|_{\mathbf{y}=\frac{\eta\mathbf{x}}{\varepsilon}}\,d\mathbf{x}d\nu(\mathbf{v})dt = 0 \ . \quad (7.47)$$

To pass to the limit, we shall apply the energy method introduced by Tartar (see [34] and the other references about homogenization quoted in the introduction). This method here consists in testing the equation against suitably chosen functions, depending on the time and space variables, and also on the fast variable $\frac{\eta\mathbf{x}}{\varepsilon}$. The intermediate results obtained by applying these test functions are summarized in the following proposition.

*Proposition.* (*i*) For any test function $\phi \in \mathcal{C}^1_{c,per}([0,T], \mathbf{R}^d_{\mathbf{x}}, \mathbf{R}^d_{\mathbf{y}}, V)$, *i.e.*, for any function that is periodic in $\mathbf{y}$, $C^1$ in the variables $(t,\mathbf{x},\mathbf{y})$, continuous in $\mathbf{v}$, and has compact support in $t$ and $\mathbf{x}$

$$\lim_{\varepsilon\ll\eta\to 0}\int_{\mathbf{R}^+}\int_{\mathbf{R}^d}\int_V f^{\varepsilon,\eta}(t,\mathbf{x},\mathbf{v})Q^*[\mathbf{y}](\phi(t,\mathbf{x},\mathbf{y},\cdot))|_{\mathbf{y}=\frac{\eta\mathbf{x}}{\varepsilon}}(\mathbf{v})dxd\nu(\mathbf{v})dt = 0 \ , \quad (7.48)$$

$$\lim_{\varepsilon\ll\eta\to 0}\int_{\mathbf{R}^+}\int_{\mathbf{R}^d}\int_V f^{\varepsilon,\eta}(t,\mathbf{x},\mathbf{v})T^{*\eta}(\phi(t,\mathbf{x},\cdot,\cdot))(\mathbf{y},\mathbf{v})|_{\mathbf{y}=\frac{\eta\mathbf{x}}{\varepsilon}}\,dxd\nu(\mathbf{v})dt = 0 \ . \quad (7.49)$$

(*ii*) If $J^{\varepsilon,\eta} = \displaystyle\int_V \mathbf{v} f^{\varepsilon,\eta} d\nu(\mathbf{v})$ then

$$\lim_{\varepsilon\ll\eta\to 0}\int_{\mathbf{R}^+}\int_{\mathbf{R}^d} J^{\varepsilon,\eta}(t,\mathbf{x})\cdot\nabla_{\mathbf{x}}\varphi(t,\mathbf{x})dtdx = 0 \ , \quad (7.50)$$

for any test function $\varphi \in C^0_c([0,T],\mathbf{R}^d)$ .

The notation $\varepsilon << \eta \to 0$ signifies that the limit is obtained along any sequence $\varepsilon$, $\eta$, with $\varepsilon \to 0$, $\eta \to 0$, and $\varepsilon/\eta \to 0$.

*Démonstration.* The first identity of Item 1 is obtained by multiplication of (7.47) by $\varepsilon^2$, and passage to the limit, using the $L^1(\mathbf{R}\times\mathbf{R}^d\times V)$ bound on $f^{\varepsilon,\eta}$. The second identity of Item 1 is a simple consequence of the first. In order to verify Item 2 we consider (7.47) with $\phi(t,\mathbf{x},\mathbf{y},\mathbf{v}) = \varphi(t,\mathbf{x}) + \varepsilon\psi(t,\mathbf{x},\mathbf{y},\mathbf{v})$. After multiplication by $\varepsilon$, this leads to the limit

$$\lim_{\varepsilon\ll\eta\to 0}\int_{\mathbf{R}^+}\int_{\mathbf{R}^d}\int_V f^{\varepsilon,\eta}\left(\mathbf{v}\cdot\nabla\varphi(t,\mathbf{x}) + Q^*[\mathbf{y}](\psi(t,\mathbf{x},\mathbf{y},\cdot))\right)|_{\mathbf{y}=\frac{\eta\mathbf{x}}{\varepsilon}}(\mathbf{v})dtdxd\nu(\mathbf{v}) = 0 \ .$$

As a consequence of Item 1, we get

$$\lim_{\varepsilon\ll\eta\to 0}\int_{\mathbf{R}^+}\int_{\mathbf{R}^d}\int_V f^{\varepsilon,\eta}\left(\mathbf{v}\cdot\nabla_{\mathbf{x}}\varphi\right)dtd\mathbf{x}d\nu(\mathbf{v}) = 0 \ ,$$

which is exactly (7.50). □

*Proposition.* Given a sequence of parameters $\varepsilon$ and $\eta$ with $\varepsilon \to 0$, $\eta \to 0$ and $\frac{\varepsilon}{\eta} \to 0$, let $f^{\varepsilon,\eta}$ be the weak solution to (7.3) and $n^{\varepsilon,\eta} = \int_V f^{\varepsilon,\eta}(t,\mathbf{x},\mathbf{v})d\nu(\mathbf{v})$ . There exists a subsequence $(\varepsilon_k,\eta_k)$, a positive, regular Borel measure $\mathcal{F}(dt,dx,d\mathbf{y},d\nu(\mathbf{v})) \in \mathcal{M}([0,\infty)\times\mathbf{R}^d\times Y\times V)$ and a positive,

regular Borel measure $N(dt,dx,d\mathbf{y}) \in \mathcal{M}([0,\infty)\times\mathbf{R}^d\times Y)$ such that, for any test function $\phi \in \mathcal{C}^0_{c,per}([0,T],\mathbf{R}^d_{\mathbf{x}},\mathbf{R}^d_{\mathbf{y}},V)$ we have

$$\lim_{k\to\infty}\int_{\mathbf{R}^+}\int_{\mathbf{R}^d}\int_V \phi(t,\mathbf{x},\frac{\eta_k\mathbf{x}}{\varepsilon_k},\mathbf{v})f^{\varepsilon_k,\eta_k}(t,\mathbf{x},\mathbf{v})dt,d\mathbf{x},d\nu(\mathbf{v}) = \int_{\mathbf{R}^+}\int_{\mathbf{R}^d}\int_Y\int_V \phi(t,\mathbf{x},y,\mathbf{v})\mathcal{F}(dt,dx,d\mathbf{y},d\nu(\mathbf{v}))\ ,$$

and for any test function $\varphi \in \mathcal{C}^0_{c,per}([0,T],\mathbf{R}^d_{\mathbf{x}},\mathbf{R}^d_{\mathbf{y}})$ we have

$$\lim_{k\to\infty}\int_{\mathbf{R}^+}\int_{\mathbf{R}^d}\varphi(t,\mathbf{x},\frac{\eta_k\mathbf{x}}{\varepsilon_k})n^{\varepsilon_k,\eta_k}(t,\mathbf{x})dt,d\mathbf{x} = \int_{\mathbf{R}^+}\int_{\mathbf{R}^d}\int_Y \varphi(t,\mathbf{x},y)N(dt,dx,d\mathbf{y})\ .$$

*Démonstration.* As the measure $\mu^{\varepsilon,\eta}(dt,dx,d\mathbf{y},d\nu(\mathbf{v})) = f^{\varepsilon,\eta}(t,\mathbf{x},\mathbf{v})\delta(\mathbf{y}-\frac{\eta\mathbf{x}}{\varepsilon})dtdxd\nu(\mathbf{v})$ is bounded in the dual norm on $[C^0([0,T]\times\mathbf{R}^d\times Y\times V)]^*$, it follows from the weak$^*$ compactness of the unit ball that there exists a subsequence $\mu_k = \mu^{\varepsilon_k,\eta_k}$ and a regular Borel measure $\mathcal{F}$ so that $\mu_k \to \mathcal{F}$ weak$^*$ in $[C^0([0,T]\times\mathbf{R}^d\times Y\times V)]^*$. Thus

$$\begin{aligned}\int_{\mathbf{R}^+}\int_{\mathbf{R}^d}\int_V \phi(t,\mathbf{x},\frac{\eta_k\mathbf{x}}{\varepsilon_k},\mathbf{v})f^{\varepsilon_k,\eta_k}(t,\mathbf{x},\mathbf{v})dtd\mathbf{x}d\nu(\mathbf{v}) &= \langle\mu_k,\phi(t,\mathbf{x},\mathbf{y},\mathbf{v})\rangle\\ &\to \int_{\mathbf{R}^+}\int_{\mathbf{R}^d}\int_Y\int_V \phi(t,\mathbf{x},\mathbf{y},\mathbf{v})\mathcal{F}(t,\mathbf{x},\mathbf{y},\mathbf{v})dtd\mathbf{x}d\mathbf{y}d\nu(\mathbf{v}).\end{aligned}$$

The case of $n^{\varepsilon,\eta}$ is handled by similar arguments. □

*Remarque* 7.4.2. Proposition 7.4.1 is a compactness result. It automatically translates into a convergence result once we demonstrate that the limits $\mathcal{F}$ and $N$ are independent of the subsequence. This independence follows from the structure result we prove in the following section.

### 7.4.2 Identification of the two-scale limits. Proof of Theorem 7.1.2

We now provided a structural characterization of the limits $\mathcal{F}$ and $N$.

*Proposition.* (*i*) Given a sequence of parameters $\varepsilon$, $\eta$, with $\varepsilon\to 0$, $\eta\to$, and $\varepsilon/\eta\to 0$, the limiting measures $\mathcal{F}(dt,d\mathbf{x},d\mathbf{y},d\nu(\mathbf{v}))$ and $N(dt,d\mathbf{x},d\mathbf{y})$ from Proposition 7.4.1 satisfy $\mathcal{F}(dt,d\mathbf{x},d\mathbf{y},d\nu(\mathbf{v})) = M(\mathbf{y},\mathbf{v})d\nu(\mathbf{v})N(dt,d\mathbf{x},d\mathbf{y})$, where $M(\mathbf{y},\mathbf{v})$ is the local Maxwellian. $M$ is the unique solution to the cell problem

$$Q[\mathbf{y}](M) = 0\ ,\quad \text{with}\quad \int_V M(\mathbf{y},\mathbf{v})d\nu(\mathbf{v}) = 1\ . \tag{7.51}$$

(*ii*) The particle density $N(dt,d\mathbf{x},d\mathbf{y})$ can be written $N(dt,d\mathbf{x},d\mathbf{y}) = \rho(\mathbf{y})d\mathbf{y}n(dt,d\mathbf{x})$, where $\rho(\mathbf{y})$ is the unique periodic solution to the elliptic problem

$$\begin{aligned}&-\mathrm{div}_{\mathbf{y}}(D(\mathbf{y})\nabla_{\mathbf{y}}\rho(\mathbf{y})) + \mathrm{div}_{\mathbf{y}}(U(\mathbf{y})\rho(\mathbf{y})) = 0\ ,\quad \int_Y\rho(\mathbf{y}) = 1\ ,\\ &\text{with}\quad D(\mathbf{y}) = \int_V \mathbf{X}^*(\mathbf{y},\mathbf{v})\otimes\mathbf{v}M(\mathbf{y},\mathbf{v})d\nu(\mathbf{v}) = \int_V \mathbf{v}\otimes\mathbf{X}(\mathbf{y},\mathbf{v})d\nu(\mathbf{v})\ ,\\ &\text{and}\quad U(\mathbf{y}) = \int_V \mathbf{v}\lambda(\mathbf{y},\mathbf{v})d\nu(\mathbf{v})\ .\end{aligned} \tag{7.52}$$

Here $X^*$, $X$ and $\lambda$ are the solutions to

$$Q^*[\mathbf{y}](X^*) = -\mathbf{v}\ ,\quad Q[\mathbf{y}](X) = -\mathbf{v}M\ ,\quad \text{and}\quad Q[\mathbf{y}](\lambda) = \mathbf{v}\cdot\nabla_{\mathbf{y}}M\ ,$$

respectively.

(*iii*) The macroscopic density $n(dt,d\mathbf{x})$ has the form $n(dt,d\mathbf{x}) = n(t,x)dt\,d\mathbf{x}$ where $n(t,x)$ is the unique solution to the diffusion equation

$$\partial_t n - \mathrm{div}_{\mathbf{x}}(\mathbf{D}\nabla_{\mathbf{x}}n) = 0\quad ,\quad n(t=0,\mathbf{x}) = n_{ini}(\mathbf{x}) = \int_V f_{ini}(\mathbf{x},\mathbf{v})\,d\nu(\mathbf{v})\ , \tag{7.53}$$

$$\text{with} \qquad \mathbf{D} = \int_Y \Big( \rho(\mathbf{y}) D(\mathbf{y}) - \mathbf{\Theta}^{[-1]}(\mathbf{y}) \otimes H(\mathbf{y}) \Big) \, d\mathbf{y} \ . \tag{7.54}$$

$H$ and $\mathbf{\Theta}^{[-1]}$ are defined by

$$H(\mathbf{y}) = \mathrm{div}_{\mathbf{y}}(\rho(\mathbf{y}) D^{\top}(\mathbf{y})) + D(\mathbf{y}) \nabla_{\mathbf{y}} \rho - \rho(\mathbf{y}) U(\mathbf{y}) \ , \quad \mathbf{\Theta}^{[-1]}(\mathbf{y}) = L^{*-1}( \int M(\mathbf{y}, \mathbf{v}) (\mathbf{v} \cdot \nabla_{\mathbf{y}}) \, \mathbf{X}^*(\mathbf{y}, \mathbf{v}) d\nu(\mathbf{v}))$$

*Démonstration.* The three items of this Proposition will be proven by testing the Boltzmann equation against three types of test functions.

*Step 1.* $\mathcal{F}(dt, d\mathbf{x}, d\mathbf{y}, d\mathbf{v}) = N(dt, d\mathbf{x}, d\mathbf{y}) M(\mathbf{y}, \mathbf{v}) d\nu(\mathbf{v})$ .

First, we recall that for any test function $\phi(t, \mathbf{x}, \mathbf{y}, \mathbf{v})$,

$$\lim_{\varepsilon \ll \eta \to 0} \int_{\mathbf{R}^+} \int_{\mathbf{R}^d} \int_V f^{\varepsilon, \eta}(t, \mathbf{x}, \mathbf{v}) Q^*[\mathbf{y}](\phi(t, \mathbf{x}, \mathbf{y}, \cdot))|_{\mathbf{y} = \frac{\eta \mathbf{x}}{\varepsilon}}(\mathbf{v}) d\nu(\mathbf{v}) d\mathbf{x} dt = 0 \ .$$

Let now $\psi(t, \mathbf{x}, \mathbf{y}, \mathbf{v})$ be an arbitrary test function, and write

$$\psi(t, \mathbf{x}\, \mathbf{y}, \mathbf{v}) = \overline{\psi}(t, \mathbf{x}, \mathbf{y}) + [\psi(t, \mathbf{x}, \mathbf{y}, \mathbf{v}) - \overline{\psi}(t, \mathbf{x}, \mathbf{y})] \ , \quad \text{with} \quad \overline{\psi} = \int \psi(t, \mathbf{x}, \mathbf{y}, \mathbf{v}) M(\mathbf{y}, \mathbf{v}) \, d\nu(\mathbf{v}) \ .$$

We note that the second term $\psi(t, \mathbf{x}, \mathbf{y}, \mathbf{v}) - \overline{\psi}(t, \mathbf{x}, \mathbf{y})$ lies in $Im(Q^*)$ since

$$\int M(\mathbf{y}, \mathbf{v}) [\psi(t, \mathbf{x}\, \mathbf{y}, \mathbf{v}) - \overline{\psi}(t, \mathbf{x}, \mathbf{y})] \, d\nu(\mathbf{v}) = 0 \ .$$

Therefore, along the subsequence $\varepsilon_k$, $\eta_k$ from Proposition 7.4.1

$$\int_{\mathbf{R}^+} \int_{\mathbf{R}^d} \int_Y \int_V \psi \mathcal{F}(dt, d\mathbf{x}, d\mathbf{y}, d\nu(\mathbf{v})) \quad = \quad \lim_{k \to \infty} \int_{\mathbf{R}^+} \int_{\mathbf{R}^d} \int_V f^{\varepsilon_k, \eta_k}(t, \mathbf{x}, \mathbf{v}) \psi(t, \mathbf{x}, \frac{\eta_k \mathbf{x}}{\varepsilon_k}, \mathbf{v}) d\nu(\mathbf{v}) d\mathbf{x} dt \ .$$

By 7.48, and the fact that $\psi - \overline{\psi} = Q^*[\mathbf{y}](\phi)$ we get

$$\begin{aligned}
\int_{\mathbf{R}^+} \int_{\mathbf{R}^d} \int_Y \int_V \psi \mathcal{F} d\nu(\mathbf{v}) d\mathbf{y} dx dt \quad &= \quad \lim_{k \to \infty} \int_{\mathbf{R}^+} \int_{\mathbf{R}^d} \int_V f^{\varepsilon_k, \eta_k}(t, \mathbf{x}, \mathbf{v}) \overline{\psi}(t, \mathbf{x}, \frac{\eta_k \mathbf{x}}{\varepsilon_k}) d\nu(\mathbf{v}) d\mathbf{x} dt \\
&= \quad \lim_{k \to \infty} \int_{\mathbf{R}^+} \int_{\mathbf{R}^d} n^{\varepsilon_k, \eta_k}(t, \mathbf{x}) \overline{\psi}(t, \mathbf{x}, \frac{\eta_k \mathbf{x}}{\varepsilon_k}) d\mathbf{x} dt \\
&= \quad \int_{\mathbf{R}^+} \int_{\mathbf{R}^d} \int_Y \overline{\psi}(t, \mathbf{x}, \mathbf{y}) N(dt, d\mathbf{x}, d\mathbf{y}) \\
&= \quad \int_{\mathbf{R}^+} \int_{\mathbf{R}^d} \int_Y \int_V \psi(t, \mathbf{x}, \mathbf{y}, \mathbf{v}) M(\mathbf{y}, \mathbf{v}) d\nu(\mathbf{v}) N(dt, d\mathbf{x}, d\mathbf{y}) \ .
\end{aligned}$$

This shows that

$$\mathcal{F}(dt, d\mathbf{x}, d\mathbf{y}, d\nu(\mathbf{v})) = N(dt, d\mathbf{x}, d\mathbf{y}) M(\mathbf{y}, \mathbf{v}) d\nu(\mathbf{v}) \ ,$$

where $N(dt, d\mathbf{x}, d\mathbf{y})$ is the two-scale limit of $n^{\varepsilon_k, \eta_k}(t, \mathbf{x}) dt d\mathbf{x}$ .

*Step 2.* $N(dt, d\mathbf{x}, d\mathbf{y}) = n(dt, d\mathbf{x}) \rho(\mathbf{y}) d\mathbf{y}$ .

Let us consider the weak formulation (7.47) with the following family of test functions. For any $\psi(t, \mathbf{x}, \mathbf{y})$, define

$$z = \mathbf{X}^* \cdot \nabla_{\mathbf{y}} \psi \ , \quad w = \mathbf{X}^* \cdot \nabla_{\mathbf{x}} \psi \ ,$$

so that

$$Q^*[\mathbf{y}](\psi) = 0 \ , \quad Q^*[\mathbf{y}](z) = -\mathbf{v} \cdot \nabla_{\mathbf{y}} \psi \ , \quad Q^*[\mathbf{y}](w) = -\mathbf{v} \cdot \nabla_{\mathbf{x}} \psi \ .$$

Then introduce

$$\phi^{\varepsilon, \eta}(t, \mathbf{x}, \mathbf{y}, \mathbf{v}) = \psi(t, \mathbf{x}, \mathbf{y}) + \eta z(t, \mathbf{x}, \mathbf{y}, \mathbf{v}) + \varepsilon w(t, \mathbf{x}, \mathbf{y}, \mathbf{v}) \ .$$

We have

$$\int_V \int_{\mathbf{R}^d} f_{ini}\phi^{\varepsilon,\eta}(0,\mathbf{x},\frac{\eta\mathbf{x}}{\varepsilon},\mathbf{v})d\mathbf{x}d\nu(\mathbf{v}) + \int_{\mathbf{R}^+}\int_V \int_{\mathbf{R}^d} f^{\varepsilon,\eta} S^{\varepsilon,\eta}(t,\mathbf{x},\frac{\eta\mathbf{x}}{\varepsilon},\mathbf{v})\, d\mathbf{x}d\nu(\mathbf{v})dt = 0 \ ,$$

where

$$\begin{aligned} S^{\varepsilon,\eta}(t,\mathbf{x},\mathbf{y},\mathbf{v}) &= \partial_t\psi + \eta\partial_t z + \varepsilon\partial_t w + \frac{1}{\varepsilon}\mathbf{v}\cdot\nabla_{\mathbf{x}}\psi + \frac{\eta}{\varepsilon}\mathbf{v}\cdot\nabla_{\mathbf{x}} z + \mathbf{v}\cdot\nabla_{\mathbf{x}} w \\ &\quad + \frac{\eta}{\varepsilon^2}\mathbf{v}\cdot\nabla_{\mathbf{y}}\psi + \frac{\eta^2}{\varepsilon^2}\mathbf{v}\cdot\nabla_{\mathbf{y}} z + \frac{\eta}{\varepsilon}\mathbf{v}\cdot\nabla_{\mathbf{y}} w + \frac{1}{\varepsilon^2}Q^*(\psi) + \frac{\eta}{\varepsilon^2}Q^*(z) + \frac{1}{\varepsilon}Q^*(w) \\ &= \partial_t\psi + \eta\partial_t z + \varepsilon\partial_t w + \frac{\eta}{\varepsilon}\mathbf{v}\cdot\nabla_{\mathbf{x}} z + \mathbf{v}\cdot\nabla_{\mathbf{x}} w + \frac{\eta^2}{\varepsilon^2}\mathbf{v}\cdot\nabla_{\mathbf{y}} z + \frac{\eta}{\varepsilon}\mathbf{v}\cdot\nabla_{\mathbf{y}} w \ . \end{aligned}$$

This leads to

$$\begin{aligned} \int_V \int_{\mathbf{R}^d} f_{ini}\phi^{\varepsilon,\eta}(0,\mathbf{x},\frac{\eta\mathbf{x}}{\varepsilon},\mathbf{v})d\mathbf{x}d\nu(\mathbf{v}) + \int_{\mathbf{R}^+}\int_V \int_{\mathbf{R}^d} f^{\varepsilon,\eta}\Big(\partial_t\psi + \eta\partial_t z + \varepsilon\partial_t w + \frac{\eta}{\varepsilon}\mathbf{v}\cdot\nabla_{\mathbf{x}} z \\ + \mathbf{v}\cdot\nabla_{\mathbf{x}} w + \frac{\eta^2}{\varepsilon^2}\mathbf{v}\cdot\nabla_{\mathbf{y}} z + \frac{\eta}{\varepsilon}\mathbf{v}\cdot\nabla_{\mathbf{y}} w)\Big) d\mathbf{x}d\nu(\mathbf{v})dt = 0 \ . \end{aligned}$$

Multiplication of this equation by $\varepsilon^2/\eta^2$, and passage to the limit along the sequence $\varepsilon_k$ , $\eta_k$ now yields

$$\int_{\mathbf{R}^+}\int_{\mathbf{R}^d}\int_Y \int_V \mathbf{v}\cdot\nabla_{\mathbf{y}} z\, \mathcal{F}(dt,d\mathbf{x},d\mathbf{y},d\nu(\mathbf{v})) = 0 \ .$$

Replacing $\mathcal{F}(dt,d\mathbf{x},d\mathbf{y},d\nu(\mathbf{v}))$ by $N(dt,d\mathbf{x},d\mathbf{y})M(\mathbf{y},\mathbf{v})d\nu(\mathbf{v})$, and $z$ by $\mathbf{X}^*\cdot\nabla_{\mathbf{y}}\psi$, we obtain

$$\int_{\mathbf{R}^+}\int_{\mathbf{R}^d}\int_Y \int_V M(\mathbf{y},\mathbf{v})(\mathbf{v}\cdot\nabla_{\mathbf{y}})(\mathbf{X}^*(\mathbf{y},\mathbf{v})\cdot\nabla_{\mathbf{y}}\psi(t,\mathbf{x},\mathbf{y}))d\nu(\mathbf{v})N(dt,dx,d\mathbf{y}) = 0 \ , \tag{7.55}$$

which is equivalent to

$$\int_{\mathbf{R}^+}\int_{\mathbf{R}^d}\int_Y L^*(\psi(t,\mathbf{x},\mathbf{y}))\, N(dt,dx,d\mathbf{y}) = 0 \ . \tag{7.56}$$

For any test function $\phi(t,\mathbf{x},\mathbf{y})$ , the function

$$\chi(t,\mathbf{x},\mathbf{y}) = \phi(t,\mathbf{x},\mathbf{y}) - \int_Y \rho(\mathbf{y})\phi(t,\mathbf{x},\mathbf{y})d(\mathbf{y}) \quad \text{ lies in } \mathrm{Im}L^* \ ,$$

since $\int_Y \rho(\mathbf{y})\chi(t,\mathbf{x},\mathbf{y})dy = 0$ . As a consequence of this and (7.56)

$$\int_{\mathbf{R}^+}\int_{\mathbf{R}^d}\int_Y \phi(t,\mathbf{x},\mathbf{y})\, N(dt,dx,d\mathbf{y}) = \int_{\mathbf{R}^+}\int_{\mathbf{R}^d}\int_Y \rho(\mathbf{y})\phi(t,\mathbf{x},\mathbf{y})\, d\mathbf{y}\int_Y N(dt,dx,d\mathbf{y}') \ .$$

Therefore $N(dt,d\mathbf{x},d\mathbf{y}) = n(dt,d\mathbf{x})\rho(\mathbf{y})d\mathbf{y}$, where $\rho(\mathbf{y})$ is the generator of the kernel of $L$ and $n(dt,dx) = \int_Y N(dt,d\mathbf{x},d\mathbf{y}')$ .

*Step 3. $n(dt,d\mathbf{x})$ solves a diffusion equation.*

Up to now, we did not use the expansion of $F^\eta$ and $\mathbf{X}^{\eta*}$ in the proof. We shall use it in this step in order to exhibit the equation satisfied by $n(dt,d\mathbf{x})$. Given a smooth function $\varphi(t,\mathbf{x})$, let $\psi^\eta(t,\mathbf{x},\mathbf{y},\mathbf{v})$ be defined by

$$\psi^\eta = T^{\eta*-1}(\mathbf{v}\cdot\nabla_{\mathbf{x}}\varphi) = \mathbf{X}^{\eta*}\cdot\nabla_{\mathbf{x}}\varphi \ ,$$

so that

$$-\eta\mathbf{v}\cdot\nabla_{\mathbf{y}}\psi^\eta - Q^*(\psi^\eta) = \mathbf{v}\cdot\nabla_{\mathbf{x}}\varphi \ .$$

The two-scale weak formulation (7.47 ) with

$$\phi^{\varepsilon,\eta}(t,\mathbf{x},\mathbf{y},\mathbf{v}) = \varphi(t,\mathbf{x}) + \varepsilon\psi^\eta(t,\mathbf{x},\mathbf{y},\mathbf{v})$$

now takes the form

$$
\begin{aligned}
&-\int_V\int_{\mathbf{R}^d} f_{ini}\phi^{\varepsilon,\eta}(0,\mathbf{x},\frac{\eta\mathbf{x}}{\varepsilon},\mathbf{v})d\mathbf{x}d\nu(\mathbf{v})\\
&\quad=\int_{\mathbf{R}^+}\int_V\int_{\mathbf{R}^d} f^{\varepsilon,\eta}[\partial_t\varphi+\varepsilon\partial_t\psi^\eta+\frac{1}{\varepsilon}\mathbf{v}\cdot\nabla_{\mathbf{x}}\varphi+\\
&\qquad\qquad+\mathbf{v}\cdot\nabla_{\mathbf{x}}\psi^\eta+\frac{\eta}{\varepsilon}\mathbf{v}\cdot\nabla_{\mathbf{y}}\psi^\eta+\frac{1}{\varepsilon}Q^*(\psi^\eta)]|_{\mathbf{y}=\frac{\eta\mathbf{x}}{\varepsilon}}d\mathbf{x}d\nu(\mathbf{v})dt\\
&\quad=\int_{\mathbf{R}^+}\int_V\int_{\mathbf{R}^d} f^{\varepsilon,\eta}[\partial_t\varphi+\varepsilon\partial_t\psi^\eta+\mathbf{v}\cdot\nabla_{\mathbf{x}}(\mathbf{X}^{\eta*}(\mathbf{y},\mathbf{v})\cdot\nabla_{\mathbf{x}}\varphi)]|_{\mathbf{y}=\frac{\eta\mathbf{x}}{\varepsilon}}d\mathbf{x}d\nu(\mathbf{v})dt\\
&\quad=\int_{\mathbf{R}^+}\int_V\int_{\mathbf{R}^d} f^{\varepsilon,\eta}[\partial_t\varphi+\varepsilon\partial_t\psi^\eta+\mathbf{v}\cdot\nabla_{\mathbf{x}}(\mathbf{X}^*_{[0]}(\mathbf{y},\mathbf{v})\cdot\nabla_{\mathbf{x}}\varphi)]|_{\mathbf{y}=\frac{\eta\mathbf{x}}{\varepsilon}}d\mathbf{x}d\nu(\mathbf{v})dt\\
&\qquad+\frac{1}{\eta}\int_{\mathbf{R}^+}\int_V\int_{\mathbf{R}^d} f^{\varepsilon,\eta}\mathbf{v}\cdot\nabla_{\mathbf{x}}(\mathbf{\Theta}^{[-1]}(\mathbf{y})\cdot\nabla_{\mathbf{x}}\varphi)|_{\mathbf{y}=\frac{\eta\mathbf{x}}{\varepsilon}}d\mathbf{x}d\nu(\mathbf{v})dt+O(\eta)\ ,
\end{aligned}
$$

since $||\mathbf{X}^{\eta*}-\frac{1}{\eta}\mathbf{\Theta}^{[-1]}(\mathbf{y})-\mathbf{X}^*_{[0]}(\mathbf{y},\mathbf{v})||_{L^\infty(Y\times V)}=0(\eta)$. Passing to the limit in this identity along the sequence $\varepsilon_k\,,\ \eta_k$, we obtain

$$
\begin{aligned}
&-\int_{\mathbf{R}^d} n_{ini}(\mathbf{x})\varphi(0,\mathbf{x})d\mathbf{x}\\
&\quad=\int_{\mathbf{R}^+}\int_{\mathbf{R}^d}\int_Y\int_V \rho(\mathbf{y})M(\mathbf{y},\mathbf{v})[\partial_t\varphi(t,\mathbf{x})+\mathbf{v}\cdot\nabla_{\mathbf{x}}(\mathbf{X}^*_{[0]}(\mathbf{y},\mathbf{v})\cdot\nabla_{\mathbf{x}}\varphi(t,\mathbf{x}))]d\mathbf{y}d\nu(\mathbf{v})\,n(dt,d\mathbf{x})\\
&\qquad+\lim_{k\to\infty}\frac{1}{\eta_k}\int_{\mathbf{R}^+}\int_V\int_{\mathbf{R}^d} f^{\varepsilon_k,\eta_k}\mathbf{v}\cdot\nabla_{\mathbf{x}}(\mathbf{\Theta}^{[-1]}(\mathbf{y})\cdot\nabla_{\mathbf{x}}\varphi)|_{\mathbf{y}=\frac{\eta_k\mathbf{x}}{\varepsilon_k}}d\mathbf{x}d\nu(\mathbf{v})dt\ ,
\end{aligned}
\tag{7.57}
$$

with $n_{ini}(\mathbf{x})=\int_V f_{ini}(\mathbf{x},\mathbf{v})d\nu(\mathbf{v})$. Let us now evaluate the last term of the the above identity. For this purpose we introduce $\Psi(x,\mathbf{y},\mathbf{v})=\mathbf{X}^*\cdot\nabla_{\mathbf{x}}(\mathbf{\Theta}^{[-1]}(\mathbf{y})\cdot\nabla_{\mathbf{x}}\varphi)$, which satisfies

$$
Q^*[\mathbf{y}](\Psi)=-\mathbf{v}\cdot\nabla_{\mathbf{x}}(\mathbf{\Theta}^{[-1]}(\mathbf{y})\cdot\nabla_{\mathbf{x}}\varphi)\ .
$$

Using $\Psi$ as a test function in (7.47), we immediately obtain the identity

$$
\begin{aligned}
&\frac{1}{\eta}\int_{\mathbf{R}^+}\int_{V\times\mathbf{R}^d} f^{\varepsilon,\eta}\mathbf{v}\cdot\nabla_{\mathbf{x}}(\mathbf{\Theta}^{[-1]}(\mathbf{y})\cdot\nabla_{\mathbf{x}}\varphi)|_{\mathbf{y}=\frac{\eta\mathbf{x}}{\varepsilon}}d\mathbf{x}d\nu(\mathbf{v})dt\\
&\quad=-\frac{1}{\eta}\int_{\mathbf{R}^+}\int_{V\times\mathbf{R}^d} f^{\varepsilon,\eta}Q^*[\mathbf{y}](\Psi)d\mathbf{x}d\nu(\mathbf{v})dt\\
&\quad=\frac{\varepsilon^2}{\eta}\int_{V\times\mathbf{R}^d} f_{ini}(\mathbf{x},\mathbf{v})\Psi(0,\mathbf{x},\mathbf{y},\mathbf{v})|_{\mathbf{y}=\frac{\eta\mathbf{x}}{\varepsilon}}d\mathbf{x}d\nu(\mathbf{v})\\
&\qquad+\textstyle\int_{\mathbf{R}^+}\int_{V\times\mathbf{R}^d} f^{\varepsilon,\eta}(\frac{\varepsilon^2}{\eta}\partial_t\Psi+\frac{\varepsilon}{\eta}\mathbf{v}\cdot\nabla_{\mathbf{x}}\Psi+\mathbf{v}\cdot\nabla_{\mathbf{y}}\Psi)|_{\mathbf{y}=\frac{\eta\mathbf{x}}{\varepsilon}}d\mathbf{x}d\nu(\mathbf{v})dt\ .
\end{aligned}
$$

Therefore, along the sequence $\varepsilon_k\,,\ \eta_k$ (with $\varepsilon_k\to 0\,,\ \eta_k\to 0$ and $\varepsilon_k/\eta_k\to 0$)

$$
\begin{aligned}
&\lim_{k\to\infty}\frac{1}{\eta_k}\int_{\mathbf{R}^+}\int_V\int_{\mathbf{R}^d} f^{\varepsilon_k,\eta_k}\mathbf{v}\cdot\nabla_{\mathbf{x}}(\mathbf{\Theta}^{[-1]}(\mathbf{y})\cdot\nabla_{\mathbf{x}}\varphi)|_{\mathbf{y}=\frac{\eta_k\mathbf{x}}{\varepsilon_k}}d\mathbf{x}d\nu(\mathbf{v})dt\\
&\qquad=\int_{\mathbf{R}^+}\int_{\mathbf{R}^d}\int_V\int_Y M(\mathbf{y},\mathbf{v})\rho(\mathbf{y})(\mathbf{v}\cdot\nabla_{\mathbf{y}})(\mathbf{X}^*\cdot\nabla_{\mathbf{x}}(\mathbf{\Theta}^{[-1]}(\mathbf{y})\cdot\nabla_{\mathbf{x}}\varphi(t,\mathbf{x})))d\mathbf{y}d\nu(\mathbf{v})\,n(dt,d\mathbf{x})\ .
\end{aligned}
$$

Inserting this identity into (7.57), and using the formula $\mathbf{X}^*_{[0]}=\mathbf{X}^*+(\mathbf{X}^*\cdot\nabla_{\mathbf{y}})\mathbf{\Theta}^{[-1]}(\mathbf{y})+\mathbf{\Theta}^{[0]}(\mathbf{y})$, we find

$$\int_{\mathbf{R}^d} n_{ini}(\mathbf{x})\varphi(0,\mathbf{x})d\mathbf{x} + \int_{\mathbf{R}^+}\int_{\mathbf{R}^d} \partial_t\varphi(t,\mathbf{x})n(dt,d\mathbf{x})$$
$$+ \int_{\mathbf{R}^+}\int_{\mathbf{R}^d}\int_Y\int_V \rho(\mathbf{y})M(\mathbf{y},\mathbf{v})(\mathbf{v}\cdot\nabla_{\mathbf{x}})[(\mathbf{X}^*+(\mathbf{X}^*\cdot\nabla_{\mathbf{y}})\mathbf{\Theta}^{[-1]}(\mathbf{y}))\cdot\nabla_{\mathbf{x}}\varphi(t,\mathbf{x})]d\mathbf{y}d\nu(\mathbf{v})\, n(dt,d\mathbf{x})$$
$$+ \int_{\mathbf{R}^+}\int_{\mathbf{R}^d}\int_Y\int_V \rho(\mathbf{y})M(\mathbf{y},\mathbf{v})(\mathbf{v}\cdot\nabla_{\mathbf{y}})(\mathbf{X}^*\cdot\nabla_{\mathbf{x}}(\mathbf{\Theta}^{[-1]}(\mathbf{y})\cdot\nabla_{\mathbf{x}}\varphi(t,\mathbf{x})))d\mathbf{y}d\nu(\mathbf{v})\, n(dt,d\mathbf{x}) = 0\,.$$

This is exactly a weak formulation of the diffusion problem

$$\partial_t n - \mathrm{div}_{\mathbf{x}}(\mathbf{D}\nabla_{\mathbf{x}})n = 0, \quad n(0,\mathbf{x}) = n_{ini}(\mathbf{x}) = \int f_{ini}(\mathbf{x},\mathbf{v})\, d\nu(\mathbf{v})\ ,$$

with

$$\begin{aligned}\mathbf{D} &= \int_Y\int_V (\mathbf{X}^*(\mathbf{y},\mathbf{v}) + \mathbf{X}^*(\mathbf{y},\mathbf{v})\cdot\nabla_{\mathbf{y}}\mathbf{\Theta}^{[-1]}(\mathbf{y}))\otimes\mathbf{v}\rho(\mathbf{y})M(\mathbf{y},\mathbf{v})\\ &\qquad -\mathbf{\Theta}^{[-1]}(\mathbf{y})\otimes\mathbf{X}^*(\mathbf{y},\mathbf{v})\mathbf{v}\cdot\nabla_{\mathbf{y}}(\rho(\mathbf{y})M(\mathbf{y},\mathbf{v}))\, d\nu(\mathbf{v})d\mathbf{y}\\ &= \int_Y\int_V \mathbf{X}^*(\mathbf{y},\mathbf{v})\otimes\mathbf{v}\rho(\mathbf{y})M(\mathbf{y},\mathbf{v}) - \mathbf{\Theta}^{[-1]}(\mathbf{y})\otimes\Big[\mathrm{div}_{\mathbf{y}}(\mathbf{v}\otimes\mathbf{X}^*(\mathbf{y},\mathbf{v})M(\mathbf{y},\mathbf{v})\rho(\mathbf{y}))\\ &\qquad +\mathbf{X}^*(\mathbf{y},\mathbf{v})\otimes\mathbf{v}M(\mathbf{y},\mathbf{v})\nabla_{\mathbf{y}}\rho(\mathbf{y}) + \mathbf{X}^*(\mathbf{y},\mathbf{v})\mathbf{v}\cdot\nabla_{\mathbf{y}}M(\mathbf{y},\mathbf{v})\rho(\mathbf{y})\Big]\, d\nu(\mathbf{v})d\mathbf{y}\\ &= \int_Y D(\mathbf{y})\rho(\mathbf{y}) - \mathbf{\Theta}^{[-1]}(\mathbf{y})\otimes[\mathrm{div}_{\mathbf{y}}(D^\top(\mathbf{y})\rho(\mathbf{y})) + D(\mathbf{y})\nabla_{\mathbf{y}}\rho(\mathbf{y}) - U(\mathbf{y})\rho(\mathbf{y})]d\mathbf{y}\ .\end{aligned}$$

We note that $\mathbf{D}$ is as introduced in (7.7). We also note that, due to the positivity of the constant matrix $\mathbf{D}$ (see Remark 7.4.5), $n(dt,d\mathbf{x})$ is unique, and it has the form $n(t,\mathbf{x})dt\,d\mathbf{x}$, where the function $n(t,\mathbf{x})$ is a classical solution to the diffusion problem.
□

In view of the Definition 7.4.1, *The proof of Theorem 7.1.2* now follows by a simple combination of the results in Proposition 7.4.1 and Proposition 7.4.2. The uniquenss of the limit (Proposition 7.4.2) implies convergence along any sequence with $\varepsilon\to 0$, $\eta\to 0$ and $\varepsilon/\eta\to 0$.

*Remarque* 7.4.3. We need a uniformly convergent asymptotic formula for $\mathbf{X}^{\eta *}$, because we do not work in the $L^2$ framework but in the $L^1$ framework. Indeed, when the Maxwellian profile depends on $y$, the usual dissipation relation does not give a $L^2$ bound for $f^{\varepsilon,\eta}$. We could obtain a weighted $L^2$ norm $||\frac{f^{\varepsilon,\eta}}{F^\eta}||_{L^2(\mathbf{R}\times\mathbf{R}^d\times V)}$ and prove that $F^\eta$ is bounded from below independently of $\eta$. It would certainly be possible to show this for sufficiently small $\eta$, by demonstrating $C^0$ convergence of $F^\eta$ towards $\rho M$, but it would imply the same technical difficulties (and the same kind of assumptions) as required for the expansion of $\mathbf{X}^{\eta *}$.

*Remarque* 7.4.4. Note that, as we suggested in the section about the strategy proof, one could formally obtain the same limiting result, starting from the work of [2] or [17], and introducing the parameter $\eta$ in the effective diffusion equation. In that case one arrives at an $\eta$ dependent density $n^\eta(t,\mathbf{x})$ satisfying

$$\partial_t n^\eta - \mathrm{div}_{\mathbf{x}}\mathbf{D}^\eta\nabla_{\mathbf{x}}n^\eta = 0 \quad \text{with} \quad \mathbf{D}^\eta = \int_V\int_Y \mathbf{X}^{\eta *}(\mathbf{y},\mathbf{v})\otimes\mathbf{v}F^\eta(\mathbf{y},\mathbf{v})d\mathbf{y}d\nu(\mathbf{v})\ .$$

By expanding $F^\eta$ and $\mathbf{X}^{\eta *}$ as above, and passing to the limit $\eta \to 0$,

$$\begin{aligned}
\mathbf{D}^\eta &= \int_V \int_Y \left[ \frac{1}{\eta} \mathbf{\Theta}^{[-1]}(\mathbf{y}) + \mathbf{X}^* + \mathbf{X}^* \cdot \nabla_{\mathbf{y}} \mathbf{\Theta}^{[-1]}(\mathbf{y}) + \mathbf{\Theta}^{[0]}(\mathbf{y}) + O(\eta) \right] \\
&\qquad \otimes_{\mathbf{v}} \left[ \rho(\mathbf{y}) M(\mathbf{y}, \mathbf{v}) + \eta Q^{-1}(\mathbf{v} \cdot \nabla_{\mathbf{y}} (\rho(\mathbf{y}) M(\mathbf{y}, \mathbf{v}))) + O(\eta^2) \right] d\mathbf{y} d\nu(\mathbf{v}) \\
&= \int_V \int_Y \Bigg[ (\mathbf{X}^* + \mathbf{X}^* \cdot \nabla_{\mathbf{y}} \mathbf{\Theta}^{[-1]}(\mathbf{y})) \otimes \mathbf{v} \rho(\mathbf{y}) M(\mathbf{y}, \mathbf{v}) \\
&\qquad + \mathbf{\Theta}^{[-1]}(\mathbf{y}) \otimes \mathbf{v} Q^{-1}(\mathbf{v} \cdot \nabla_{\mathbf{y}} (\rho(\mathbf{y}) M(\mathbf{y}, \mathbf{v}))) \Bigg] d\mathbf{y} d\nu(\mathbf{v}) + O(\eta) \\
&= \int_V \int_Y \Bigg[ (\mathbf{X}^* + \mathbf{X}^* \cdot \nabla_{\mathbf{y}} \mathbf{\Theta}^{[-1]}(\mathbf{y})) \otimes \mathbf{v} \rho(\mathbf{y}) M(\mathbf{y}, \mathbf{v}) \\
&\qquad - \mathbf{\Theta}^{[-1]}(\mathbf{y}) \otimes \mathbf{X}^* \mathbf{v} \cdot \nabla_{\mathbf{y}} (\rho(\mathbf{y}) M(\mathbf{y}, \mathbf{v})) \Bigg] d\mathbf{y} d\nu(\mathbf{v}) + O(\eta) \ ,
\end{aligned}$$

in other words

$$\lim_{\eta \to 0} \mathbf{D}^\eta = \mathbf{D} \ ,$$

and one recovers, formally, the result of our main theorem.

*Remarque* 7.4.5. The fact that

$$\mathbf{D}\xi \cdot \xi = \lim_{\eta \to 0} \mathbf{D}^\eta \xi \cdot \xi$$

(see the previous Remark) in combination with the dissipation relation

$$\mathbf{D}^\eta \xi \cdot \xi = \frac{1}{2} \int_V \int_V \int_Y \sigma(\mathbf{y}, \mathbf{v}, \mathbf{v}') F^\eta(\mathbf{y}, \mathbf{v}) |\mathbf{X}^{\eta *}(\mathbf{y}, \mathbf{v}') \cdot \xi - \mathbf{X}^{\eta *}(\mathbf{y}, \mathbf{v}) \cdot \xi|^2 d\nu(\mathbf{v}') d\nu(\mathbf{v}) d\mathbf{y}$$

(see [17]) immediately yields

$$\mathbf{D}\xi \cdot \xi = \frac{1}{2} \lim_{\eta \to 0} \int_V \int_V \int_Y \sigma(\mathbf{y}, \mathbf{v}, \mathbf{v}') F^\eta(\mathbf{y}, \mathbf{v}) |\mathbf{X}^{\eta *}(\mathbf{y}, \mathbf{v}') \cdot \xi - \mathbf{X}^{\eta *}(\mathbf{y}, \mathbf{v}) \cdot \xi|^2 d\nu(\mathbf{v}') d\nu(\mathbf{v}) d\mathbf{y} \ . \quad (7.58)$$

Thanks to Corollary 3.7 we have

$$\begin{aligned}
&|\mathbf{X}^{\eta *}(\mathbf{y}, \mathbf{v}') \cdot \xi - \mathbf{X}^{\eta *}(\mathbf{y}, \mathbf{v}) \cdot \xi|^2 \\
&\quad = |((\mathbf{X}^{\eta *}(\mathbf{y}, \mathbf{v}') - \frac{1}{\eta} \Theta^{[-1]}(y) - \Theta^{[0]}(y)) \cdot \xi - (\mathbf{X}^{\eta *}(\mathbf{y}, \mathbf{v}) - \frac{1}{\eta} \Theta^{[-1]}(y) - \Theta^{[0]}(y)) \cdot \xi|^2 \\
&\quad \to |(\mathbf{X}^*(\mathbf{y}, \mathbf{v}') + \mathbf{X}^*(\mathbf{y}, \mathbf{v}') \cdot \nabla_{\mathbf{y}} \Theta^{[-1]}(\mathbf{y})) \cdot \xi - (\mathbf{X}^*(\mathbf{y}, \mathbf{v}) + \mathbf{X}^*(\mathbf{y}, \mathbf{v}) \cdot \nabla_{\mathbf{y}} \Theta^{[-1]}(\mathbf{y})) \cdot \xi|^2 \ ,
\end{aligned}$$

in $L^\infty_{per}(\mathbf{R}^d_{\mathbf{y}} \times V)$ and, thanks to Proposition 7.3.2, we have

$$F^\eta(\mathbf{y}, \mathbf{v}) \to \rho(\mathbf{y}) M(\mathbf{y}, \mathbf{v}) \quad \text{in } L^2(Y \times V) \ , \quad \text{as } \eta \to 0 \ .$$

In view of these two convergence statements, and the fact that $\sigma(\mathbf{y}, \mathbf{v}, \mathbf{v}') \rho(\mathbf{y}) M(\mathbf{y}, \mathbf{v})$ is positive, the identity (7.58) implies

$$\begin{aligned}
\mathbf{D}\xi \cdot \xi = \frac{1}{2} \int_V \int_V \int_Y \sigma(\mathbf{y}, \mathbf{v}, \mathbf{v}') \rho(\mathbf{y}) M(\mathbf{y}, \mathbf{v}) |(\mathbf{X}^*(\mathbf{y}, \mathbf{v}') + \mathbf{X}^*(\mathbf{y}, \mathbf{v}') \cdot \nabla_{\mathbf{y}} \Theta^{[-1]}(\mathbf{y})) \cdot \xi \quad (7.59) \\
- (\mathbf{X}^*(\mathbf{y}, \mathbf{v}) + \mathbf{X}^*(\mathbf{y}, \mathbf{v}) \cdot \nabla_{\mathbf{y}} \Theta^{[-1]}(\mathbf{y})) \cdot \xi|^2 d\nu(\mathbf{v}') d\nu(\mathbf{v}) d\mathbf{y} \geq 0 \ .
\end{aligned}$$

Furthermore, it follows that if $\mathbf{D}\xi \cdot \xi$ vanishes, then the expression

$$(\mathbf{X}^*(\mathbf{y}, \mathbf{v}) + \mathbf{X}^*(\mathbf{y}, \mathbf{v}) \cdot \nabla_{\mathbf{y}} \Theta^{[-1]}(\mathbf{y})) \cdot \xi$$

must be independent of $\mathbf{v}$, and so it must belong to the kernel of $Q^*[\mathbf{y}]$. An application of $Q^*[\mathbf{y}]$ now yields

$$(\mathbf{v} + \mathbf{v} \cdot \nabla_{\mathbf{y}} \Theta^{[-1]}(\mathbf{y})) \cdot \xi = 0 \ ,$$

which, after integration in $\mathbf{y}$, gives

$$\mathbf{v} \cdot \xi = 0 \quad , \text{ for a.e. } \mathbf{v} \in V \ .$$

We conclude that $\xi = 0$ and, as a consequence of (7.59), that

$$\mathbf{D}\xi \cdot \xi \geq \beta |\xi|^2 \quad \text{ for some } \beta > 0 \ .$$

This positivity property of the (constant) matrix $\mathbf{D}$ was used to ensure the uniquenss of the density $n(dt, d\mathbf{x})$.

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## 7.6 Postscript

Throughout much of the work on this project Naoufel Ben Abdallah was the moving force. However, the manuscript was completed after he passed away on July the fifth, 2010. We (Marjolaine Puel and Michael Vogelius) very much hope he would have approved of this final version.

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# Chapitre 8

# A corrector result in diffusion and homogenization limits of linear transport equations

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**abstract**

This paper concerns the diffusion-homogenization of transport equations when both the adimensionalized scale of the heterogeneities $\alpha$ and the adimensionalized mean-free path $\varepsilon$ converge to 0. When $\alpha = \varepsilon$, it is well known that the heterogeneous transport solution converges to a homogenized diffusion solution. We are interested here in the situation where $0 < \varepsilon \ll \alpha \ll 1$ and in the respective rates of convergences to the homogenized limit and to the diffusive limit. Our main result is an approximation to the transport solution with an error term that is negligible compared to the maximum of $\alpha$ and $\frac{\varepsilon}{\alpha}$. After establishing the diffusion-homogenization limit to the transport solution, we show that the correction is dominated by an error to homogenization when $\alpha^2 \ll \varepsilon$ and by an an error to diffusion when $\varepsilon \ll \alpha^2$.

Our regime of interest involves singular perturbations in the small parameter $\eta = \frac{\varepsilon}{\alpha}$. Disconnected local equilibria at $\eta = 0$ need to be reconnected to provide a global equilibrium on the cell of periodicity when $\eta > 0$. This reconnection between local and global equilibria is shown to hold when sufficient *no-drift* conditions are satisfied. The Hilbert expansion methodology followed in this paper builds on corrector theories for the result developed in [9].

## 8.1 Setting of the problem

This paper studies the interaction between the convergence to a homogenized limit and the convergence to a diffusive limit in the context of linear transport equations, or linear Boltzmann equations [12, 6], that model the propagation of particles in scattering environments. These two phenomena have been widely studied in the past [11, 1, 25, 23, 21, 22, 24, 14]. The homogenization limit typically arises when the underlying coefficients oscillate at a scale $\alpha \ll 1$ much smaller than the macroscopic scale at which phenomena are observed. The diffusion limit arises in highly scattering environments in which an equilibrium emerges in the velocity variable and a diffusion equation models the spatial behavior of the probability density of particles. High scattering is modeled by a mean free path $\varepsilon \ll 1$, where $\varepsilon$ is the (adimensionalized) main distance between successive collisions with the underlying medium.

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We assume here that the oscillations in the medium are periodic with period $\alpha \ll 1$. When both $\alpha$ and $\varepsilon$ converge to 0, we expect the transport solution to converge to the solution of a homogenized diffusion equation. How fast convergence may occur and what are the main contributions to the error between the heterogeneous transport solution and its homogenized limit are the main problems of interest in this paper. We restrict ourselves to the case where the mean free path is (much) smaller than the correlation length $\alpha$. We thus set $\eta = \frac{\varepsilon}{\alpha}$ and we assume that $\eta \ll 1$. The equation for the particle density takes the form :

$$v \cdot \nabla f_\alpha^\varepsilon + \varepsilon f_\alpha^\varepsilon + \frac{1}{\varepsilon} Q_\alpha f_\alpha^\varepsilon = \varepsilon S_\alpha(x,v) := \varepsilon S(x, \frac{x}{\alpha}, v), \tag{8.1}$$

in an infinite domain $\mathbb{R}^d \times V$ where $V$ is a smooth compact domain in the velocity space such that $0 \notin V$. This equation could also be seen as an evolution equation by a change of variables $u_\alpha^\varepsilon = e^{-t} f_\alpha^\varepsilon$ in which the source $S$ would play the role of an initial condition. We restrict ourselves to the time independent setting and to the case of constant absorption to simplify.

The source term $S$ is $1-$periodic in $y = \frac{x}{\alpha}$. We define the collision operator as

$$Q_\alpha f = \Sigma(\frac{x}{\alpha}, v) f(x,v) - \int_V \sigma(\frac{x}{\alpha}, v', v) f(x,v') d\nu(v'). \tag{8.2}$$

The choice of $\Sigma$ is such that the above operator is conservative in the following sense. Define

$$Qf(y,v) = \Sigma(y,v) f(y,v) - \int_V \sigma(y,v',v) f(y,v') d\nu(v'), \quad y \in Y, \tag{8.3}$$

where $Y = [0,1]^d$ is the unit cell. Then we assume the existence of $C^{-1} \geq \psi^\eta(y,v) \geq C > 0$ such that

$$(\eta v \cdot \nabla_y + Q)\psi^\eta = 0, \qquad Y \times V. \tag{8.4}$$

We also define

$$(-\eta v \cdot \nabla_y + Q^*)\psi^{\eta *} = 0, \qquad Y \times V, \tag{8.5}$$

where $Q^*$ is the adjoint operator to $Q$ defined for a.e. $y \in Y$ and $g \in L^2(V)$ as

$$Q^* g(v) = \Sigma(y,v) g(v) - \int_V \sigma(y,v,v') g(v') d\nu(v'). \tag{8.6}$$

The simultaneous limit when the mean free path and the correlation length go to zero together with $\alpha = \varepsilon$ has been considered in several recent papers [2, 3, 4, 5, 10, 17, 18, 19, 20, 25]. In all those paper, only the case $\eta = 1$ is considered except in [25], where both $\psi^\eta$ and $\psi^{\eta *}$ are space independent and hence also independent of $\eta$. A first result in the case $\eta \ll 1$ was obtained in [9] for spatially dependent $\psi^\eta$ in the setting where $\psi^{*,\eta} \equiv 1$. In that paper, a two scale convergence result is established in the setting where the limiting behavior involves the homogenization of a heterogeneous diffusion. The proof of that result is based on the weak formulation of the transport equation and on the method of moments.

The aim of the present work is to implement the Hilbert expansion method, which leads to a strong convergence result under appropriate smoothness conditions in the (restricted) setting where $\psi^{*,\eta} = \psi^*(v)$. This strong convergence has two advantages. First, we are able to compute the correction terms in the expansion of the transport solution up to a term that is negligible compared to the maximum of $\alpha$ and $\eta = \frac{\varepsilon}{\alpha}$. Second, we expect this result also to be the first step in the study of the nonlinear Boltzmann equation given by the Fermi-Dirac equation in the spirit of [7, 8]. Note that in the case where $\eta$ goes to infinity, we expect the limit to be the diffusion approximation to the homogenized transport equation obtained in [15].

The main mathematical difficulty of the present work arises because the limit $\eta \to 0$ in (8.4) is singular. In the limit $\eta = 0$, local equilibria are obtained as a function of $v \in V$ for *each* position $y \in Y$. For $\eta > 0$, a global equilibrium on $Y \times V$ emerges, as in the standard procedure obtained when $\eta = 1$ [2, 19, 17, 21, 22, 25]. The passage from the local equilibria to the global equilibrium may in fact be arbitrarily complicated. Several conditions need to be imposed in order for a well-defined macroscopic equilibrium to arise. Even in the case $\eta = 1$ do we need to impose a *no-drift* condition. In

the presence of drift, advection dominates scattering and entirely different phenomena arise (see, e.g., [5] and the derivation of Euler equations when advection is dominant). Under appropriate sufficient symmetry assumptions similar to (though more constraining than) those in [2], we are able to verify the necessary no-drift conditions used in our derivation.

The rest of the paper is structured as follows. The main hypotheses of regularity and no-drift as well as the main results of this paper are described in section 8.2. The main result is Theorem 8.2.1 below. The rest of the paper is devoted to its proof. Global a priori estimates are formulated and proved in section 8.3. The expansion of the transport solution in powers of $\varepsilon$ is treated in section 8.4. The expansion in $\eta$ of several cell solutions is given in section 8.5. The definition and $\eta$-dependence of spatial density terms are given in section 8.6. These results are combined to finish the proof of the theorem in section 8.7. Several technical results obtained in [9] and the generalization of their proofs if necessary are collected in the Appendix.

## 8.2 Main result

Under regularity assumptions recalled below, a standard application of a Banach fixed point theorem [13] ensures that (8.1) admits a unique solution in $H^k(\mathbb{R}^d, L^2(V))$. The limit of $f^{\varepsilon,\eta}(x,v) := f^\varepsilon_\alpha(x,v)$ as $\eta \to 0$, however, involves singular perturbations. The reason is that the local equilibria in (8.4) and (8.5) become degenerate in the limit $\eta \to 0$. In this limit, equilibria at different points $y \in Y$ become disconnected and this can result in a very large effect at the macroscopic scale $x$. We consider here situations where the equilibria remain smooth in the $y$ variable and generate no *drift*. Drift effects are ubiquitous in the homogenization of transport equations, with drastic effects as may be seen in, e.g., [5]. Diffusion limits arise under sufficient *no-drift* conditions as in, e.g., [2], which ensure that transport is not in an advection-dominated regime. Our analysis in this paper shows that diffusion-like regimes are still valid in the singular limit $\eta \to 0$ under appropriate no-drift assumptions. We show that these assumptions are consequences of symmetries of the transport coefficients, which we now define.

Our first main assumptions is that $\psi^{*,\eta}(y,v)$ is independent of $y$ and hence of $\eta$. Note that $Q$ in (8.4) is independent of $\eta$ and it is therefore not clear why non-trivial solutions would exist for all values of $\eta$. Here, we assume that a.e. $y \in Y$, we have an equilibrium $\psi^*(v)$ solution of

$$Q^*(\psi^*) = 0 \qquad \text{in } V. \tag{8.7}$$

The equilibrium solution is allowed to depend on $v$ but is independent of $y \in Y$. When $\sigma(y,v',v)$ and $\Sigma(y,v)$ are continuous functions bounded above and below by positive constants, then $\psi^*$ can be chosen positive and normalized so that $\int_V (\psi^*)^2(y,v)d\nu(v) = 1$ a.e. $y \in Y$. Moreover, up to normalization, $\psi^*(v)$ is the unique solution to (8.7) as an application of the Krein Rutman theorem [2, 13] for the compact operators defined for a.e. $y \in Y$ :

$$f \mapsto \frac{1}{\Sigma(y,v)} \int_V \sigma(y,v,v')f(v')d\nu(v'), \tag{8.8}$$

which preserve the (solid) cone of positive continuous functions.

The no-drift conditions mentioned above will be verified under sufficient symmetry assumptions. We first assume that :

$$\sigma(y,v',v) = \sigma(y,-v',-v) = \sigma(-y,v',v), \qquad \Sigma(y,v) = \Sigma(y,-v) = \Sigma(-y,v). \tag{8.9}$$

We deduce that $\psi^*(-v) = \psi^*(v)$. Note that a method to construct local equilibria consists of selecting $\sigma$ satisfying the above symmetries, $\psi^*$ arbitrary (uniformly positive) such that $\psi^*(-v) = \psi^*(v)$, and finally define $\Sigma(y,v)$ by (8.7), which also satisfies (8.9).

We also obtain the existence of a unique, bounded, positive, solution $\psi(y,v)$ of the adjoint equation

$$Q(\psi(y,\cdot)) = 0, \qquad \text{in } V, \tag{8.10}$$

normalized so that $\int_V \psi(y,v)\psi^*(y,v)d\nu(v) = 1$ a.e. $y \in Y$. It is not difficult to observe that $\psi(y,-v) = \psi(y,v) = \psi(-y,v)$ when (8.9) holds.

The Krein Rutman theorem for (8.8) (all eigenvalues not equal to 1 have modulus strictly smaller than 1) shows that 0 is a simple eigenvalue associated to $Q$ and $Q^*$ and that all other eigenvalues

of $Q$ and $Q^*$ have strictly positive real part [2, 13]. On the vector space of functions $f \in L^2(V)$ such that $\int_V f(v)\psi^*(v)d\nu(v) = 0$, we define

$$Q^{-1}f := -\int_0^\infty e^{-rQ} f dr, \tag{8.11}$$

which converges thanks to the spectral gap just mentioned. Note that $(Q^{-1}f, \psi^*)_{L^2(V)} = 0$ by construction. The inverse operator $Q^{-*} := (Q^*)^{-1}$ is defined similarly. We verify that under (8.9), both $Q^{-1}$ and $Q^{-*}$ preserve the subspaces of even and odd functions in the variable $v$.

With $\psi^*$ seen as a normalized solution of $(-\eta v \cdot \nabla_y + Q^*)\psi^* = 0$ on $Y \times V$, we also obtain the existence of unique, bounded, positive, solutions $\psi^\eta(y, v)$ of (8.4) normalized such that $\int_{Y\times V} \psi^\eta(y, v)\psi^*(v)d\nu(v) = 1$. Upon introducing $T^\eta = \eta v \cdot \nabla_y + Q$, we also define the inverse operator

$$T^{\eta-1}f = -\int_0^\infty e^{-rT^\eta} f dr, \qquad f \in L^2(Y \times V) \text{ s.t. } \int_{Y\times V} f(y, v)\psi^*(v)dyd\nu(v) = 0. \tag{8.12}$$

Finally, we observe that under (8.9), $\psi^\eta(y, -v)$ and $\psi^\eta(-y, v)$ are also solutions of (8.4). Once properly normalized, since $\psi^*(-v) = \psi^*(v)$, we deduce that $\psi^\eta(y, v) = \psi^\eta(y, -v) = \psi^\eta(-y, v)$.

Let us collect our main **Assumptions** :

(H1) The velocity variables $v$ lies in a compact, symmetric set $V$ of $\mathbb{R}^d$ equiped with a symmetric probability measure $\nu$. There exist constants $C$, $\gamma > 0$ such that $\nu(\{v \in V,\ |v \cdot \xi| \le h\}) \le Ch^\gamma$, for all $\xi \in S^{d-1}$, $h > 0$ . In particular, $\nu(\{v \in V,\ |v \cdot \xi| = 0\}) = 0$ for all $\xi \neq 0$ so that

$$v \cdot \xi = 0 \quad \text{a.e. in } v \text{ implies } \xi = 0 \ .$$

(H2) The source term $S \in H^4(\mathbb{R}^d(C^\infty(Y), L^2(V)))$

(H3) The scattering coefficient $\sigma$ is in $\mathcal{C}^0(V_v \times V_{v'}; \mathcal{C}^\infty_{per}(\mathbb{R}^d_y))$. It is bounded from above and below by positive constants and is 1-periodic with respect to the variable $y$. The coefficient $\Sigma(y, v)$ is defined implicitly in (8.7).

(H4) The symmetry relations (8.9) hold so that the uniquely defined (after proper normalization) solutions $\psi^*(v)$, $\psi(y, v)$ and $\psi^\eta(y, v)$ of (8.10), (8.7), and (8.4), respectively, satisfy $\psi^*(-v) = \psi^*(v)$ and $\psi(y, v) = \psi(-y, v) = \psi(y, -v)$ as well as $\psi^\eta(y, v) = \psi^\eta(-y, v) = \psi^\eta(y, -v)$.

We want to stress again that the limit $\eta \to 0$ is singular. The limit of $\psi^\eta$ as $\eta \to 0$ is therefore not necessarily equal to $\psi(y, v)$. We are now ready to state our main results on the Hilbert expansion of the solution $f^{\varepsilon,\eta} := f^\eta_\alpha$ of (8.1). The main result of this paper is the following theorem, which provides a strong convergence result for the corrector to homogenization theory :

*Theorem* 8.2.1. Let $f^{\varepsilon,\eta}$ be the solution of (8.1). Then the following expansion holds

$$\begin{aligned} \Big\| f^{\varepsilon,\eta} &- n^{0,0}(x)\rho^0(\tfrac{\eta x}{\varepsilon})\psi(\tfrac{\eta x}{\varepsilon}, v) \\ &- \eta\Big[n^{0,0}(x)Q^{-1}\big(-v \cdot \nabla_y(\rho^0(\tfrac{\eta x}{\varepsilon})\psi(\tfrac{\eta x}{\varepsilon}, v))\big) + n^{0,1}(x)\rho^0(\tfrac{\eta x}{\varepsilon})\psi(\tfrac{\eta x}{\varepsilon}, v)\Big] \\ &- \frac{\varepsilon}{\eta}\Big[\theta^{-1}(y)\psi(\tfrac{\eta x}{\varepsilon}, v) \cdot \nabla_x n^{0,0}(x) + n^{1,-1}(x)\rho^0(\tfrac{\eta x}{\varepsilon})\psi(\tfrac{\eta x}{\varepsilon}, v)\Big]\Big\|_{L^2(\mathbb{R}^d\times V)} = o(\eta + \tfrac{\varepsilon}{\eta}) \end{aligned} \tag{8.13}$$

where $\psi(y, v)$ and $\psi^*(v)$ are solutions of (8.10) and (8.7), respectively, and $Q^{-1}$ is defined in (8.11). The microscopic density $\rho^0(y)$ is the unique solution of the elliptic equation

$$L(\rho^0) = 0 \quad \text{with the normalization} \quad \int_Y \rho^0(y)dy = 1$$

where

$$L(\rho) = -\int_V \psi^*(v)v \cdot \nabla_y(Q^{-1}(v \cdot \nabla_y(\psi(y, v)\rho(y))))d\nu(v).$$

The function $\theta^{-1}$ is given by

$$\theta^{-1} = L^{-1}\left(\int_V (vQ^{-1}(-v \cdot \nabla_y\rho^0(y)\psi(y, v)) - v \cdot \nabla_y(Q^{-1}(v\psi(y, v))\rho^0(y)))\psi^*(v)d\nu(v)\right)$$

with $L^{-1}$ defined in Proposition 8.8 below on functions in $L^2(Y)$ that average to 0 on $Y$ and the macroscopic density is given by the diffusion equation

$$n^{0,0}(x) - \nabla_x \cdot (\mathbf{D} \cdot \nabla_x n^{0,0}(x)) \quad = \quad \int_Y \int_V S(x,y,v)\psi^*(v)dyd\nu(v).$$

The diffusion coefficient in the preceding equation is given by the expression

$$\mathbf{D} = \int_V \int_Y (\overline{\chi}^{*0}(y,v) \otimes v\rho^0(y)\psi(y,v) + \theta^{*-1}(y)\psi^*(v) \otimes vQ^{-1}(-v\nabla_y(\rho^0(y)\psi(y,v)))d\nu(v)dy$$

in which we have defined

$$\begin{aligned} \overline{\chi}^{*0} &= Q^{*-1}\big(v\psi^* + v \cdot \nabla_y(\theta^{*-1}\psi^*)\big) \\ \theta^{*-1} &= L^{*-1}\Big(\int_V \psi(y,v)v \cdot \nabla_y Q^{*-1}\big(v\psi^*(v)\big)d\nu(v)\Big). \end{aligned}$$

where $L^{*-1}$ is also defined in Proposition 8.8. The correctors $n^{0,1}(x)$ and $n^{1,-1}(x)$ satisfy the same elliptic equation as $n^{0,0}(x)$ with different source terms. Their expression is given explicitly in Proposition 8.6 below.

Before proving this theorem, we make a few remarks.

*Remark* 8.2.1. The leading term in the expansion of $f_\alpha^\varepsilon$ is thus given by $n^{0,0}(x)\rho^0(\frac{x}{\alpha})\psi(\frac{x}{\alpha},v)$. The behavior at the macroscopic level is given by $n^{0,0}(x)$, solution of a standard diffusion equation. The microscopic level is given by the product of two terms. The first contribution to the product is the standard local solution $\psi(y,v)$, which indicates how particles are distributed in the $v$ variable for each $y \in Y$. The second, less standard, contribution is given by $\rho^0(\frac{x}{\alpha})$ and indicates how the local (for each $y$) equilibria are related to one-another to generate a global equilibrium at the level of the cell $Y$.

*Remark* 8.2.2. The above expansion implies that when $\eta \ll \frac{\varepsilon}{\eta}$, then the corrector is given by

$$\theta^{-1}\psi(\frac{\eta x}{\varepsilon},v) \cdot \nabla_x n^{0,0}(x) + n^{1,-1}(x)\rho^0(\frac{\eta x}{\varepsilon})\psi(\frac{\eta x}{\varepsilon},v).$$

This is a regime of (relatively) low scattering where the corrector to homogenization (characterized by a term linear in $\nabla n^{0,0}$) dominates. The contribution $n^{1,-1}$ provides a correction to the influence of the local equilibria at each $y \in Y$ to a global equilibrium on $Y$.

When $\frac{\varepsilon}{\eta} \ll \eta$, the corrector is given instead by

$$n^{0,0}(x)Q^{-1}\big(-v \cdot \nabla_y(\rho^0(\frac{\eta x}{\varepsilon})\psi(\frac{\eta x}{\varepsilon},v))\big) + n^{0,1}(x)\rho^0(\frac{\eta x}{\varepsilon})\psi(\frac{\eta x}{\varepsilon},v).$$

This is the regime of high scattering, where the correction to approximating the transport solution by a diffusion approximation dominates the correction coming from the homogenization procedure. The passage from local to global equilibria on $Y$ generates a corrector described by $n^{0,1}(x)$.

The rest of the paper is devoted to the proof of the theorem.

## 8.3 A priori estimates

We start with an estimate that controls the remainder terms in the Hilbert expansion :

*Proposition.* Let $f_\alpha^\varepsilon$ be the solution of (8.1). Then we have :

$$\|f_\alpha^\varepsilon\| + \frac{1}{\varepsilon}\|f_\alpha^\varepsilon - \psi_\alpha^\eta \frac{\overline{f_\alpha^\varepsilon}}{\psi_\alpha^\eta}\| \leq C\Big(\|\varepsilon S_\alpha\| + \|\overline{S_\alpha \psi_\alpha^{\eta *}}\|\Big), \tag{8.14}$$

where $\|\cdot\|$ is the $L^2(\mathbb{R}^d \times V)$-norm and for $u \in L^2(\mathbb{R}^d \times V)$ we have defined

$$\overline{u}(x) = \frac{1}{|V|}\int_V u(x,v)d\nu(v). \tag{8.15}$$

*Démonstration.* We verify that

$$(\eta v\cdot\nabla_y+Q^\eta)1=0,\qquad (-\eta v\cdot\nabla_y+Q^{\eta*})(\psi^\eta\psi^{\eta*})=0,\qquad Y\times V, \tag{8.16}$$

where we have defined the rescaled transport operator

$$Q^\eta u(y,v)=\int_V\frac{\sigma(y,v',v)\psi^\eta(y,v')}{\psi^\eta(y,v)}\big[u(y,v)-u(y,v')\big]d\nu(v'). \tag{8.17}$$

The reason for introducing the operator $Q^\eta$ is that

$$(v\cdot\nabla_x+\frac{1}{\varepsilon}Q^\eta_\alpha+\varepsilon)u=\varepsilon F_\alpha(x,v):=\varepsilon F(x,\frac{x}{\alpha},v), \tag{8.18}$$

where we have defined

$$u(x,v)=\frac{f^\varepsilon_\alpha(x,v)}{\psi^\eta(\frac{x}{\alpha},v)},\qquad F(x,y,v)=\frac{S(x,y,v)}{\psi^\eta(y,v)}, \tag{8.19}$$

and

$$Q^\eta_\alpha u=\int_V\frac{\sigma(\frac{x}{\alpha},v',v)\psi^\eta(\frac{x}{\alpha},v')}{\psi^\eta(\frac{x}{\alpha},v)}\big[u(x,v)-u(x,v')\big]d\nu(v'). \tag{8.20}$$

Define the operator

$$T^\eta_\alpha=v\cdot\nabla_x+\frac{1}{\varepsilon}Q^\eta_\alpha. \tag{8.21}$$

Then we recast (8.18) as $(\varepsilon+T^\eta_\alpha)u=\varepsilon F_\alpha$. We verify that

$$\begin{aligned}\mathcal{Q}^\eta_\alpha(h) &:= (T^\eta_\alpha h,\psi^\eta_\alpha\psi^{\eta*}_\alpha h)\\ &= \frac{1}{\varepsilon}\int\sigma(\frac{x}{\alpha},v',v)\psi^\eta_\alpha(x,v')\psi^{\eta*}_\alpha(x,v)\frac{|h(x,v)-h(x,v')|^2}{2}dxd\nu(v)d\nu(v)'.\end{aligned} \tag{8.22}$$

This comes from the fact that $v\cdot\nabla_x(\psi^\eta_\alpha\psi^{\eta*}_\alpha)=\frac{1}{\varepsilon}Q^{\eta*}_\alpha(\psi^\eta_\alpha\psi^{\eta*}_\alpha)$ so that

$$\mathcal{Q}^\eta_\alpha(h)=\frac{1}{\varepsilon}(Q^\eta_\alpha(h)h-Q^\eta_\alpha\frac{h^2}{2},\psi^\eta_\alpha\psi^{\eta*}_\alpha).$$

Let $\rho=\rho(x)$ and $s=u-\rho$ for an arbitrary $\rho(x)$ independent of $v$. Then we find that $\mathcal{Q}^\eta_\alpha(\rho)=0$ and more importantly that $\mathcal{Q}^\eta_\alpha(u)=\mathcal{Q}^\eta_\alpha(s)$. Now for $s$ such that $\int_V sd\nu(v)=0$, we find that

$$\mathcal{Q}^\eta_\alpha(s)\geq\frac{\beta}{\varepsilon}\|s\|^2, \tag{8.23}$$

for some $\beta>0$ as is clear from (8.22) provided that $\psi^\eta_\alpha\psi^{\eta*}_\alpha$ is bounded from below by a positive constant uniformly in $\eta$. We thus define $\rho=\overline{u}$ and $s=u-\overline{u}$ so that $\overline{u}$ is independent of $v$ and $s$ mean zero in $v$. Multiplying (8.18) by $u\psi^\eta_\alpha\psi^{\eta*}_\alpha$ and integrating yields

$$\varepsilon(u,u\psi^\eta_\alpha\psi^{\eta*}_\alpha)+\mathcal{Q}^\eta_\alpha(s)=(\varepsilon F_\alpha,u\psi^\eta_\alpha\psi^{\eta*}_\alpha). \tag{8.24}$$

As a consequence,

$$\varepsilon\|u\|^2+\frac{1}{\varepsilon}\|s\|^2\leq|(\varepsilon F_\alpha,s\psi^\eta_\alpha\psi^{\eta*}_\alpha)|+|(\varepsilon\overline{F_\alpha\psi^\eta_\alpha\psi^{\eta*}_\alpha},\rho)|. \tag{8.25}$$

From this, we deduce the a priori estimate

$$\|u\|+\frac{1}{\varepsilon}\|u-\overline{u}\|\leq C\Big(\|\varepsilon F_\alpha\|+\|\overline{F_\alpha\psi^\eta_\alpha\psi^{\eta*}_\alpha}\|\Big). \tag{8.26}$$

In the variables $f^\varepsilon_\alpha$, this is equivalent to (8.14). □

## 8.4 Expansion in $\varepsilon$

To emphasize the dependency in $\eta$, let us denote $f^{\varepsilon,\eta} := f^{\varepsilon}_{\alpha}$ the solution of (8.1). As in the standard derivation of diffusion approximations, we first expand $f^{\varepsilon,\eta}$ in powers of $\varepsilon$ at a fixed (arbitrary) value of $\eta$. We prove the following proposition.

*Proposition.* The solution $f^{\varepsilon,\eta}$ can be expanded as follows

$$f^{\varepsilon,\eta} = n^{0,\eta}(x)\psi^{\eta}(\frac{\eta x}{\varepsilon}, v) + \varepsilon f^{1,\eta}(x, \frac{\eta x}{\varepsilon}, v) + \varepsilon^2 f^{2,\eta}(x, \frac{\eta x}{\varepsilon}, v) + \varepsilon^3 f^{3,\eta}(x, \frac{\eta x}{\varepsilon}, v) + r^{\varepsilon,\eta}(x, v) \tag{8.27}$$

where we have defined :

$$\begin{aligned} f^{1,\eta} &= T^{\eta-1}(-v\cdot\nabla_x(n^{0,\eta}\psi^{\eta})) + n^{1,\eta}\psi^{\eta} \\ f^{2,\eta} &= T^{\eta-1}(-v\cdot\nabla_x n^{1,\eta}\psi^{\eta}) + \overline{f}^{2,\eta} \\ \overline{f}^{2,\eta} &= T^{\eta-1}(S - v\cdot\nabla_x T^{\eta-1}(n^{0,\eta}\psi^{\eta}) - f^{0,\eta}) \\ f^{3,\eta} &= T^{\eta-1}(-v\cdot\nabla_x f^{2,\eta} - f^{1,\eta}). \end{aligned}$$

The operator $T^{\eta-1}$ is defined in (8.12). The density $n^{0,\eta}$ satisfies the diffusion equation

$$n^{0,\eta} - \nabla_x \cdot \int_{Y\times V} v\chi^{\eta}(y,v)\psi^{*}(v)d\nu(v)dy\,\nabla_x n^{0,\eta} = \int_{Y\times V} S(x,y,v)\psi^{\eta *}(v)d\nu(v)dy \tag{8.28}$$

where $\chi^{\eta} = T^{\eta-1}(v\psi^{\eta})$ and the density for the correcting term $n^{1,\eta}$ satisfies

$$n^{1,\eta} - \nabla_x \cdot \int_{Y\times V} v\chi^{\eta}(y,v)\psi^{*}(v)d\nu(v)dy\,\nabla_x n^{1,\eta} = -\int_{Y\times V} v\cdot\nabla_x \overline{f}^{2,\eta}(x,y,v)\psi^{*}(v)d\nu(v)dy. \tag{8.29}$$

The remainder term $r^{\varepsilon,\eta}$ satisfies the following estimate :

$$\begin{aligned} ||r^{\varepsilon,\eta}||_{L^2_{xv}} \quad \leq \quad & \varepsilon^2||\overline{v\cdot\nabla_x f^{3,\eta}\psi^{\eta *}}||_{L^2_x L^\infty_y L^2_v} + \varepsilon^3||v\cdot\nabla_x f^{3,\eta}||_{L^2_x L^\infty_y L^2_v} \\ & + \varepsilon^2||f^{2,\eta}||_{L^2_x L^\infty_y L^2_v} + \varepsilon^3||f^{3,\eta}||_{L^2_x L^\infty_y L^2_v}. \end{aligned} \tag{8.30}$$

*Démonstration.* We propose the following ansatz for $f^{\varepsilon,\eta}$ :

$$f^{\varepsilon,\eta} = f^{0,\eta}(x, \frac{\eta x}{\varepsilon}, v) + \varepsilon f^{1,\eta}(x, \frac{\eta x}{\varepsilon}, v) + \varepsilon^2 f^{2,\eta}(x, \frac{\eta x}{\varepsilon}, v) + \varepsilon^3 f^{3,\eta}(x, \frac{\eta x}{\varepsilon}, v) + r^{\varepsilon,\eta}(x, v).$$

We plug the ansatz into the transport equation and equate like powers of $\varepsilon$ to obtain the following sequence of equations. At the leading order, we obtain

$$T^{\eta} f^{0,\eta} = 0 \quad \text{which yields that} \quad f^{0,\eta} = n^{0,\eta}(x)\psi^{\eta}(y,v).$$

We recall that $T^{\eta} = \eta v\cdot\nabla_y + Q$. At the next order, $f^{1,\eta}$ satisfies

$$T^{\eta} f^{1,\eta} = -v\cdot\nabla_x f^{0,\eta}(x,y,v) = -\psi^{\eta}(y,v)v\cdot\nabla_x n^{0,\eta}(x).$$

We can then rewrite $f^{1,\eta}$ as

$$f^{1,\eta}(x,y,v) = -\chi^{\eta}(y,v)\cdot\nabla_x n^{0,\eta}(x) + n^{1,\eta}(x)\psi^{\eta}(y,v)$$

by defining $\chi^{\eta} = T^{\eta,-1}(v\psi^{\eta})$, the unique solution to

$$T^{\eta}\chi^{\eta} = v\psi^{\eta} \quad \text{such that} \quad \int_V\int_Y \chi^{\eta}(y,v)\psi^{*}(v)dyd\nu(v) = 0.$$

We can apply the inverse operator $T^{\eta,-1}$ defined in (8.12) to the source term $v\psi^{\eta}$ because the following standard *no-drift* condition is satisfied :

$$\int_V\int_Y v\psi^{\eta}(y,v)\psi^{*}(v)d\nu(v)dy = 0,$$

thanks to assumption (H4).

The second-order equation reads

$$T^\eta f^{2,\eta} = -f^{0,\eta}(x,y,v) + S(x,y,v) - v\cdot\nabla_x f^{1,\eta}(x,y,v)$$

and the compatibility equation that the right-hand side must satisfy to be in the domain of definition of $T^{\eta,-1}$ gives the diffusion equation (8.28) for $n^{0,\eta}$.

Introduce $\overline{f}^{2,\eta}$ the solution to

$$T^\eta \overline{f}^{2,\eta} = -f^{0,\eta}(x,y,v) + S(x,y,v) - v\cdot\nabla_x(-\chi^\eta(y,v)\cdot\nabla_x n^{0,\eta}(x)).$$

The third-order equation is the following

$$T^\eta f^{3,\eta} = -v\cdot\nabla_x f^{2,\eta}(x,y,v) - f^{1,\eta}(x,y,v).$$

and the corresponding compatibility equation gives a diffusion equation for $n^{1,\eta}$

$$n^{1,\eta}(x) - \nabla_x\cdot\int_{Y\times V} v\chi^\eta(y,v)\psi^*(v)d\nu(v)dy\nabla_x n^{1,\eta}(x) = -\int_{Y\times V} v\cdot\nabla_x\overline{f}^{2,\eta}(x,y,v)\psi^*(v)d\nu(v)dy$$

since

$$\int_Y\int_V \psi^*(v)\chi^\eta(y,v)\cdot\nabla_x n^{0,\eta}(x) = 0.$$

With the above expressions, we verify that remainder term now satisfies that :

$$r^{\varepsilon,\eta} + \frac{1}{\varepsilon}v\cdot\nabla_x r^{\varepsilon,\eta} + \frac{1}{\varepsilon^2}Q(r^{\varepsilon,\eta}) = -\varepsilon^2 f^{2,\eta}(x,\frac{\eta x}{\varepsilon},v) - \varepsilon^3 f^{3,\eta}(x,\frac{\eta x}{\varepsilon},v) - \varepsilon^2 v\cdot\nabla_x f^{3,\eta}(x,\frac{\eta x}{\varepsilon},v).$$

Thanks to the estimate (8.14) proved in the previous section, we obtain (8.30). □

In order to analyze the behavior of the terms $f^{0,\eta}$, $f^{1,\eta}$, $f^{2,\eta}$ and $f^{3,\eta}$ as $\eta\to 0$, we need to study some auxiliary equations. Such studies are conducted in the following section.

## 8.5 Expansion of the auxiliary functions

This section is devoted to the asymptotic expansion as $\eta\to 0$ of the three cell functions $\psi^\eta$, $\chi^\eta$, and $\chi^{\eta *}$.

### 8.5.1 Expansion of $\psi^\eta$

Recall that $\psi^\eta$ satisfies

$$\eta v\cdot\nabla_y\psi^\eta + Q(\psi^\eta) = 0 \quad\text{and}\quad \int_Y\int_V \psi^\eta(y,v)\psi^*(v)d\nu(v)dy = 1.$$

We prove the following expansion

*Proposition.* The function $\psi^\eta$, solution to (8.4) satisfies

$$\psi^\eta(y,v) = \psi^0(y,v) + \eta\psi^1(y,v) + \eta^2\psi^2(y,v) + \tilde{r}_p^\eta(y,v), \quad ||\tilde{r}_p^\eta||_{H^k(Y,L^2(V))} \le C\eta^3, \tag{8.31}$$

with

$$\begin{array}{rcl} \psi^0(y,v) &=& \rho^0(y)\psi(y,v), \\ \psi^1(y,v) &=& Q^{-1}(-v\cdot\nabla_y\psi^0(y,v)) \\ \psi^2(y,v) &=& Q^{-1}(-v\cdot\nabla_y\psi^1(y,v)) + \rho^2(y)\psi(y), \end{array} \tag{8.32}$$

where

$$L(\rho^0) = 0 \text{ with } \int\rho^0(y)dy = 1 \tag{8.33}$$

$$\rho^2 = -L^{-1}\left(\int_V \psi^*(v)v\cdot\nabla_y(Q^{-1}(-v\cdot\nabla_y(Q^{-1}(-v\cdot\nabla_y\psi^1(y,v)))))d\nu(v)\right). \tag{8.34}$$

The operator $L$ and its inverse defined on functions in $L^2(Y)$ with vanishing average over $Y$ are defined in Proposition 8.8 in the appendix. From this expansion, we deduce that :

$$||\psi^\eta||_{L^2_v L^\infty_y} \leq C \quad \text{and} \quad ||\int_V v\psi^\eta \psi^* d\nu(v)||_{L^\infty_y} \leq C\eta.$$

Moreover, we find that $\psi^0$ and $\psi^2$ are even functions and $\psi^1$ is an odd function in the variable $v$.

*Démonstration.* The plug the ansatz into the equation for $\psi^\eta$ and equate like powers of $\eta$. The resulting series of equations is the following. First we have

$$Q(\psi^0) = 0,$$

which implies that $\psi^0 = \rho^0(y)\psi(y,v)$. Next, we obtain

$$Q(\psi^1) = -v \cdot \nabla_y(\rho^0(y)\psi(y,v)).$$

Since thanks to assumption (H4) the no-drift condition $\int_V v\psi(y,v)\psi^*(v)d\nu(v) = 0$ is satisfied, the compatibility condition for the previous equation is fulfilled and we can solve the equation for $\psi^1$ to obtain

$$\psi^1 = Q^{-1}(-v \cdot \nabla_y(\rho^0(y)\psi(y,v))).$$

The compatibility condition for the third equation gives the equation for $\rho^0$ since for

$$Q(\psi^2) = -v \cdot \nabla_y(\psi^1(y,v)) \quad \text{i.e.,} \quad \psi^2 = Q^{-1}(-v \cdot \nabla_y \psi^1(y,v)) + \rho^2(y)\psi(y,v)$$

to have a solution, we need

$$\int_V \psi^*(v)v \cdot \nabla_y(\psi^1(y,v))d\nu(v) = 0.$$

This relation is satisfied if

$$L(\rho^0) = 0 \quad \text{with} \quad L(\rho) = -\int_V \psi^*(v)v \cdot \nabla_y(Q^{-1}(v \cdot \nabla_y(\psi(y,v)\rho(y))))d\nu(v).$$

The precise study of $L$ and its adjoint $L^*$ defined by

$$L^*(\rho) = -\int_V \psi(y,v)v \cdot \nabla_y(Q^{*-1}(v \cdot \nabla_y(\psi^*(v)\rho))d\nu(v)$$

is conducted in the appendix. The next equation is

$$Q(\psi^3) = -v \cdot \nabla_y(\psi^2).$$

The corresponding compatibility equation is satisfied since $\psi^2$ is an even function of $v$. Then we write

$$Q(\psi^4) = -v \cdot \nabla_y(\psi^3).$$

Its compatibility condition gives the following equation for $\rho^2$ :

$$L(\rho^2) = -\int_V \psi^*(v)v \cdot \nabla_y(Q^{-1}(-v \cdot \nabla_y(Q^{-1}(-v \cdot \nabla_y \psi^1(y,v)))))d\nu(v). \tag{8.35}$$

Recall that $L^*(1) = 0$. By the Fredholm alternative, equation (8.35) thus admits a unique mean zero solution since

$$\int_Y \int_V \psi^*(v)v \cdot \nabla_y(Q^{-1}(-v \cdot \nabla_y(Q^{-1}(-v \cdot \nabla_y \psi^1(y,v)))))d\nu(v)dy = 0.$$

The remainder term $r_p^\eta$ satisfies

$$\eta v \cdot \nabla r_p^\eta + Q(r_p^\eta) = -\eta^5 v \cdot \nabla_y(\psi^4)$$

and if we impose $\int_y \rho^0(y)dy = 1$ and $\int_Y \rho^2(y)dy = 0$, we get

$$\int_Y \int_V r_p^\eta(y,v)\psi^*(v)d\nu(v)dy = 0.$$

Then using the technical regularity estimate (8.51) presented and proved in the appendix below, we obtain

$$||r_p^\eta||_{H^k(Y),L^2(V))} \leq \eta^3 ||\psi^4||_{H^{k+1}(Y,L^2(V))}.$$

The remainder involved in the proposition is given by $\tilde{r}_p^\eta = \eta^3\psi^3 + \eta^4\psi^4 + r_p^\eta$ and then satisfies

$$||\tilde{r}_p^\eta||_{H^k(Y),L^2(V))} \leq \eta^3.$$

By taking $k$ large enough, we get the desired $L_y^\infty$ bounds. □

### 8.5.2 Expansion of $\chi^\eta$

We want to expand $\chi^\eta$ satisfying

$$\eta v \cdot \nabla_y \chi^\eta + Q(\chi^\eta) = v\psi^\eta \quad \text{and} \quad \int_Y \int_V \chi^\eta(y,v)\psi^*(v)d\nu(v)dy = 0.$$

*Proposition.* We prove the following expansion

$$\chi^\eta(y,v) = \frac{1}{\eta}\theta^{-1}(y)\psi(y,v) + Q^{-1}(v\psi)\rho^0(y) - Q^{-1}(v\cdot\nabla_y(\theta^{-1}\psi(y,v))) + \tilde{r}_\chi^\eta(y,v),$$

with $||\tilde{r}_\chi^\eta||_{H^k(Y,L^2(V))} \leq C\eta$, where we have

$$L(\rho^0) = 0 \quad \text{with} \quad \int_Y \rho^0(y)dy = 1$$
$$L\theta^{-1} = \int_V (vQ^{-1}(-v\cdot\nabla_y\rho^0(y)\psi(y,v))\psi^*(v) - v\psi^*(v)\cdot\nabla_y(Q^{-1}(v\psi)(y,v)\rho^0(y)))d\nu(v).$$

Moreover

$$||\chi^\eta||_{L_v^2 L_y^\infty} \leq \frac{C}{\eta} \quad \text{and} \quad ||\int_V v\chi^\eta(y,v)\psi^*(v)d\nu(v)||_{L_y^\infty} \leq C. \tag{8.36}$$

*Démonstration.* In view of estimate (8.52) obtained in the appendix below, we expect the expansion of $\chi^\eta$ to start at the order $\frac{1}{\eta}$. Thus, we begin with

$$Q(\chi^{-1}) = 0 \quad \text{or} \quad \chi^{-1}(y,v) = \theta^{-1}(y)\psi(y,v),$$

Then at the order $O(\eta^0)$ we get

$$Q(\chi^0) = v\psi^0 - v\cdot\nabla_y\chi^{-1}.$$

Thanks to the no-drift condition and (H4), the compatibility condition for this equation is satisfied. Next, we write

$$Q(\chi^1) = v\psi^1 - v\cdot\nabla_y\chi^0 \quad \text{i.e.,} \quad \chi^1 = Q^{-1}(v\psi^1 - v\cdot\nabla_y\chi^0)(y,v) + \theta^1(y)\psi(y,v).$$

The compatibility condition for the equation giving $\chi^1$ gives the expression of $\theta^{-1}$ since it satisfies the following elliptic equation

$$L(\theta^{-1}) = \int_V (v\psi^1(y,v)\psi^*(v) - v\psi^*(v)\cdot\nabla_y(\chi(y,v)\rho^0(y)))d\nu(v).$$

This equation has a solution since

$$\int_Y \int_V (v\psi^1(y,v)\psi^*(v) - v\psi^*(v)\cdot\nabla_y(\chi(y,v)\rho^0(y)))d\nu(v)dy = 0$$

because $\psi^1$ defined in (8.32) is odd in $y$ as a gradient of an even function and the second contribution can be written as a divergence in $y$ and hence averages to 0 over the cell $Y$. Note that $\theta^{-1}$ is odd in the $y$ variable and that therefore

$$\int_Y \int_V \chi^{-1}(y,v)\psi^*(v)d\nu(v)dy = 0.$$

Finally, we write

$$Q(\chi^2) = v\psi^2 - v\cdot\nabla_y\chi^1$$

for which the compatibility condition holds since $\psi^2$ is even with respect to $v$ and $\chi^1$ is even. Indeed, $\psi^1$ is odd with repect to $v$ (even with respect to $(y,v)$) and so is $\chi^0$.

The remainder term satisfies

$$\eta v\cdot\nabla_y r_\chi^\eta + Q(r_\chi^\eta) = -\eta^3 v\cdot\nabla_y\chi^2 + vr_p^\eta$$

We chose $\theta^1$ such that $\int_V(-\eta^3 v\cdot\nabla_y\chi^2(y,v) + vr_p^\eta(y,v))\psi^*(v)d\nu(v) = 0$ for any $y\in Y$. Note that it is possible since $\int_V\int_Y(-\eta^3 v\cdot\nabla_y\chi^2 + vr_p^\eta)\psi^* d\nu(v)dy = 0$ because $r_p^\eta$ is even with respect to $(y,v)$ whereas $\chi^2$ is odd. The fact that $\chi^2$ and $vr_p^\eta$ are odd with respect to $(y,v)$ implies that $\theta^1$ is odd and therefore $\int_Y\int_V\chi^1(y,v)\psi^*(v)d\nu(v)dy = 0$ which finally gives

$$\int_Y\int_V r_\chi^\eta(y,v)\psi^*(v)d\nu(v)dy = 0.$$

By using estimate (8.51), since $||-\eta^3 v\cdot\nabla_y\chi^2 + vr_p^\eta||_{H^k(Y,L^2(V))} \le C\eta^2$ then

$$||r_\chi^\eta||_{H^k(Y,L^2(V))} \le C\eta.$$

This concludes the proof of the result. □

### 8.5.3 Expansion of $\chi^{\eta*}$

*Proposition.* Let $\chi^{\eta*}$ be the solution of

$$-\eta v\cdot\nabla_y\chi^{\eta*} + Q^*(\chi^{\eta*}) = v\psi^*.$$

The following expansion holds

$$\chi^{\eta*} = \frac{1}{\eta}\theta^{*-1}(y)\psi^*(v) + \chi^{*0}(y,v) + \eta\chi^{*1}(y,v) + \tilde{r}_{\chi^*}^\eta(y,v)$$

with $||\tilde{r}_{\chi^*}^\eta||_{H^k(Y,L^2(V))} \le C\eta^2$, where

$$\chi^{*0}(y,v) = Q^{*-1}(v\psi^* + v\cdot\nabla_y\theta^{*-1}\psi^*)(y,v) + \theta^{*0}(y)\psi^*(v) \tag{8.37}$$

$$L^*(\theta^{*-1}) = \int_V \psi(y,v)v\cdot\nabla_y Q^{*-1}(v\psi^*(v))d\nu(v), \tag{8.38}$$

$$L^*(\theta^{*0}) = \int_V \psi(y,v)v\cdot\nabla_y Q^{*-1}(v\cdot\nabla_y Q^{*-1}(v\psi^* + v\cdot\nabla_y\chi^{*-1}))d\nu(v), \tag{8.39}$$

$$\chi^{*1}(y,v) = Q^{*-1}(v\cdot\nabla_y\chi^{*0}(y,v)) + \theta^{*1}(y)\psi^*(y,v), \tag{8.40}$$

$$L^*(\theta^{*1}) = \int_V \psi(y,v)v\cdot\nabla_y Q^{*-1}(v\cdot\nabla_y Q^{*-1}(\chi^{*0}))d\nu(v). \tag{8.41}$$

*Démonstration.* The proof is a straightforward adaptation of the proof of the same result in [9]. We briefly sketch it to be self-contained. As in the expansion of $\chi^\eta$, we start at the order $\frac{1}{\eta}$ and write

$$\chi^{\eta*} = \frac{1}{\eta}\chi^{*-1} + \chi^{*0} + \eta\chi^{*1} + \eta^2\chi^{*2} + \eta^3\chi^{*3} + r_{\chi^*}^\eta$$

with

$$Q^*(\chi^{*-1}) = 0 \text{ so that } \chi^{*-1}(y,v) = \theta^{*-1}(y)\psi^*(v).$$

Then we have

$$Q^*(\chi^{*0}) = v\psi^* + v \cdot \nabla_y \chi^{*-1}$$

which admits a solution thanks to the no-drift condition given by

$$\chi^{*0}(y,v) = Q^{*-1}(v\psi^* + v \cdot \nabla_y \chi^{*-1})(y,v) + \theta^{*0}(y)\psi^*(v).$$

The first order equation is

$$Q^*(\chi^{*1}) = v \cdot \nabla_y \chi^{*0}.$$

The compatibility equation for this equation gives

$$L^*(\theta^{*-1}) := -\int_V \psi(y,v) v \cdot \nabla_y Q^{*-1}(v \cdot \nabla_y(\psi^*(v)\theta^{*-1}) d\nu(v) = \int_V \psi(y,v) v \cdot \nabla_y Q^{*-1}(v\psi^*(v)) d\nu(v).$$

This equation has a solution since

$$\int_Y \rho^0(y) \int_V \psi(y,v) v \cdot \nabla_y Q^{*-1}(v\psi^*(v)) d\nu(v) dy = 0$$

because $\rho^0\psi$ is even with respect to the variable $(y,v)$ whereas $v\psi^*$ is odd. Then $\chi^{*1}$ can be written as

$$\chi^{*1}(y,v) = Q^{*-1}(v \cdot \nabla_y \chi^{*0}(y,v)) + \theta^{*1}(y)\psi^*(y,v).$$

The compatibility condition for

$$Q^*(\chi^{*2}) = v \cdot \nabla_y \chi^{*1}$$

gives the equation determining $\theta^{*0}$ since we need

$$L^*(\theta^{*0}) = \int_V \psi(y,v) v \cdot \nabla_y Q^{*-1}(v \cdot \nabla_y Q^{*-1}(v\psi^* + v \cdot \nabla_y \chi^{*-1})) d\nu(v).$$

Note that the parity properties ensure that the previous equation has a solution. The final equation is

$$Q^*(\chi^{*3}) = v \cdot \nabla_y \chi^{*2}$$

whose compatibility condition gives the equation for $\theta^{*1}$

$$L^*(\theta^{*1}) = \int_V \psi(y,v) v \cdot \nabla_y Q^{*-1}(v \cdot \nabla_y Q^{*-1}(\chi^{*0})) d\nu(v).$$

The remainder term satisfies

$$\eta v \cdot \nabla_y r^\eta_{\chi^*} - Q^*(r^\eta_{\chi^*}) = \eta^4 v \cdot \nabla \chi^{*3}.$$

As for $\chi^\eta$, $\theta^{*i}$ are odd for $i = -1, 0, 1$ and then, $\int_V \int_Y r^\eta_{\chi^*} \psi^* d\nu(v) dy = 0$, and then, thanks to estimate (8.55), we get

$$||r^\eta_{\chi^*}||_{H^k(Y,L^2(V))} \leq C\eta^2$$

which concludes the proof. □

## 8.6 Estimates on the densities

We are now ready to analyze the expansion in $\eta$ of the densities $n^{k,\eta}$ for $k = 0, 1$ obtained in Proposition 8.4. We start with the following simple lemma :

*Lemma* 8.6.1. Let $n^\eta$ be the solution of

$$n^\eta(x) - \nabla_x \cdot (D^\eta \cdot \nabla_x n^\eta(x)) = S^\eta(x)$$

with

$$\begin{aligned} D^\eta &= D^0 + \eta D^1 + \eta \tilde{D}^\eta, \quad \text{where} \quad |||\tilde{D}^\eta||| \to_{\eta \to 0} 0, \quad ||| \cdot ||| \text{ a norm on } d \times d \text{ matrices} \\ S^\eta &= S^0 + \eta S^1 + \eta \tilde{S}^\eta \quad \text{where} \quad S^0 \in H^k(\mathbb{R}^d),\ S^1 \in H^k(\mathbb{R}^d) \text{ and } ||\tilde{S}^\eta||_{H^k(\mathbb{R}^d)} \to_{\eta \to 0} 0. \end{aligned}$$

We decompose

$$n^\eta = n^0 + \eta n^1 + \eta \tilde{n}^\eta$$

with

$$n^0(x) - \nabla_x \cdot (D^0 \nabla_x n^0(x)) = S^0(x), \quad ||n^0||_{H^{k+1}(\mathbb{R}^d)} \leq C||S^0||_{H^k(\mathbb{R}^d)}$$

and

$$n^1(x) - \nabla_x \cdot (D^0 \nabla_x n^1(x)) = S^1(x) + \nabla_x \cdot (D^1 \cdot \nabla_x n^0(x)), \ ||n^1||_{H^{k+1}(\mathbb{R}^d)} \leq C||S^0||_{H^k(\mathbb{R}^d)} + ||S^1||_{H^k(\mathbb{R}^d)}.$$

Then $\tilde{n}^\eta$ satisfies

$$||\tilde{n}^\eta||_{H^{k+1}(\mathbb{R}^d)} \to 0 \quad \text{when} \quad \eta \to 0.$$

*Démonstration.* Indeed $\tilde{n}^\eta$ satisfies the equation

$$\tilde{n}^\eta - \nabla_x \cdot (D^0 \nabla_x \tilde{n}^\eta) - \nabla_x \cdot [(\eta D^1 + \eta \tilde{D}^\eta) \nabla_x \tilde{n}^\eta] = \tilde{S}^\eta + \nabla_x \cdot [\tilde{D}^\eta \nabla_x n^0 + (\eta D^1 + \eta \tilde{D}^\eta) \nabla_x n^1].$$

A standard a priori estimate shows that $||\tilde{n}^\eta||_{H^1(\mathbb{R}^d)} \to 0$ when $\eta \to 0$. Upon differentiating the equation $k$ times, we obtain the result in $H^{k+1}(\mathbb{R}^d)$. □

We apply this lemma to derive a convergence result for $n^{0,\eta}$ and $n^{1,\eta}$.

*Proposition.* Assume $S \in H^k(\mathbb{R}^d)$. First, we have that

$$||n^{0,\eta} - n^{0,0} - \eta n^{0,1}||_{H^{k+1}(\mathbb{R}^d)} \to 0$$

where $n^{0,0} \in H^{k+1}(\mathbb{R}^d)$ is solution to

$$n^{0,0} - \nabla_x \cdot (\mathbf{D} \cdot \nabla_x n^{0,0}) = \int_Y \int_V S(x,y,v) \psi^*(v) dy d\nu(v)$$

and $n^{0,1} \in H^{k+1}(\mathbb{R}^d)$ is solution to

$$n^{0,1} - \nabla_x \cdot (\mathbf{D} \cdot \nabla_x n^{0,1}) = \nabla_x \cdot (\mathbf{D^1} \cdot \nabla_x n^{0,0}),$$

$$\begin{aligned}
\mathbf{D} &= \int_V \int_Y (\chi^{*0}(y,v) \otimes v\psi^0(y,v) + \chi^{*-1}(y,v) \otimes v\psi^1(y,v)) d\nu(v) dy \\
&= \int_V \int_Y Q^{*-1}(v\psi^* + v \cdot \nabla_y \theta^{*-1} \psi^*)(y,v) \otimes v\rho^0(y)\psi(y,v) \\
&\qquad + \theta^{*-1}(y)\psi^*(v) \otimes vQ^{-1}(-v \cdot \nabla_y(\rho^0(y)\psi(y,v))) d\nu(v) dy
\end{aligned}$$

$$\begin{aligned}
\mathbf{D^1} &= \int_V \int_Y (\chi^{*-1}(y,v) \otimes v\psi^2(y,v) + \chi^{*0}(y,v) \otimes v\psi^1(y,v) + \chi^{*1}(y,v) \otimes v\psi^0(y,v)) d\nu(v) dy \\
&= \int_V \int_Y (\theta^{*-1}(y)\psi^*(v) \otimes vQ^{-1}(-v \cdot \nabla_y(Q^{-1}(-v \cdot \nabla_y(\rho^0(y)\psi(y,v)))))) d\nu(v) dy \\
&\quad \int_V \int_Y [Q^{*-1}(v\psi^* + v \cdot \nabla_y \theta^{*-1}\psi^*) + \theta^{*0}\psi^*] \otimes vQ^{-1}(-v \cdot \nabla_y(\rho^0(y)\psi(y,v))) d\nu(v) dy \\
&\quad \int_V \int_Y Q^{*-1}(v \cdot \nabla_y Q^{*-1}(v\psi^* + v \cdot \nabla_y \theta^{*-1}\psi^*) + \theta^{*1}(y)\psi^*(y,v)) \otimes v\rho^0(y)\psi(y,v) d\nu(v) dy.
\end{aligned}$$

Second, we have that

$$||n^{1,\eta} - \frac{1}{\eta} n^{1,-1}||_{H^{k-1}(\mathbb{R}^d)} \to_{\eta \to 0} 0 \tag{8.42}$$

with $n^{1,-1} \in H^{k-1}(\mathbb{R}^d)$ the solution of

$$n^{1,-1} - \nabla_x \cdot (\mathbf{D} \cdot \nabla_x n^{1,-1}) = S^{1,-1}(x) \tag{8.43}$$

where $S^{1,-1} \in H^{k-2}(\mathbb{R}^d)$ is given by

$$\begin{aligned}
S^{1,-1}(x) &= -\nabla_x \cdot \int_{Y \times V} [-n^{0,0}(x)\psi(y,v) + S(x,y,v) + v \cdot \nabla_x(\chi^0(y,v) \cdot \nabla_x n^{0,0}(x))] \chi^{*-1}(y,v) d\nu(v) dy \\
&\quad -\nabla_x \cdot \int_{Y \times V} [-v \cdot \nabla_x(-\chi^{-1}(y,v) \cdot \nabla_x n^{0,0}(x))] \chi^{*0}(y,v) d\nu(v) dy.
\end{aligned}$$

*Démonstration.* Recall that $n^{0,\eta}$ satisfies

$$n^{0,\eta} + \nabla_x \cdot (D^\eta \cdot \nabla_x n^{0,\eta}) = \int_Y \int_V S(x,y,v)\psi^*(v)d\nu(v)dy.$$

By using the definition of $D^\eta = \int_V \int_Y \chi^{\eta*}(y,v) \otimes v\psi^\eta(v)dyd\nu(v)$ and the expansions of $\psi^\eta$ and $\chi^{\eta*}$ obtained in the preceding section, we obtain that

$$D^\eta = \mathbf{D} + \eta\mathbf{D^1} + \eta\tilde{D}^\eta \quad \text{with} \quad |||\tilde{D}^\eta||| \to_{\eta\to 0} 0.$$

Then Lemma 8.6.1 gives the first result. On the another hand, $n^{1,\eta}$ satisfies

$$n^{1,\eta} - \nabla_x \cdot (D^\eta \cdot \nabla_x n^{1,\eta}) = S^{1,\eta}(x)$$

with

$$\begin{aligned}
S^{1,\eta}(x) &= -\nabla_x \cdot \int_Y \int_V (\overline{f}^{2,\eta}(x,y,v))v\psi^{\eta*}(v)d\nu(v)dy \\
&= -\nabla_x \int_Y \int_V \overline{f}^{2,\eta} T^{\eta*}\chi^{\eta*}(y,v)d\nu(v)dy = -\nabla_x \int_Y \int_V T^\eta \overline{f}^{2,\eta}\chi^{\eta*}(y,v)d\nu(v)dy \\
&= -\nabla_x \int_{Y\times V} [-n^{0,\eta}(x)\psi^\eta(y,v) + S(x,y,v) + v\cdot\nabla_x(\chi^\eta(y,v)\cdot\nabla_x n^{0,\eta}(x))]\chi^{\eta*}(y,v)d\nu(v)dy \\
&= \frac{1}{\eta}S^{1,-1}(x) + o(\frac{1}{\eta}).
\end{aligned}$$

Since $S^{1,\eta}$ is of order $\frac{1}{\eta}$, we apply Lemma 8.6.1 to $\eta^2 n^{1,\eta}$ to obtain (8.42). This concludes the proof of the result. □

## 8.7 Proof of Theorem 8.2.1

We are now in a position to conclude the proof of our main result. We first obtain an estimate for the remainder term $r^{\varepsilon,\eta}$ in (8.27).

*Proposition.* The different terms involved in the estimate of the remainder term $r^{\varepsilon,\eta}$ are bounded as follows

$$||f^{0,\eta}||_{H^2_x L^2_v L^\infty_y} \leq C \tag{8.44}$$

$$||f^{k,\eta}||_{H^{4-k}_x L^2_v L^\infty_y} \leq \frac{C}{\eta^k} \quad \text{and} \quad ||\overline{v\cdot\nabla_x f^{k,\eta}\psi^*}||_{H^{3-k}_x L^2_v L^\infty_y} \leq \frac{C}{\eta^{k-1}} \quad 1 \leq k \leq 3. \tag{8.45}$$

These result yield that

$$||r^{\varepsilon,\eta}||_{L^2_{x,v}} = o(\frac{\varepsilon}{\eta}).$$

*Démonstration.* Let us start with $f^{0,\eta}$. Recall that $f^{0,\eta}(x,y,v) = n^{0,\eta}(x)\psi^\eta(y,v)$ and that

$$||\psi^\eta||_{L^2_v L^\infty_y} \leq C \quad \text{and} \quad ||n^{0,\eta}||_{H^2_x} \leq C.$$

This gives (8.44). Concerning $f^{1,\eta} = -\chi^\eta(y,v)\cdot\nabla_x n^{0,\eta}(x) + n^{1,\eta}(x)\psi^\eta(y,v)$, we get

$$||f^{1,\eta}||_{H^3_x L^2_v L^\infty_y} \leq C[||n^{0,\eta}||_{H^4_x}||\chi^\eta||_{L^2_v L^\infty_y} + ||n^{1,\eta}||_{H^3_x}||\psi^\eta||_{L^2_v L^\infty_y}]$$

and

$$||\overline{v\cdot\nabla_x f^{1,\eta}\psi^*}||_{H^2_x L^2_v L^\infty_y} \leq ||n^{0,\eta}||_{H^4_x}|||\overline{v\chi^\eta\psi^*}||_{H^2_x L^2_v L^\infty_y} + ||n^{1,\eta}||_{H^3_x}||\overline{v\psi^\eta\psi^*}||_{H^2_x L^2_v L^\infty_y} \leq C.$$

This yields (8.45) for $k = 1$. Let us now consider $f^{2,\eta}$. Since $\int_V \int_Y f^{2,\eta}\psi^*(v)d\nu(v)dy = 0$, $||T^\eta(f^{2,\eta})||_{H^2_x L^2_v L^\infty_y} \leq \frac{C}{\eta}$ and $||\int_V T^\eta(f^{2,\eta})\psi^*(v)d\nu(v)dv||_{H^2_x L^\infty_y} \leq C$, we get the first point of (8.45) for $k = 2$. Indeed, we decompose

$$f^{2,\eta} = T^{\eta-1}[\int_V T^\eta(f^{2,\eta})\psi^*(v)d\nu(v)dv\psi(y,v)] + T^{\eta-1}[T^\eta(f^{2,\eta}) - \int_V T^\eta(f^{2,\eta})\psi^*(v)d\nu(v)dv\psi(y,v)]$$

and use (8.51) and (8.52), respectively, obtained in the appendix below.

Concerning the second point, we write after defining $\chi^* = Q^{*-1}(v\psi^*)$,

$$\begin{aligned}\overline{v\cdot\nabla_x f^{2,\eta}\psi^{\eta *}} &= \int_V v\cdot\nabla_x f^{2,\eta}\psi^* d\nu(v) = \int_V \nabla_x f^{2,\eta}\cdot Q^*(\chi^*(y,v))d\nu(v)\\ &= \int_V \nabla_x Q(f^{2,\eta})\cdot\chi^*(y,v)d\nu(v)\\ &= \int_V \nabla_x(-\eta v\nabla_y f^{2,\eta} - v\nabla_x f^{1,\eta}(x,y,v) - f^{0,\eta} + S)\cdot\chi^*(y,v)d\nu(v),\end{aligned}$$

so that

$$||\overline{v\cdot\nabla_x f^{2,\eta}\psi^{\eta *}}||_{H^1_x L^2_v L^\infty_y} \leq C[\eta||f^{2,\eta}||_{H^2_x L^2_v L^\infty_y} + ||f^{1,\eta}||_{H^3_x L^2_v L^\infty_y} + ||f^{0,\eta}||_{H^2_x L^2_v L^\infty_y} + ||S||_{H^2_x L^2_v L^\infty_y}]$$

is bounded by $C\eta^{-1}$. In a same way, since $\int_V\int_Y f^{3,\eta}\psi^*(v)d\nu(v)dy = 0$ and $||T^\eta(f^{3,\eta})||_{H^1_x L^2_v L^\infty_y} \leq \frac{C}{\eta}$ and $||\int_V T^\eta(f^{3,\eta})\psi^*(v)d\nu(v)dv||_{H^1_x L^\infty_y} \leq \frac{C}{\eta}$, we get the first point of (8.45) for $k=3$.

Finally, we write

$$\begin{aligned}\overline{v\cdot\nabla_x f^{3,\eta}\psi^{\eta *}} &= \int_V v\cdot\nabla_x f^{3,\eta}\psi^* d\nu(v) = \int_V \nabla_x f^{3,\eta}\cdot Q^*(\chi^*(y,v))d\nu(v)\\ &= \int_V \nabla_x Q(f^{3,\eta})\cdot\chi^*(y,v)d\nu(v)\\ &= \int_V \nabla_x(-\eta v\nabla_y f^{3,\eta} - v\nabla_x f^{2,\eta}(x,y,v) - f^{1,\eta})\cdot\chi^*(y,v)d\nu(v),\end{aligned}$$

so that

$$\begin{aligned}||\overline{v\cdot\nabla_x f^{3,\eta}\psi^{\eta *}}||_{L^2_x L^2_v L^\infty_y} &\leq C[\eta||f^{3,\eta}||_{H^1_x L^2_v L^\infty_y} + ||f^{2,\eta}||_{H^2_x L^2_v L^\infty_y} + ||f^{1,\eta}||_{H^1_x L^2_v L^\infty_y}]\\ &\leq \frac{C}{\eta^2}\end{aligned}$$

which is bounded by $C\eta^{-2}$. This concludes the proof of (8.45) for $k=3$.

In view of the estimate (8.30) and (8.45), we get

$$||r^{\varepsilon,\eta}||_{L^2_{x,v}} \leq C(\frac{\varepsilon^2}{\eta^2} + \frac{\varepsilon^3}{\eta^3}).$$

This concludes the proof of the proposition. □

Now, the rest of the proof of the main theorem simply consists of collecting the expression for $f^{\varepsilon,\eta}$ given in (8.27), the above estimate for $r^{\varepsilon,\eta}$, the expansions for the densities in section 8.6 and the expansions for the auxiliary functions $\psi^\eta$ and $\chi^\eta = T^{\eta-1}(v\psi^\eta)$ given in section 8.5. This yields (8.13) and concludes the proof of Theorem 8.2.1.

## 8.8 Previous results

In this appendix, we improve useful propositions proved in [9]. Their proofs in the case where $\psi^*$ depends on $v$ is a straightforward adaptation of the proofs obtained in [9] when $\psi^* = 1$, except for the estimates (8.51) and (8.52) which are new.

*Proposition.* [9] Let $L$ be the unbounded operator on $L^2_{per}(\mathbb{R}^d_y)$ with domain $H^2_{per}(\mathbb{R}^d_y)$, defined by

$$\begin{aligned}L(\rho) &= -\int_V \psi^*(v)v\cdot\nabla_y(Q^{-1}(v\cdot\nabla_y(\psi(y,v)\rho(y))))d\nu(v),\\ &= -\mathrm{div}_y(D\nabla_y\rho) + \nabla_y\cdot(U(y)\rho)\end{aligned}$$

and let $L^*$ be its adjoint defined by

$$L^*(n) = -\mathrm{div}_y(D^\top(y)\nabla_y n) - U(y)\cdot\nabla_y n \quad \text{with periodic boundary conditions.} \tag{8.46}$$

The matrix-valued function $D$ and the vector field $U$ are given by

$$D(y) = \int_V \psi^* v \otimes Q^{-1}(v\psi) d\nu(v), \qquad U(y) = \int_V \psi^* v \otimes Q^{-1}(v \cdot \nabla_y \psi) d\nu(v).$$

The following statements hold :
(i) $Im(L) = \{u \in L^2_{per}(\mathbb{R}^d_y), \quad s.t. \quad \int_Y u\, dy = 0\}$ .
(ii)There exists a unique, nonnegative function $\rho(y) \in H^2_{per}(\mathbb{R}^d_y)$ such that $L(\rho) = 0$ and $\int_Y \rho\, dy = 1$ .
(iii) For any function $v \in Im(L)$, there exists a unique solution to $L(u) = v$, $\int_Y u\, dy = 0$ with $u \in H^2_{per}(\mathbb{R}^d_y)$. This solution will be denoted $u = L^{-1}(v)$, and $L^{-1}$ will be referred to as the pseudo inverse of $L$.
(iv) $Im(L^*) = \{u \in L^2_{per}(\mathbb{R}^d_y), \quad s.t. \quad \int_Y u(y)\rho(y)\, dy = 0\}$ .
(v) The kernel of $L^*$ is the set of constant functions (in the variable $y$).
(vi) For any function $v^* \in Im(L^*)$, there exists a unique solution to $L^*(u^*) = v^*$, $\int_Y u^*(y)\, dy = 0$ with $u^* \in H^2_{per}(\mathbb{R}^d_y)$. This solution will be denoted $u^* = L^{*-1}(v^*)$, and $L^{*-1}$ will be referred to as the pseudo-inverse of $L^*$.
(vii) $D(y)$, $U(y)$ and $\rho(y)$ have the following symmetry properties :
- The diffusion matrix $y \mapsto D(y)$ is even. The flux $y \mapsto U(y)$ is odd. The equilibrium function $y \mapsto \rho(y)$ is even.
- The sets of even and odd functions in $y$ are invariant under $L^{-1}$ and $L^{*-1}$.

(viii) The diffusion matrix $D(y)$ is in $C^\infty_{per}(\mathbb{R}^d_y)$ and satisfies

$$D(y)\xi \cdot \xi = 0 \iff \xi = 0 \quad \text{and furthermore} \quad D(y)\xi \cdot \xi \geq \beta |\xi|^2 \ , \tag{8.47}$$

for some constant $\beta > 0$.
(ix) There exists a constant $C$ such that

$$\begin{cases} \|L^{*-1}v^*\|_{H^2_{per}} \leq C\|v^*\|_{L^2_{per}}, \qquad \|L^{-1}v\|_{H^2_{per}} \leq C\|v\|_{L^2_{per}}, \\ \|L^{*-1}v^*\|_{L^2_{per}} \leq C\|v^*\|_{H^{-2}_{per}}, \qquad \|L^{-1}v\|_{L^2_{per}} \leq C\|v\|_{H^{-2}_{per}} \ , \end{cases} \tag{8.48}$$

for all $v \in Im(L)$ and $v^* \in Im(L^*)$.

*Remark* 8.8.1. We need the fact that $\psi^*$ does not depend on $y$ to identify zero as the first eigenvalue of $L^*$ and then of $L$. When $\psi^*$ is allowed to depend on $y$, more complex global equilibria (or possibly the lack of such a global equilibrium) needs to be analyzed on $Y$.

*Proposition.* The operator $T^\eta = \eta v \cdot \nabla_y + Q[y](\cdot)$ is an unbounded operator on $L^2_{per}(\mathbb{R}^d_y \times V)$ with domain

$$\mathcal{D} = \{u \in L^2_{per}(\mathbb{R}^d_y \times V), \quad \text{such that } v \cdot \nabla_y u \in L^2_{per}(\mathbb{R}^d_y \times V)\ \} \ .$$

Let $T^{\eta^*} = -\eta v \cdot \nabla_y + Q^*[y](\cdot)$ be the adjoint of $T^\eta$. Then
(i) The kernel of $T^\eta$ is a one dimensional space spanned by a positive function $\psi^\eta(y, v)$ normalized by $\int_{V\times Y} \psi^\eta \psi^*(v) d\nu(v) dy = 1$. The function $\psi^\eta$ satisfies

$$\psi^\eta(-y, -v) = \psi^\eta(y, v), \quad a.e. \text{ in } y \text{ and } v \ . \tag{8.49}$$

(ii) The range of $T^\eta$ is the set of functions $g \in L^2_{per}(\mathbb{R}^d_y \times V)$ such that we have $\int_{V\times Y} g(y, v)\psi^*(v)\, d\nu(v) dy = 0$ .
(iii) The adjoint $T^{\eta *}$ has the same domain $\mathcal{D}$. Its range is the set of functions $g$ such that $\int_{V\times Y} \psi^\eta(y, v)\, g(y, v) d\nu(v) dy = 0$. Its kernel is spaned by $\psi^*$.

For $g \in L^2_{per}(\mathbb{R}^d_y \times V)$ satisfying $\int_{V\times Y} g(y, v)\psi^*(v) d\nu(v) dy = 0$, there exists a unique function $R^\eta \in \mathcal{D}$ such that

$$T^\eta(R^\eta) = g \quad \text{and} \quad \int_{V\times Y} R^\eta(y, v)\psi^*(v) d\nu(v) dy = 0 \ . \tag{8.50}$$

We denote $R^\eta = (T^\eta)^{-1}(g)$. There exists $\eta_0 > 0$ such that

$$||R^\eta||_{H^k(Y,L^2(V))} \leq \frac{C}{\eta^2} ||g||_{H^k(Y,L^2(V))} \ , \quad 0 < \eta < \eta_0 \ . \tag{8.51}$$

If moreover, $\int_V g\psi^*(v)d\nu(v) = 0$, then

$$||R^\eta||_{H^k(Y,L^2(V))} \leq \frac{C}{\eta}||g||_{H^k(Y,L^2(V))}. \tag{8.52}$$

Moreover, the following symmetry implications hold true

$$\begin{cases} \text{If } g(-y,-v) = g(y,v) \quad a.e. \quad \text{then} \quad R^\eta(-y,-v) = R^\eta(y,v) \quad a.e. \\ \text{If } g(-y,-v) = -g(y,v) \quad a.e. \quad \text{then} \quad R^\eta(-y,-v) = -R^\eta(y,v) \quad a.e. \end{cases} \tag{8.53}$$

(iv) For $g^* \in L^2_{per}(\mathbb{R}^d_y \times V)$ satisfying $\int_{V\times Y} \psi^\eta(y,v)g^*(y,v)d\nu(v)dy = 0$ there exists a unique function $R^{\eta*} \in \mathcal{D}$ such that

$$T^{\eta*}(R^{\eta*}) = g^* \quad \text{and} \quad \int_{V\times Y} R^{\eta*}(y,v)\psi^\eta(y,v)d\nu(v)dy = 0\ . \tag{8.54}$$

We denote $R^{\eta*} = (T^{\eta*})^{-1}(g^*)$. There exists $\eta_0 > 0$ such that

$$||R^{\eta*}||_{H^k(Y,L^2(V))} \leq \frac{C}{\eta^2}||g^*||_{H^k(Y,L^2(V))}\ , \quad 0 < \eta < \eta_0\ . \tag{8.55}$$

If moreover, $\int_V g^*\psi d\nu(v) = 0$, then

$$||R^{\eta^*}||_{H^k(Y,L^2(V))} \leq \frac{C}{\eta}||g^*||_{H^k(Y,L^2(V))}. \tag{8.56}$$

Finally, the following symmetry implications hold true

$$\begin{cases} \text{If } g^*(-y,-v) = g^*(y,v) \quad a.e. \quad \text{then} \quad R^{\eta*}(-y,-v) = R^{\eta*}(y,v) \quad a.e. \\ \text{If } g^*(-y,-v) = -g^*(y,v) \quad a.e. \quad \text{then} \quad R^{\eta*}(-y,-v) = -R^{\eta*}(y,v) \quad a.e. \end{cases} \tag{8.57}$$

*Démonstration.* **Step 1 :** For the proof of estimate (8.52), we argue by contradiction. Assume that $||\tilde{R}^\eta|| = 1$ and $||\tilde{g}^\eta|| \to_{\eta\to 0} 0$ with

$$\eta v\cdot\nabla_y\tilde{R}^\eta + Q(\tilde{R}^\eta) = \eta\tilde{g}^\eta.$$

Decompose $\tilde{R}^\eta = \gamma^\eta(y)\psi(y,v) + \delta^\eta(y,v)$, we get by the dissipation property

$$\begin{aligned} ||\delta^\eta||^2_{L^2(Y\times V)} &\leq C\int_Y\int_V Q(\tilde{R}^\eta)\frac{\tilde{R}^\eta}{\psi}\psi^* d\nu(v)dy \\ &\leq \int_Y\int_V(\eta\tilde{g}^\eta - \eta v\cdot\nabla_y\tilde{R}^\eta)\frac{\tilde{R}^\eta}{\psi}\psi^* d\nu(v)dy \\ &\leq C(\eta||\tilde{R}^\eta||_{L^2(Y\times V)}||\tilde{g}^\eta||_{L^2(Y\times V)} + \eta||\tilde{R}^\eta||^2_{L^2(Y\times V)}) \end{aligned}$$

which implies that $||\delta^\eta||_{L^2(Y\times V)} \to 0$ when $\delta$ goes to zero.

On the other hand

$$\eta v\cdot\nabla_y\delta^\eta + Q(\delta^\eta) = \eta\tilde{g}^\eta - \eta v\cdot\nabla_y(\gamma^\eta\psi)$$

and an integration with respect to $v$ against $\psi^*$ gives

$$\mathrm{div}_y\int_V v\delta^\eta\psi^* d\nu(v) = \int\tilde{g}^\eta\psi^* d\nu(v) = 0.$$

But

$$\begin{aligned} &\mathrm{div}_y\int_V v\delta^\eta\psi^* d\nu(v) = -\mathrm{div}_y(\int_V\chi^* Q(\delta^\eta)d\nu(v)) \\ = \ &\eta\mathrm{div}_y(\int_V\chi^* v\cdot\nabla_y\delta^\eta d\nu(v)) + \eta\mathrm{div}_y(\int_V\chi^* v\cdot\nabla_y(\gamma^\eta\psi)) - \eta\mathrm{div}_y(\int_V\chi^*\tilde{g}^\eta d\nu(v)) \end{aligned}$$

which leads to the following elliptic equation for $\gamma^\eta$

$$L(\gamma^\eta) = \mathrm{div}_y(\int_V \chi^* v \cdot \nabla_y \delta^\eta d\nu(v)) - \mathrm{div}_y(\int_V \chi^* \tilde{g}^\eta d\nu(v)).$$

Since the right-hand side goes to zero in $H^{-2}(Y)$, we obtain that $\gamma^\eta - C^\eta \rho^0(y) \to 0$. But

$$C^\eta - \int_Y \gamma^\eta dy = -\int_Y L^{-1}(\mathrm{div}_y(\int_V \chi^* v \cdot \nabla_y \delta^\eta d\nu(v)) - \mathrm{div}_y(\int_V \chi^* \tilde{g}^\eta d\nu(v))) \to 0.$$

Moreover

$$\begin{aligned}\int_Y \gamma^\eta(y)dy &= \int_{V\times Y} \gamma^\eta(y)\psi(y,v)\psi^*(v)dyd\nu(v) = \int_{V\times Y} (R^\eta(y) - \delta^\eta(y,v))\psi^*(v)dyd\nu(v) \\ &= -\int_{V\times Y} \delta^\eta(y,v)\psi^*(v)dyd\nu(v) \longrightarrow 0 \text{ when } \eta \to 0.\end{aligned}$$

Finally, we obtained that $\tilde{R}^\eta \to 0$ strongly which leads to a contradiction.

**Step 2 :** Proof of estimate (8.51). First we know that $||R^\eta||_{L^2(Y,V)} \leq C$. Note that a direct application of the estimate proved in [9] would make us loose two powers of $\eta$ for each derivative. This is the reason why we need additional computations.

We prove the result by induction. Assume that for any multi-indice $i$ satisfying $|i| < |k|$, we have $||\partial_y^i R^\eta|| \leq C$. To prove that $||\partial_y^k R^\eta||_{L^2(Y,V)} \leq C$, we argue by contradiction.

First, we note that for any $f$ and any multi-indice $k = (k_1, k_2, \cdots, k_d)$, we have

$$\partial_y^k(Q^*(f)) = \sum_{i_1}^{|k_1|} \cdots \sum_{i_d}^{|k_d|} \Pi_{l=1}^d (C_{k_l}^{i_l}) \partial_{y_1}^{k_1 - i_1} \cdots \partial_{y_d}^{k_d - i_d} Q(\partial_{y_1}^{i_1} \cdots \partial_{y_d}^{i_d} f).$$

Thus, we write

$$\eta v \cdot \nabla_y \partial_y^k \tilde{R}^\eta + Q(\partial_y^k \tilde{R}^\eta) = \eta^2 \partial_y^k \tilde{g}^\eta - \sum_{|i|=0}^{|k|-1} \mathbf{C}_{\mathbf{k}}^{\mathbf{i}} \partial_y^{k-i} Q(\partial_y^i \tilde{R}^\eta)$$

where $\mathbf{C}_{\mathbf{k}}^{\mathbf{i}} = \Pi_{l=1}^d (C_{k_l}^{i_l})$ and where the sum runs over $i = (i_1, i_2, \cdots, i_d)$ such that $|i| \leq |k| - 1$ and $i_l \leq k_l$ for $1 \leq l \leq d$. To obtain a contradiction, we choose a renormalization that implies $||\partial_y^k \tilde{R}^\eta|| = 1$ and $||\partial_y^k \tilde{g}^\eta|| \to 0$ and $||\partial_y^i \tilde{R}^\eta|| \to 0$ for any $i$ satisfying $|i| < |k|$.

As in the proof of the previous Lemma, we decompose $\partial_y^k \tilde{R}^\eta$ into two parts as follows

$$\partial_y^k \tilde{R}^\eta = \gamma_k^\eta(y)\psi(y,v) + \delta_k^\eta(y,v).$$

We get from the dissipation property

$$\begin{aligned}||\delta_k^\eta||^2_{L^2(Y,V)} &\leq -C \int_Y \int_V Q(\partial_y^k \tilde{R}^\eta) \frac{\partial_y^k \tilde{R}^\eta}{\psi} \psi^* d\nu(v)dy \\ &= C \int_Y \int_V (\eta^2 \partial_y^k \tilde{g}^\eta - \sum_{|i|=0}^{|k|-1} \mathbf{C}_{\mathbf{k}}^{\mathbf{i}} \partial_y^{k-i} Q(\partial_y^i \tilde{R}^\eta) - \eta v \cdot \nabla_y \partial_y^k \tilde{R}^\eta) \frac{\partial_y^k \tilde{R}^\eta}{\psi} \psi^* d\nu(v)dy\end{aligned}$$

which gives

$$||\delta_k^\eta||_{L^2(Y,V)} \longrightarrow_{\eta \to 0} 0.$$

On the other hand, $\gamma_k^\eta$ satisfies the following elliptic equation

$$\begin{aligned}L(\gamma_k^\eta) &= \int_V \partial_y^k \tilde{g}^\eta \psi^* + \mathrm{div}_y(\int_V \chi^* v \cdot \nabla_y(\delta_k^\eta)) - \eta \mathrm{div}_y \int_V \partial_y^k \tilde{g}^\eta \chi^* \\ &\quad + \frac{1}{\eta} \mathrm{div}_y \int_V \sum_{|i|=0}^{|k|-1} \mathbf{C}_{\mathbf{k}}^{\mathbf{i}} \partial_y^{k-i} Q(\partial_y^i \tilde{R}^\eta) \chi^*.\end{aligned}$$

We have to deal with the last term which a priori is of order $\frac{1}{\eta}$.

For that purpose, we write

$$\int_V \partial_y^{k-i} Q(\partial_y^i \tilde{R}^\eta)\chi^* = \int_V \partial_y^i \tilde{R}^\eta \partial_y^{k-i} Q^*(\chi^*).$$

Moreover, since $\psi^*$ does not depend on $y$

$$\partial_y^k(Q^*(\chi^*)) = \sum_{i_1}^{|k_1|} \cdots \sum_{i_d}^{|k_d|} \Pi_{l=1}^d (C_{k_l}^{i_l}) \partial_{y_1}^{k_1-i_1} \cdots \partial_{y_d}^{k_d-i_d} Q(\partial_{y_1}^{i_1} \cdots \partial_{y_d}^{i_d} \chi^*) = 0$$

and then

$$\int_V \sum_{|i|=0}^{|k|-1} \mathbf{C_k^i} \partial^{k-i} Q(\partial^i \tilde{R}^\eta)\chi^* = -\int_V \sum_{i=0}^{|k|-1} \mathbf{C_k^i} \partial^i \tilde{R}^\eta \sum_{|j|=1}^{|k-i|} \mathbf{C_{k-i}^j} \partial_y^{k-i-j} Q^*(\partial_j \chi^*)$$

which gives by rearranging the sums

$$\begin{aligned}\int_V \sum_{|i|=0}^{|k|-1} \mathbf{C_k^i} \partial^{k-i} Q(\partial^i \tilde{R}^\eta)\chi^* &= -\int_V \sum_{|j|=1}^{|k|} \partial_j \chi^* \sum_{|i|=0}^{|k-j|} \mathbf{C_{k-i}^j C_k^i} \partial_y^{k-i-j} Q(\partial^i \tilde{R}^\eta) \\ &= -\int_V \sum_{|j|=1}^{|k|} \mathbf{C_k^j} \partial_j \chi^* \sum_{|i|=0}^{|k-j|} \mathbf{C_{k-j}^i} \partial_y^{k-i-j} Q(\partial^i \tilde{R}^\eta).\end{aligned}$$

Finally, it leads to

$$\int_V \sum_{|i|=0}^{|k|-1} \mathbf{C_k^i} \partial^{k-i} Q(\partial^i \tilde{R}^\eta)\chi^* = -\int_V \sum_{|j|=1}^{|k|} \mathbf{C_k^j} \partial_j \chi^* \partial_y^{k-j}(Q(\tilde{R}^\eta))$$

and then

$$\frac{1}{\eta}\mathrm{div}_y \int_V \sum_{|i|=0}^{|k|-1} \mathbf{C_k^i} \partial^{k-i} Q(\partial^i \tilde{R}^\eta)\chi^* = \frac{1}{\eta}\mathrm{div}_y \int_V \sum_{|j|=1}^{|k|} \mathbf{C_k^j} \partial_j \chi^* (-\eta v \cdot \nabla_y \partial_y^{k-j} \tilde{R}^\eta + \eta^2 \partial_y^{k-j} \tilde{g}^\eta).$$

This term converges to zero in $H^{-2}(Y, L^2(V))$ when $\eta \to 0$. The same argument as above leads to a contradiction. □

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# Chapitre 9

# Anomalous diffusion limit for kinetic equations with degenerate collision frequency

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**abstract**

This paper is devoted to hydrodynamic limits for collisional linear kinetic equations. It is a classical result that under certain conditions on the collision operator, the long time/small mean free path asymptotic behavior of the density of particles can be described by diffusion type equations. We are interested in situations in which the degeneracy of the collision frequency for small velocities causes this limit to break down. We show that the appropriate asymptotic analysis leads to an anomalous diffusion regime.

## 9.1 Introduction

In this paper we investigate diffusion regimes for the following kinetic equation :

$$\begin{cases} \varepsilon^\alpha \partial_t f^\varepsilon + \varepsilon v \cdot \nabla_x f^\varepsilon = L(f^\varepsilon) & \text{for all } (x,v) \in \mathbf{R}^N \times \mathbf{R}^N \text{ and } t > 0 \\ f^\varepsilon(x,v,0) = f_0(x,v) & \text{for all } (x,v) \in \mathbf{R}^N \times \mathbf{R}^N \end{cases} \tag{9.1}$$

when the operator $L$ is a linear collision operator of the form :

$$L(f) := \int_{\mathbf{R}^N} b(v,v')[f(v')M(v) - f(v)M(v')]\,dv'. \tag{9.2}$$

The thermodynamical equilibrium function $M(v)$ is a given function and it is normalized so that

$$\int_{\mathbf{R}^N} M(v)\,dv = 1.$$

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Typically, one can take $M$ to be a Maxwellian distribution function $M(v) = \frac{1}{(2\pi T)^{N/2}} e^{-\frac{v^2}{2T}}$, (for some temperature $T > 0$) though other distribution functions could be considered without difficulties. The collision kernel $b(v, v')$ in (9.2) is usually assumed to satisfy

$$b(v, v') = b(v', v) \geq 0 \tag{9.3}$$

and in this paper, we are going to limit ourself to a very simple case where we have

$$b(v, v') = a(v)a(v') \tag{9.4}$$

for some function $a(v) \geq 0$. Of course, our result also applies to more general collision kernels, with minor modifications.

When (9.4) holds, the operator $L$ reads :

$$L(f) = a(v) \int a(v')f(v')\, dv' M(v) - \nu(v)f(v)$$

where

$$\nu(v) := \int b(v, v')M(v')\, dv' = a(v) \int a(v')M(v')\, dv'$$

denotes the collision frequency. We can also write $L$ as follows (and we will use mainly this notation from now on) :

$$L(f) = \nu(v)[\rho_\nu M(v) - f(v)] \qquad \text{with} \quad \rho_\nu := \frac{\displaystyle\int \nu(v)f(v)\, dv}{\displaystyle\int \nu(v)M(v)\, dv} \tag{9.5}$$

(note that $\rho_\nu$ is not the usual density of $f$, except when $f(x, v, t) = \rho(x, t)M(v)$).

The parameter $\varepsilon$ in (9.1) is the Knudsen number. It is defined as the ratio of the mean free path of the particles to the typical macroscopic length. We are interested in the small mean free path ($\varepsilon \ll 1$) long time ($t \sim \varepsilon^{-\alpha}$) regime, which leads to the scaling in (9.1). The derivation of hydrodynamic limits for kinetic equations such as (9.1) was first investigated by E. Wigner [10], A. Bensoussan, J.L. Lions and G. Papanicolaou [4] and E.W. Larsen and J.B. Keller [10] and it has been the topic of many papers since (see in particular C. Bardos, R. Santos and R. Sentis [1] and P. Degond, T. Goudon and F. Poupaud [7] and references therein). It is well known (see [7] for instance) that for $\alpha = 2$, the distribution function $f^\varepsilon$ converges, as $\varepsilon \to 0$, to a function $\rho(x, t)M(v)$ with $\rho$ solution of the diffusion equation

$$\partial_t \rho - \ \text{div}\ (D\nabla\rho) = 0$$

where

$$D = \int v \otimes v \frac{M(v)}{\nu(v)}\, dv. \tag{9.6}$$

We are interested in a situation in which this limit fails because of the degeneracy of the collision frequency. More precisely, we will assume that $\nu(0) = 0$ and

$$\nu(v) \sim \nu_0 |v|^{N+2+\beta}, \qquad \text{as } |v| \to 0 \quad \text{for some } \beta > 0. \tag{9.7}$$

The restriction $\beta > 0$ and the fact that $M(0) > 0$ implies that the diffusion matrix $D$ above is infinite, because of the non integrable singularity at $v = 0$. In that case, the usual diffusion limit breaks down, which can be interpreted by saying that the time scale $t \sim \varepsilon^{-2}$ was too long. We will thus show that for an appropriate choice of $\alpha < 2$ (depending on $\beta$), the asymptotic regime $\varepsilon \to 0$ in (9.1) gives rise to anomalous diffusion regimes (the term anomalous refers to the fact that the mean squared displacement of a particle is no longer a linear function of time).

In previous works, we investigated similar anomalous diffusion regimes, that were due to the large velocity ($|v| \to \infty$) behavior of the equilibrium function $M$ and the collision frequency $\nu$. In particular it is shown, in [14, 13, 3], that if $\nu$ is bounded and non-degenerate and $M(v)$ is a heavy tail function satisfying

$$M(v) \sim |v|^{-N-\alpha} \quad \text{as } |v| \to \infty, \qquad \text{for some } \alpha \in (0, 2)$$

(the restriction $\alpha < 2$ implies that the diffusion matrix $D$ above is infinite because of the behavior of the integrand for $|v| \to \infty$) then the solution of (9.1) converges to $\rho(x,t)M(v)$ with $\rho$ solution of a fractional diffusion equation of order $\alpha$.

The goal of this paper is to show that a similar phenomenon can arise when the equilibrium function is, for instance, the usual Maxwellian distribution, but the collision frequency satisfies (9.7).

In order to make the computations a little bit simpler, we will assume that there exists $\delta > 0$, $\beta > 0$ and $\nu_0 > 0$ such that

$$\begin{aligned} &\nu(v) = \nu_0 |v|^{N+2+\beta}, && \text{for } |v| \le \delta \\ &M(v) = M_0 > 0, && \text{for } |v| \le \delta. \end{aligned} \tag{9.8}$$

Neither of these conditions are really necessary, and it is easy to check that they could be replaced by

$$\lim_{|v|\to 0} \frac{\nu(v)}{|v|^{N+2+\beta}} = \nu_0 > 0, \qquad \lim_{|v|\to 0} M(v) = M_0 > 0.$$

Next, as usual with diffusion limits, it is very important to assume that the flux associated to $M(v)$ vanishes :

$$\int_{\mathbf{R}^N} vM(v)\,dv = 0 \tag{9.9}$$

(this is usually a consequence of the fact that $M(-v) = M(v)$). Finally, we need the following technical assumptions (which clearly holds for a wide range of functions $\nu(v)$ if $M(v)$ is a Maxwellian distribution function) :

$$\int_{|v|\ge\delta} \frac{|v|^2}{\nu(v)} M(v)\,dv < \infty, \qquad \int_{\mathbf{R}^N} \nu(v)^2 M(v)\,dv < \infty \tag{9.10}$$

(the first inequality - in which we recognize the integrant of the diffusion matrix (9.6) - asserts that the only problem in the break down of the usual diffusion limit is due to the behavior of $\nu$ and $M$ for small velocity. The second inequality will be used to show that $\rho_\nu^\varepsilon$ is bounded in $L^2$).

We can now state our main result :

**Theorem 9.1.1.** *Assume that (9.8), (9.9) and (9.10) hold and let*

$$\alpha = \frac{\beta + 2N + 2}{\beta + N + 1} = 2 - \frac{\beta}{\beta + N + 1}. \tag{9.11}$$

*Then the solution $f^\varepsilon$ of (9.1) converges weakly in $L^2_{\nu M^{-1}}(\mathbf{R}^N \times \mathbf{R}^N \times (0,T))$ to a function $\rho(x,t)M(v)$ where $\rho$ solves*

$$\begin{cases} \partial_t \rho + \kappa(-\Delta)^{\alpha/2}\rho = 0 \\ \rho(x,0) = \rho_0 \end{cases} \tag{9.12}$$

*with*

$$\kappa = \frac{1}{c_{N,\alpha}} \frac{1}{\beta + N + 1} \frac{M_0}{\nu_0^{\alpha-1}} \int_0^\infty z^\alpha e^{-z}\,dz.$$

The constant $c_{N,\alpha}$ is the constant that appears in the definition of the fractional Laplace operator as a singular integral :

$$(-\Delta)^{\alpha/2} u = c_{N,\alpha} \text{P.V.} \int_{\mathbf{R}^N} \frac{u(x) - u(x+y)}{|y|^{N+\alpha}}\,dy$$

Note that when $\beta > 0$, $\alpha$ defined by (9.11) satisfies

$$1 < \alpha < 2.$$

The critical case $\beta = 0$, which would give $\alpha = 2$ is also interesting. The standard diffusion limit (with time scale $\varepsilon^2$) fails in that case, and following [14], it can be checked that one needs to consider an anomalous diffusion scaling in (9.1) ($\varepsilon^2 \ln(\varepsilon^{-1})$ instead of $\varepsilon^\alpha$ in (9.1)) but that the limiting density $\rho$ will solve a standard diffusion equation.

It is also worth noticing that unlike the anomalous diffusion regime due to power tail equilibrium functions, the degenerate collision frequency framework cannot lead to fractional diffusion equation of order less than one (there were no such restriction in [14, 13, 3]).

Finally, we point out that in [6], Jara, Komorowski and Olla obtained a result equivalent to Theorem 9.1.1, using a purely probabilistic approach, when studying the behavior of a chain of harmonic oscillators.

The proof of Theorem 9.1.1 follows classical steps : Derivation of a priori estimate, multiplication of (9.1) by the solution of an appropriate auxiliary equation and passage to the limit. Those steps are detailed in the next section.

**Acknowledgment :** Part of this work was done while A. Mellet was visiting the Department of Mathematics at the Université Paul Sabatier. As we were finishing this paper, Professor N. Ben Abdallah passed away unexpectedly. We hope that the paper meets his high standards, though we are responsible for this final version.

## 9.2 Proof of Theorem 9.1.1

### 9.2.1 A priori estimates

It is a classical result that operators of the form (9.2) satisfy

$$-\int_{\mathbf{R}^N} L(f)\frac{f}{M}\,dv \geq 0.$$

Under some conditions on the collision kernel (see for instance [14]) one can usually also show that

$$-\int_{\mathbf{R}^N} L(f)\frac{f}{M}\,dv \geq c||f-\rho M||^2_{L^2_{M^{-1}}}$$

where $\rho = \int_{\mathbf{R}^N} f\,dv$. When the collision frequency is degenerate, however, such an estimate does not seem to hold, but a simple computation yields the following lemma :

**Lemma 9.2.1.** *Assume that $b(v,v') = a(v)a(v')$. Then*

$$-\int L(f)\frac{f}{M}\,dv = \int \nu(v)\frac{|f-\rho_\nu M|^2}{M}\,dv$$

*where*

$$\rho_\nu = \frac{\displaystyle\int \nu(v)f(v)\,dv}{\displaystyle\int \nu(v)M(v)\,dv}.$$

Note that a similar result holds for more general collision kernel that do not satisfy (9.4). Indeed, we have :

**Lemma 9.2.2.** *Assume that $L$ is given by (9.2) where $b(v,v')$ satisfies (9.3) and*

$$\sup_{v\in\mathbf{R}^N}\int_{\mathbf{R}^N}\frac{\nu(v)\nu(v')^2}{b(v,v')}\,M(v')\,dv' \leq C < \infty. \tag{9.13}$$

*Then,*

$$-\int L(f)\frac{f}{M}\,dv \geq \frac{\left(\int \nu M\,dv\right)^2}{2C}\int \nu(v)\frac{|f-\rho_\nu M|^2}{M}\,dv$$

*for all $f$.*

This lemma is the only technical lemma that is needed to generalize Theorem 9.1.1 to more general collision operators (still given by (9.2)) and it is also of independent interest. We thus give its proof in Appendix 9.3, even though it is not required for the proof of Theorem 9.1.1.

Multiplying (9.1) by $f^\varepsilon M^{-1}$ and integrating with respect to $x$ and $v$, we deduce the following equalities :

$$\begin{aligned}\varepsilon^\alpha \frac{d}{dt}\int_{\mathbf{R}^{2N}} \frac{(f^\varepsilon)^2}{2} M^{-1}\,dvdx &= \int_{\mathbf{R}^{2N}} L(f^\varepsilon)\, f^\varepsilon\, M^{-1}\,dvdx \\ &= -\int_{\mathbf{R}^{2N}} \nu(v)|f^\varepsilon - \rho_\nu^\varepsilon M|^2 M^{-1}\,dvdx \end{aligned} \tag{9.14}$$

We thus introduce the spaces $L^2_{M^{-1}}$ and $L^2_{\nu M^{-1}}$ equipped with the norms :

$$||f||_{L^2_{M^{-1}}} = \int_{\mathbf{R}^N}\int_{\mathbf{R}^N} |f(x,v)|^2 M(v)^{-1}\,dv\,dx$$

and

$$||f||_{L^2_{\nu M^{-1}}} = \int_{\mathbf{R}^N}\int_{\mathbf{R}^N} |f(x,v)|^2 \nu(v) M(v)^{-1}\,dv\,dx.$$

Integrating (9.14) with respect to $t$, we deduce the following proposition :

**Proposition 9.2.3.** *The solution $f^\varepsilon$ of (9.1) satisfies the following estimates :*

$$\sup_t ||f^\varepsilon(t)||_{L^2_{M^{-1}}} \leq ||f_0||_{L^2_{M^{-1}}} \tag{9.15}$$

$$||g^\varepsilon||_{L^2(0,\infty;L^2_{\nu M^{-1}})} \leq \varepsilon^{\alpha/2}||f_0||_{L^2_{M^{-1}}} \tag{9.16}$$

*where $g^\varepsilon = f^\varepsilon - \rho_\nu^\varepsilon M$ with $\rho_\nu^\varepsilon = \frac{\int \nu f^\varepsilon\,dv}{\int \nu M\,dv}$.*

Note that Cauchy-Schwarz inequality also gives :

$$\begin{aligned}\rho_\nu^\varepsilon(t,x) &= \frac{1}{\int \nu M\,dv}\int_{\mathbf{R}^N} \nu \frac{f^\varepsilon}{M^{1/2}} M^{1/2}\,dv \\ &\leq \frac{\left(\int \nu^2 M\,dv\right)^{1/2}}{\int \nu M\,dv}\left(\int_{\mathbf{R}^N}\frac{(f^\varepsilon)^2}{M}\,dv\right)^{1/2},\end{aligned}$$

and so (using (9.10))

$$\sup_{t\geq 0}\int_{\mathbf{R}^N} \rho_\nu^\varepsilon(t,.)^2\,dx \leq C\sup_{t\geq 0}\|f^\varepsilon\|^2_{L^2_{M^{-1}}} \leq C\|f_0\|^2_{L^2_{M^{-1}}}. \tag{9.17}$$

We deduce

*Corollary* 9.2.1. The sequence $f^\varepsilon$ converges weakly in $L^\infty(0,T;L^2_{M^{-1}}(\mathbf{R}^N\times\mathbf{R}^N))$ to a function $\rho(x,t)F(x,v)$ where $\rho$ is the weak limit of $\rho_\nu^\varepsilon$ in $L^\infty(0,T;L^2(\mathbf{R}^N))$.

### 9.2.2 An auxiliary equation

The main tool in the proof of Theorem 9.1.1 is the introduction of an appropriate auxiliary equation. Proceeding as in [13], we consider $\chi^\varepsilon(x,v,t)$ solution of

$$\nu(v)\chi^\varepsilon(v) - \varepsilon v\cdot\nabla_x\chi^\varepsilon = \nu(v)\varphi(x,t), \tag{9.18}$$

where $\varphi(x,t)$ is a test function in $\mathcal{D}(\mathbf{R}^N\times[0,\infty))$. A simple computation shows that $\chi^\varepsilon$ is given by the following formula :

$$\chi^\varepsilon = \int_0^\infty e^{-\nu(v)z}\nu(v)\varphi(x+\varepsilon vz,t)\,dz,$$

and we will make repeated use of the following formula :

$$\chi^\varepsilon - \varphi = \int_0^\infty e^{-\nu(v)z}\nu(v)[\varphi(x+\varepsilon vz,t)-\varphi(x,t)]\,dz. \tag{9.19}$$

In the next section, we will use $\chi^\varepsilon$ as a test function in (9.1). We thus point out that $\chi^\varepsilon$ is smooth and bounded in $L^\infty$. Moreover, we have

$$\begin{aligned} |\chi^\varepsilon - \varphi| &= \left| \int_0^\infty e^{-\nu(v)z}\nu(v)\big[\varphi(x+\varepsilon vz,t)-\varphi(x,t)\big]\,dz \right| \\ &\leq ||D\varphi||_{L^\infty}\varepsilon|v|, \end{aligned}$$

and thus

$$\chi^\varepsilon(x,v,t) \longrightarrow \varphi(x,t) \qquad \text{as } \varepsilon \to 0$$

uniformly with respect to $x$ and $t$. However, this convergence is not uniform with respect to $v$, so we will need the following lemma :

**Lemma 9.2.4.** *Let $\varphi \in \mathcal{D}(\mathbf{R}^N \times [0,\infty))$, and define $\chi^\varepsilon$ by (9.19). Then*

$$\chi^\varepsilon \longrightarrow \varphi \qquad \textit{strongly in } L^\infty(0,\infty;L^2_M(\mathbf{R}^N\times\mathbf{R}^N))$$

*and*

$$\partial_t\chi^\varepsilon \longrightarrow \partial_t\varphi \qquad \textit{strongly in } L^\infty(0,\infty;L^2_M(\mathbf{R}^N\times\mathbf{R}^N))$$

*Démonstration.* Using (9.19) we get

$$\begin{aligned} ||\chi^\varepsilon - \varphi||^2_{L^2_M} &= \int_{\mathbf{R}^{2N}} M(v) \left| \int_0^\infty e^{-\nu(v)z}\nu(v)[\varphi(x+\varepsilon vz)-\varphi(x)]\,dz \right|^2 dx\,dv \\ &\leq \int_{\mathbf{R}^{2N}} M(v) \int_0^\infty e^{-\nu(v)z}\nu(v)[\varphi(x+\varepsilon vz)-\varphi(x)]^2\,dz\,dx\,dv \\ &\leq \int_{\mathbf{R}^N}\int_0^\infty M(v)e^{-\nu(v)z}\nu(v)||\varphi(\cdot+\varepsilon vz)-\varphi||^2_{L^2(\mathbf{R}^N)}\,dz\,dv. \end{aligned}$$

It is well know that

$$\lim_{\varepsilon\to 0} ||\varphi(\cdot+\varepsilon vz)-\varphi||^2_{L^2(\mathbf{R}^N)} = 0 \qquad \text{for all } v \text{ and } z.$$

Furthermore,

$$M(v)e^{-\nu(v)z}\nu(v)||\varphi(\cdot+\varepsilon vz)-\varphi||^2_{L^2(\mathbf{R}^N)} \leq 2||\varphi||^2_{L^2(\mathbf{R}^N)}M(v)e^{-\nu(v)z}\nu(v)$$

Since $M(v)e^{-\nu(v)z}\nu(v)$ is in $L^1(\mathbf{R}^N\times(0,\infty))$ (in fact, its integral is equal to 1), Lebesgue dominated convergence theorem gives the result.

A similar proof holds for $\partial_t\chi^\varepsilon$ ($t$ is just a parameter here). □

### 9.2.3 Weak formulation and passage to the limit

Now, we use $\chi^\varepsilon$ as a test function in (9.1) : Multiplying (9.1) by $\chi^\varepsilon$ and integrating with respect to $x, v, t$, we get :

$$\begin{aligned} -\varepsilon^\alpha \int_0^\infty \int_{\mathbf{R}^{2N}} f^\varepsilon \partial_t\chi^\varepsilon\,dx\,dv\,dt - \varepsilon^\alpha \int_{\mathbf{R}^{2N}} f_0(x,v)\chi^\varepsilon(x,v,0)\,dx\,dv \\ = \int_0^\infty \int_{\mathbf{R}^{2N}} \nu\rho^\varepsilon_\nu M\chi^\varepsilon - \nu f^\varepsilon\chi^\varepsilon + f^\varepsilon\varepsilon v\cdot\nabla_x\chi^\varepsilon\,dx\,dv\,dt \end{aligned}$$

which, using the auxiliary equation (9.18), yields :

$$\begin{aligned} -\int_0^\infty \int_{\mathbf{R}^{2N}} f^\varepsilon\partial_t\chi^\varepsilon\,dx\,dv\,dt - \int_{\mathbf{R}^{2N}} f_0(x,v)\chi^\varepsilon(x,v,0)\,dx\,dv \\ = \varepsilon^{-\alpha}\int_0^\infty\int_{\mathbf{R}^{2N}} \nu\rho^\varepsilon_\nu M\chi^\varepsilon - \nu f^\varepsilon\varphi(x,t)\,dx\,dv\,dt \\ = \varepsilon^{-\alpha}\int_0^\infty\int_{\mathbf{R}^{2N}} \nu\rho^\varepsilon_\nu M\chi^\varepsilon\,dx\,dv\,dt - \int \rho^\varepsilon_\nu \left(\int \nu M\,dv\right)\varphi(x,t)\,dx\,dt. \end{aligned}$$

We deduce :

$$-\int_0^\infty \int_{\mathbf{R}^{2N}} f^\varepsilon \partial_t \chi^\varepsilon \, dx\, dv\, dt - \int_{\mathbf{R}^{2N}} f_0(x,v)\chi^\varepsilon(x,v,0)\, dx\, dv$$
$$= \varepsilon^{-\alpha} \int_0^\infty \int_{\mathbf{R}^N} \rho_\nu^\varepsilon \int_{\mathbf{R}^N} \nu(v) M(v) \left[\chi^\varepsilon(x,v,t) - \varphi(x,t)\right] \, dv\, dx\, dt$$

which we write as

$$-\int_0^\infty \int_{\mathbf{R}^{2N}} f^\varepsilon \partial_t \chi^\varepsilon \, dx\, dv\, dt - \int_{\mathbf{R}^{2N}} f_0(x,v)\chi^\varepsilon(x,v,0)\, dx\, dv$$
$$= \int_0^\infty \int_{\mathbf{R}^N} \rho_\nu^\varepsilon \mathcal{L}^\varepsilon(\varphi) dx\, dt \tag{9.20}$$

with

$$\mathcal{L}^\varepsilon(\varphi) = \varepsilon^{-\alpha} \int_{\mathbf{R}^N} \nu(v) M(v) \left[\chi^\varepsilon(x,v,t) - \varphi(x,t)\right] \, dv.$$

The rest of the proof consists in passing to the limit $\varepsilon \to 0$ in (9.20). We immediately check that the left hand side converges to

$$-\int_0^\infty \int_{\mathbf{R}^N} \rho \, \partial_t \varphi \, dx\, dt - \int_{\mathbf{R}^N} \rho_0(x)\varphi(x,0)\, dx$$

(this is an immediate consequence of Lemma 9.2.4 and Corollary 9.2.1). Passing to the limit in the right hand side of (9.20) is the most interesting part of the proof since the nonlocal operator should now appear in the limit of $\mathcal{L}^\varepsilon$ (it is also the part of the proof that differs the most from [13]). More precisely, we have to prove :

**Proposition 9.2.5.** *Assume that the conditions of Theorem 9.1.1 hold and that $\chi^\varepsilon$ is defined by (9.18). Then*

$$\mathcal{L}^\varepsilon(\varphi) := \varepsilon^{-\alpha} \int_{\mathbf{R}^N} \nu(v) M(v) \left[\chi^\varepsilon(x,v,t) - \varphi(x,t)\right] \, dv \tag{9.21}$$

*converges strongly in $L^2(\mathbf{R}^N \times (0,\infty))$ as $\varepsilon$ goes to zero to*

$$-\kappa(-\Delta)^{\alpha/2}(\varphi) = \kappa \, c_{N,\alpha} \, PV \int_{\mathbf{R}^N} \frac{\varphi(x+y) - \varphi(x)}{|y|^{N+\alpha}} \, dy$$

*with $\kappa$ given in Theorem 9.1.1 (note that we also have uniform convergence with respect to $x$ and $t$).*

Proposition 9.2.5 and the weak convergence of $\rho^\varepsilon$ allow us to pass to the limit in the right hand side of (9.20). We deduce :

$$-\int \rho \, \partial_t \varphi \, dx\, dt - \int \rho_0(x)\varphi(x,0)\, dx = -\int \rho \kappa (-\Delta)^{\alpha/2} \varphi \, dx\, dt$$

which is the weak formulation of (9.12). This completes the proof of Theorem 9.1.1, and it only remains to prove Proposition 9.2.5.

*Proof of Proposition 9.2.5.* We split the integral in (9.21) into two parts by writing

$$\mathcal{L}^\varepsilon(\varphi) = I_1^\varepsilon(x,t) + I_2^\varepsilon(x,t)$$

with

$$I_1^\varepsilon(x,t) = \varepsilon^{-\alpha} \int_{|v|\geq\delta} \nu(v) M(v) \left[\chi^\varepsilon(x,v,t) - \varphi(x,t)\right] \, dv$$
$$I_2^\varepsilon(x,t) = \varepsilon^{-\alpha} \int_{|v|\leq\delta} \nu(v) M(v) \left[\chi^\varepsilon(x,v,t) - \varphi(x,t)\right] \, dv.$$

We first show that only the small values of $|v|$ matter in (9.21), by showing that $I_1^\varepsilon(x,t)$ goes to zero : Using formula (9.19) and integrations by parts, we can write

$$\begin{aligned}\nu(v)[\chi^\varepsilon(x,v,t)-\varphi(x,t)] &= \int_0^\infty \nu(v)^2 e^{-\nu(v)z}[\varphi(x+\varepsilon vz,t)-\varphi(x,t)]\,dz \\ &= \int_0^\infty \nu(v) e^{-\nu(v)z}\varepsilon v\cdot\nabla_x\varphi(x+\varepsilon vz,t)\,dz \\ &= \varepsilon v\cdot\nabla_x\varphi(x)+\varepsilon^2\int_0^\infty e^{-\nu(v)z} v^T\cdot D_x^2\varphi(x+\varepsilon vz,t)\cdot v\,dz.\end{aligned}$$

Using the null flux condition (9.9) (and the fact that $M=M_0$ for $|v|\le\delta$), we deduce

$$I_1^\varepsilon(x,t)=\varepsilon^{2-\alpha}\int_{|v|\ge\delta}\int_0^\infty M(v)e^{-\nu(v)z}v^T\cdot D_x^2\varphi(x+\varepsilon vz,t)\cdot v\,dz.\,dv$$

and so

$$\begin{aligned}I_1^\varepsilon(x,t) &\le \varepsilon^{2-\alpha}\left(\int_{|v|\ge\delta}\int_0^\infty M(v)|v|^2e^{-\nu(v)z}\,dz\,dv\right)^{1/2} \\ &\quad\times\left(\int_{|v|\ge\delta}\int_0^\infty M(v)|v|^2e^{-\nu(v)z}|D_x^2\varphi(x+\varepsilon vz,t)|^2\,dz\,dv\right)^{1/2}.\end{aligned}$$

We deduce

$$\begin{aligned}||I_1^\varepsilon(t)||_{L^2(\mathbf{R}^N)} &\le \varepsilon^{2-\alpha}||D_x^2\varphi(t)||_{L^2(\mathbf{R}^N)}\left(\int_{|v|\ge\delta}\int_0^\infty M(v)|v|^2e^{-\nu(v)z}\,dz\,dv\right) \\ &\le \left(\int_{|v|\ge\delta}M(v)\frac{|v|^2}{\nu(v)}\,dv\right)||D_x^2\varphi(t)||_{L^2(\mathbf{R}^N)}\varepsilon^{2-\alpha}\end{aligned}$$

which goes to $0$ as $\varepsilon\to 0$ (note that we use (9.10) here).

So it only remains to show that $I_2^\varepsilon(x,t)$ converges strongly in $L^2(\mathbf{R}^N\times(0,\infty))$ to $-\kappa(-\Delta)^{\alpha/2}(\varphi)$. We have :

$$I_2^\varepsilon(x,t) = \varepsilon^{-\alpha}\int_{|v|\le\delta}\int_0^\infty e^{-\nu(v)z}\nu(v)^2M(v)[\varphi(x+\varepsilon vz,t)-\varphi(x,t)]\,dz\,dv.$$

Integrating by parts (with respect to $z$) and performing the change of variable $s=\nu(v)z$, we can write

$$\begin{aligned}I_2^\varepsilon(x,t) &= \varepsilon^{-\alpha}\int_{|v|\le\delta}\int_0^\infty e^{-\nu(v)z}\nu(v)M(v)\varepsilon v\cdot\nabla_x\varphi(x+\varepsilon vz,t)\,dz\,dv \\ &= \varepsilon^{-\alpha}\int_{|v|\le\delta}\int_0^\infty e^{-s}M(v)\varepsilon v\cdot\nabla_x\varphi(x+\varepsilon\frac{v}{\nu(v)}s,t)\,ds\,dv.\end{aligned}$$

We now use Condition (9.8) and the change of variable $w=\varepsilon\frac{v}{\nu_0|v|^{N+2+\beta}}$. We denote $\gamma=N+1+\beta$, so that

$$w=\varepsilon\frac{v}{\nu_0|v|^{\gamma+1}},\qquad |w|=\frac{\varepsilon}{\nu_0}|v|^{-\gamma},\qquad v=\left(\frac{\varepsilon}{\nu_0|w|^{\gamma+1}}\right)^{\frac{1}{\gamma}}w$$

and

$$dv=\frac{1}{\gamma}\left(\frac{\varepsilon}{\nu_0|w|^{\gamma+1}}\right)^{\frac{N}{\gamma}}dw.$$

We deduce (note that $\alpha = \frac{\gamma+N+1}{\gamma}$) :

$$
\begin{aligned}
&I_2^\varepsilon(x,t) \\
&\quad = M_0\varepsilon^{-\alpha}\int_{|v|\le\delta}\int_0^\infty e^{-s}\varepsilon v\cdot\nabla_x\varphi(x+\varepsilon\frac{v}{\nu_0|v|^{\gamma+1}}s,t)\,ds\,dv \\
&\quad = M_0\varepsilon^{-\alpha}\int_{|w|\ge\frac{\varepsilon}{\nu_0}\delta^{-\gamma}}\int_0^\infty e^{-s}\left(\frac{\varepsilon^{\gamma+1}}{\nu_0|w|^{\gamma+1}}\right)^{\frac{1}{\gamma}} w\cdot\nabla_x\varphi(x+ws,t)\,ds\,\frac{1}{\gamma}\left(\frac{\varepsilon}{\nu_0|w|^{\gamma+1}}\right)^{\frac{N}{\gamma}} dw \\
&\quad = \frac{\varepsilon^{-\alpha}\varepsilon^{\frac{\gamma+1+N}{\gamma}}}{\gamma}\frac{M_0}{\nu_0^{(N+1)/\gamma}}\int_{|w|\ge\frac{\varepsilon}{\nu_0}\delta^{-\gamma}}\int_0^\infty e^{-s}\frac{1}{|w|^{N+\frac{\gamma+1+N}{\gamma}}}w\cdot\nabla_x\varphi(x+ws,t)\,ds\,dw \\
&\quad = \frac{1}{\gamma}\frac{M_0}{\nu_0^{\alpha-1}}\int_{|w|\ge\frac{\varepsilon}{\nu_0}\delta^{-\gamma}}\int_0^\infty e^{-s}\frac{1}{|w|^{N+\alpha}}w\cdot\nabla_x\varphi(x+ws,t)\,ds\,dw \qquad (9.22)
\end{aligned}
$$

The integral above can also be written as

$$
J^\varepsilon = \int_{|w|\ge c(\delta)\varepsilon}\int_0^\infty e^{-s}\frac{\varphi(x+ws,t)-\varphi(x,t)}{|w|^{N+\alpha}}\,ds\,dw. \qquad (9.23)
$$

The definition of the Cauchy principal value implies that it converges (pointwise) to

$$
\begin{aligned}
J^0 &= \mathrm{PV}\int_{\mathbf{R}^N}\int_0^\infty e^{-s}\frac{\varphi(x+ws,t)-\varphi(x,t)}{|w|^{N+\alpha}}\,ds\,dw \\
&= \left(\int_0^\infty e^{-s}s^\alpha\,ds\right)\mathrm{PV}\int_{\mathbf{R}^N}\frac{\varphi(x+y,t)-\varphi(x,t)}{|y|^{N+\alpha}}\,dy
\end{aligned}
$$

for all $(x,t)$, which is what we wanted. In order to show that we have convergence in $L^2(\mathbf{R}^N)$, we recall that we can also write

$$
\begin{aligned}
J^0 &= \int_{|w|\ge1}\int_0^\infty e^{-s}\frac{\varphi(x+ws,t)-\varphi(x,t)}{|w|^{N+\alpha}}\,ds\,dw \\
&\quad+\int_{|w|\le1}\int_0^\infty\frac{e^{-s}}{|w|^{N+\alpha}}[\varphi(x+ws,t)-\varphi(x,t)-sw\cdot\nabla\varphi(x)]\,ds\,dw
\end{aligned}
$$

with all integrals being defined in the classical sense (no principal value). Proceeding similarly, we can rewrite (9.23) as follows :

$$
\begin{aligned}
J^\varepsilon &= \int_{|w|\ge1}\int_0^\infty e^{-s}\frac{\varphi(x+ws,t)-\varphi(x,t)}{|w|^{N+\alpha}}\,ds\,dw \\
&\quad+\int_{c(\delta)\varepsilon\le|w|\le1}\int_0^\infty\frac{e^{-s}}{|w|^{N+\alpha}}[\varphi(x+ws,t)-\varphi(x,t)-sw\cdot\nabla\varphi(x,t)]\,ds\,dw
\end{aligned}
$$

and so

$$
J^\varepsilon - J^0 = \int_{|w|\le c(\delta)\varepsilon}\int_0^\infty\frac{e^{-s}}{|w|^{N+\alpha}}[\varphi(x+ws,t)-\varphi(x,t)-sw\cdot\nabla\varphi(x,t)]\,ds\,dw.
$$

Finally, integrating by parts (twice) with respect to $s$, we deduce :

$$
J^\varepsilon - J^0 = \int_{|w|\le c(\delta)\varepsilon}\int_0^\infty\frac{e^{-s}}{|w|^{N+\alpha}}w^T\cdot D^2\varphi(x+ws,t)\cdot w\,ds\,dw
$$

and so using the fact that

$$
\int_{|w|\le c(\delta)\varepsilon}\int_0^\infty\frac{e^{-s}}{|w|^{N+\alpha-2}}\,ds\,dw \le C\varepsilon^{2-\alpha}
$$

we deduce

$$
\left(\int_{\mathbf{R}^N}|J^\varepsilon-J^0|^2\,dx\right)^{1/2} \le C||D^2\varphi(\cdot,t)||_{L^2(\mathbf{R}^N)}\,\varepsilon^{2-\alpha}.
$$

Since $\alpha < 2$, this implies that $I_2^\varepsilon$ converges strongly in $L^2(\mathbf{R}^N \times (0,\infty))$ to $-\kappa(-\Delta)^{\alpha/2}(\varphi)$.

Writing

$$\mathcal{L}^\varepsilon(\varphi) = I_1^\varepsilon(x,t) + I_2^\varepsilon(x,t),$$

this completes the proof of Proposition 9.2.5 □

## 9.3 Proof of Lemma 9.2.2

We stress out the fact that a similar estimate (with $\rho$ instead of $\rho_\nu$) was derived in [14] under the condition :

$$\int_{\mathbf{R}^N} M(v') \frac{\nu(v)}{b(v,v')}\, dv' \le C \quad \text{for all} \quad v \in \mathbf{R}^N,$$

which, in the simplest case $b(v,v') = a(v)a(v')$, is equivalent to

$$\int_{\mathbf{R}^N} \frac{M(v')}{a(v')}\, dv' \int a(v')M(v')\, dv' \le C.$$

Unfortunately, this condition is clearly incompatible with the degeneracy of the collision frequency (9.8). By contrast, condition (9.13) reduces, when $b(v,v') = a(v)a(v')$, to

$$\int_{RR^N} a(v)M(v)\, dv < \infty$$

and is thus clearly satisfied.

*Proof of Lemma 9.2.2.* A simple computation using (9.3) yields

$$-\int_{\mathbf{R}^N} L(f)\frac{f}{F}\, dv = \frac{1}{2}\int_{\mathbf{R}^N}\int_{\mathbf{R}^N} b(v,v')M\,M'\left[\frac{f'}{M'} - \frac{f}{M}\right]^2 dv\, dv', \tag{9.24}$$

where $M = M(v)$ and $M' = M(v')$. Next, integrating (with respect to $v'$) the identity

$$f\,\nu'\,M' - f'\,\nu'\,M = \nu' M\,M'\left[\frac{f}{M} - \frac{f'}{M'}\right]$$

we get

$$f\int \nu M\, dv - \rho_\nu M \int \nu M\, dv = \int_{\mathbf{R}^N} \nu'\,M\,M'\left[\frac{f}{M} - \frac{f'}{M'}\right] dv'$$

or

$$g\int \nu M\, dv = \int_{\mathbf{R}^N} \nu'\,M\,M'\left[\frac{f}{M} - \frac{f'}{M'}\right] dv'$$

where $g = f - \rho_\nu M$. The Cauchy-Schwarz inequality implies

$$\begin{aligned} g^2 &\left(\int \nu M\, dv\right)^2 \\ &\le \left(\int_{\mathbf{R}^N} b(v,v')\,M\,M'\left[\frac{f}{M} - \frac{f'}{M'}\right]^2 dv'\right)\left(\int_{\mathbf{R}^N} \frac{\nu'^2}{b(v,v')}\,M\,M'\,dv'\right), \end{aligned}$$

so that

$$\begin{aligned} &\int_{\mathbf{R}^N} \frac{\nu g^2}{M}\, dv \left(\int \nu M\, dv\right)^2 \\ &\le \left(\sup_{v\in\mathbf{R}^N}\int_{\mathbf{R}^N} \frac{\nu\nu'^2}{b(v,v')}M'dv'\right)\left(\int_{\mathbf{R}^N}\int_{\mathbf{R}^N} bMM'\left[\frac{f}{M} - \frac{f'}{M'}\right]^2 dv'\, dv\right) \end{aligned}$$

and (9.24) gives the result.
□

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# Chapitre 10

# Fractional diffusion limit for collisional kinetic equations : a Hilbert expansion approach

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**abstract**

We provide in this paper a Hilbert expansion proof of the fractional diffusion approximation of the linear Boltzmann equation corresponding to heavy tail equilibria.

## 10.1 Setting of the result

### 10.1.1 Introduction

The linear Boltzmann equation is a simple kinetic equation describing the evolution of a particle distribution function. It couples free transport and scattering phenomena (due to collisions with the background). The properties of the scattering process determines the long time behavior of the particle distribution function. It fixes in particular the profile of the equilibrium velocity distribution.

Asymptotic analysis of such an equation is a very classical problem. Typically, assuming that the mean free path (i.e. distance between two collisions) is very small and the time scale is very large, it is possible to derive a hydrodynamic type equation describing the evolution of the density of particles. Often, the thermodynamical equilibrium is given by a Maxwellian distribution function, and the mean square displacement of the particle is a linear function of time. The correct time/space scaling ($x = \varepsilon x'$ and $t = \varepsilon^2 t'$) then leads to a diffusion equation for the density of particles. This type of diffusion approximation for kinetic equations has been widely studied in several papers, see for instance [1, 4, 10, 7] (and references therein).

In recent works (see [14], [13], [3]) similar asymptotic analysis are performed when the equilibrium function is not a Maxwellian distribution, but rather a heavy tail function. In this case, the diffusion coefficient appearing in the classical diffusion limit is no longer well defined and one has to modify the time scale to recover an equation for the particle density. This is known in the literature as anomalous

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diffusion phenomena (the mean square displacement of the particle is not a linear function of time), and it leads to fractional diffusion equations.

Denoting by $\varepsilon$ the ratio of the microscopic length scale (mean free path) and macroscopic length scale, we scale the space variable as $x = \varepsilon x'$ and the time variable as $t = \varepsilon^\alpha t'$. The starting point of this paper is then the following rescaled Boltzmann equation :

$$\varepsilon^\alpha \partial_t f^\varepsilon + \varepsilon v \cdot \nabla_x f^\varepsilon = Q(f^\varepsilon) \tag{10.1}$$

where the particle distribution function $f^\varepsilon(t,x,v)$ depends on the time $t > 0$, the space variable $x \in \mathbf{R}^n$ and the velocity variable $v \in \mathbf{R}^n$. The collision operator appearing in the right hand side is a linear integral operator of the form :

$$Q(f) = \int_{\mathbf{R}^n} \sigma(x,v,v')\big[f(v')F(v) - f(v)F(v')\big]dv' \tag{10.2}$$

with

$$\sigma(x,v,v') = \sigma(x,v',v).$$

Note that the operator $Q$ is the sum of a gain term

$$Q^+(f) = \int_{\mathbf{R}^n} \sigma(x,v,v')f(v')\,dv'\,F(v)$$

and a loss term

$$Q^-(f) = -\nu(v)f(v) \quad \text{ with } \nu(v) = \int_{\mathbf{R}^n} \sigma(x,v,v')F(v')dv'.$$

This operator has a one dimensional kernel spanned by the function $F(v)$ and it preserves the total mass :

$$\int_{\mathbf{R}^n} Q(f)\,dv = 0 \qquad \text{for all } f. \tag{10.3}$$

The usual diffusion limit corresponds to $\alpha = 2$. It can be studied using the so-called *Hilbert expansion method*, which is based on a formal expansion of the solution in the form

$$f^\varepsilon = f^0 + \varepsilon f^1 + \varepsilon^2 f^2 + r^\varepsilon.$$

Inserting this expansion in (10.1) and identifying the terms of same order in $\varepsilon$ yields :

$$Q(f^0) = 0 \tag{10.4}$$
$$Q(f^1) = v \cdot \nabla_x f^0 \tag{10.5}$$
$$Q(f^2) = \partial_t f^0 + v \cdot \nabla_x f^1 \tag{10.6}$$

and the remainder $r^\varepsilon$ solves

$$\partial_t r^\varepsilon + \varepsilon^{-1} v \cdot \nabla_x r^\varepsilon = \varepsilon^{-2} Q(r^\varepsilon) + \varepsilon \partial_t f^1 + \varepsilon^2 \partial_t f^2 + \varepsilon v \cdot \nabla_x f^2.$$

Equation (10.4) implies that $f^0(t,x,v) = \rho(t,x)F(v)$ and (10.5) then yields $f^1 = Q^{-1}(v \cdot \nabla_x f^0) = Q^{-1}(vF(v)) \cdot \nabla_x \rho$ (assuming that such an inverse exists, see Proposition 10.2.1). Finally, in view of (10.3), the equation (10.6) for $f^2$ gives (integrating with respect to $v$) :

$$\partial_t \rho + \nabla_x \cdot j = 0$$

where

$$j = \int_{\mathbf{R}^n} v f^1\,dv = \int_{\mathbf{R}^n} v \otimes Q^{-1}(vF(v))\,dv \nabla_x \rho.$$

We deduce that for $f^2$ to exist, $\rho$ must solve

$$\partial_t \rho - \nabla_x \cdot D \nabla_x \rho = 0$$

with diffusion matrix

$$D = -\int_{\mathbf{R}^n} v \otimes Q^{-1}(vF(v))\,dv.$$

Now, assuming that $\rho$ is a (smooth solution) of this diffusion equation, we can define $f^0$, $f^1$ and $f^2$ solutions of (10.4), (10.5) and (10.6), and show that the remainder $r^\varepsilon$ converges strongly to 0 in some $L^2$ space. The main drawback of this method, compared with the moment method, is that it requires stronger regularity assumptions on the initial data in order to get the appropriate bounds on $f^0$, $f^1$ and $f^2$. On the other hand the method yields strong convergence of the solution (rather than weak convergence for the moment method). Hilbert expansion type methods have also proved very useful to study diffusion limits for some non linear collision operators, see [12, 9, 11].

In this paper we are considering a situation in which this limit fails because the diffusion matrix $D$ above is infinite. This is clearly the case when the equilibrium function $F$, rather than a Maxwellian distribution function is a heavy tail distribution function, satisfying

$$F(v) \sim \frac{\kappa_0}{|v|^{n+\alpha}} \qquad \text{as } |v| \to \infty$$

for some $\alpha \in (0,2)$ (note that this $\alpha$ will be the same as the time scale in (10.1)). Such a profile has been mentioned in several context such as astrophysical plasmas ([17, 15] or in the study of granular plasmas ([5, 8] for inelastic Maxwell models, [16] for inelastic Kac model and [6] for mixture of gases with Maxwellian collision kernel). A similar study is undertaken in [14] using a relatively simple method based on the Fourier transform. This method does not generalize easily to more complicated situation involving space dependent collision kernel $\sigma$ or non linear operator. In [13], an alternative method is developed which is based on the weak formulation of the Boltzmann equation (10.1) and the choice of particular test functions defined via an auxiliary equation. This method is somewhat analogous to the well known *moment method* for the usual diffusion limit. Both approaches provide the weak convergence of $f^\varepsilon$ to $\rho F$. The aim of the present paper is to develop a Hilbert expansion approach for this anomalous diffusion regime. One of the advantages of this new method (compared with [14, 13]) will be to give strong convergence results. We also believe that this method can be used to address some nonlinear problems, such as models involving the Boltzmann-Pauli collision operator.

The first difficulty in developing such an expansion method is that since $\alpha \in (0,2)$, it is not obvious how one should identify the terms with same order (some terms have a fractional order). The originality of this expansion is thus the splitting of the different terms of equation (10.1) and their distribution in the different levels of the expansion. This splitting follows a heuristic based on the choice of the auxiliary function in [13]. Indeed, we notice that the advection term and the gain loss of the collision operator are of the same order whereas the gain term is of higher order (in term of $\varepsilon$). This leads to a rearrangement in the Hilbert expansion as explained in section 10.3.

We are now going to list our assumptions on $\sigma$ and $F$ and give the main result of this paper (Theorem 10.1.1 below). In Section 10.2, we will present a proof of this result in the simpler case $\sigma = 1$. In this case, the expansion is relatively simple, and we will explain why in the more general case, a more complicated expansion needs to be introduced. The main theorem is then proved in Section 10.3.

### 10.1.2 Main result

In order to simplify the computation, we will assume that there exist some constants $\sigma_0$ and $\sigma_1$ such that

$$0 < \sigma_0 \le \sigma(x,v,v') = \sigma(x,v',v) \le \sigma_1 < \infty \quad \text{for all } (x,\, v,\, v') \in \mathbf{R}^{3n}. \tag{10.7}$$

In particular, this implies that the collision frequency

$$\nu(x,v) = \int_{\mathbf{R}^n} \sigma(x,v,v')F(v')dv'$$

satisfies

$$\sigma_0 \le \nu(x,v) \le \sigma_1 \quad \text{for all } (x,v) \in \mathbf{R}^{2n}.$$

We also need to assume that the collision frequency is regular enough and has a nice asymptotic behavior for large $v$. For simplicity, we will assume :

$$\begin{aligned} &\partial_x^k \nu \in L^\infty(\mathbf{R}^n \times \mathbf{R}^n) \text{ for all multi-indice } k \text{ such that } |k| \le 4 \\ &\nu(x,v) = \nu_0(x) \quad \forall |v| > 1 \end{aligned} \tag{10.8}$$

(this last assumption can of course be replaced with $\lim_{|v|\to\infty} \nu(x,v) = \nu_0(x)$).

The most important assumption concerns the behavior of the equilibrium $F(v)$ for large $v$. We assume :

$$F(v) = \frac{\kappa}{|v|^{n+\alpha}} \qquad \text{for } |v| \geq 1 \tag{10.9}$$

for some $\alpha \in (0,2)$ and

$$F(v) \leq \kappa \qquad \text{for all } v \in \mathbf{R}^n\,. \tag{10.10}$$

Finally, we assume that the following symmetry conditions are satisfied :

$$F(-v) = F(v)\,, \qquad \nu(x,-v) = \nu(x,v) \quad \text{for all } (x\,v) \in \mathbf{R}^{2n}. \tag{10.11}$$

This implies in particular that the flux $\int_{\mathbf{R}^n} vF(v)\,dv = 0$ (when it is defined, that is when $\alpha > 1$), which we know is a necessary condition for the derivation of diffusion type equation.

We can now state our main result :

*Theorem* 10.1.1. Assume that (10.7), (10.8), (10.9), (10.10), (10.11) hold and let $f^\varepsilon$ be the solution of (10.1) with initial condition $f_{in}(x,v) = \rho_{in}(x)F(v)$ such that $\rho_{in} \in H^4(\mathbf{R}^n)$, then

$$||f^\varepsilon - \rho_0 F||_{L^\infty(0,\infty;L^2_{F^{-1}}(\mathbf{R}^n\times\mathbf{R}^n))} \longrightarrow 0, \qquad \text{as } \varepsilon \to 0, \tag{10.12}$$

where $\rho_0$ satisfies

$$\begin{cases} \partial_t\rho_0 + L(\rho_0) = 0 & \text{in } [0,\infty)\times\mathbf{R}^n, \\ \rho_0(0,x) = \rho_{in}(x) & \text{in } \mathbf{R}^n \end{cases} \tag{10.13}$$

and $L$ is an elliptic operator defined by the singular integral :

$$L(\rho) = \text{P.V.} \int_{\mathbf{R}^n} \gamma(x,y)\frac{\rho(x)-\rho(y)}{|y-x|^{n+\alpha}}\,dy. \tag{10.14}$$

with

$$\gamma(x,y) = \nu_0(x)\,\nu_0(y)\int_0^\infty z^\alpha e^{-z\int_0^1 \nu_0((1-s)x+sy)\,ds}\,dz$$

Note that $\gamma(x,y) = \gamma(y,x)$ and that condition (10.7) yields

$$0 < \gamma_1 < \gamma(x,y) \leq \gamma_2.$$

In particular the operator $L$ is a self-adjoint elliptic operator of order $\alpha$ (comparable to $(-\Delta)^{\alpha/2}$).

Before giving the proof of this theorem in the general case (Section 10.3), we will present two simpler cases that can be handled with a more direct Hilbert method based on the Fourier Transform and for which the result is obtained in a $L^1$ framework as well as in a $L^2$ framework with less regularity assumptions on the initial data.

## 10.2 Space homogeneous cross sections

### 10.2.1 Constant cross section

In this section, we prove Theorem 10.1.1 in the simpler framework where the cross section $\sigma$ is constant (for the sake of simplicity we take $\sigma = 1$). We then have

$$Q(f) = \int_{\mathbf{R}^n} f(v')\,dv'\,F(v) - f(v). \tag{10.15}$$

In that case, the use of the Fourier transform (as in [14]) allows for explicit computations and a much simpler proof.

We have the following proposition :

*Proposition.*

Let $f^\varepsilon$ be the solution of (10.1) with $Q$ given by (10.15) where $F$ satisfies (10.9-10.11) and with initial condition $f_{in}(x,v) = \rho_{in}(x)F(v)$ such that $\rho_{in} \in W^{3,1}(\mathbf{R}^n)$. Then

$$||f^\varepsilon - \rho_0 F||_{L^\infty(0,\infty;L^1(\mathbf{R}^n\times\mathbf{R}^n))} \longrightarrow 0, \qquad \text{as } \varepsilon \to 0 \tag{10.16}$$

where $\rho_0$ satisfies

$$\begin{cases} \partial_t \rho_0 + \kappa(-\Delta)^{\frac{\alpha}{2}}\rho_0 = 0 & \text{in } [0,\infty)\times\mathbf{R}^n \\ \rho_0(0,x) = \rho_{in}(x) & \text{in } \mathbf{R}^n \end{cases} \tag{10.17}$$

with

$$\kappa = \int_{\mathbf{R}^n} \frac{(w\cdot e)^2}{1+(w\cdot e)^2}\frac{1}{|w|^{n+\alpha}}\,dw. \tag{10.18}$$

If instead we have $\rho_{in} \in H^3(\mathbf{R}^n)$, then

$$||f^\varepsilon - \rho_0 F||_{L^\infty(0,\infty;L^2_{F^{-1}}(\mathbf{R}^n\times\mathbf{R}^n))} \longrightarrow 0, \qquad \text{as } \varepsilon \to 0. \tag{10.19}$$

The proof relies on the following Hilbert expansion for $f^\varepsilon$ :

$$f^\varepsilon = f_0^\varepsilon + g_1^\varepsilon + g_2^\varepsilon + r^\varepsilon,$$

with

$$Q(f_0^\varepsilon) = 0 \tag{10.20}$$

$$(1+\varepsilon v\cdot\nabla_x)g_1^\varepsilon = -\varepsilon v\cdot\nabla_x f_0^\varepsilon \tag{10.21}$$

$$Q(g_2^\varepsilon) = \varepsilon^\alpha \partial_t f_0^\varepsilon - Q^+(g_1^\varepsilon) \tag{10.22}$$

and $r^\varepsilon$ solution of

$$\partial_t r^\varepsilon + \varepsilon^{1-\alpha} v\cdot\nabla_x r^\varepsilon = \varepsilon^{-\alpha}Q(r^\varepsilon) - \partial_t g_1^\varepsilon - \partial_t g_2^\varepsilon - \varepsilon^{1-\alpha} v\cdot\nabla g_2^\varepsilon. \tag{10.23}$$

In order to investigate these equations, we recall the following result, which holds for general operators of the form (10.2), whenever (10.7) holds (see [7]) :

*Proposition.*

i) The operators $Q^+$ and $Q$ are bounded operator on $L^2_{F^{-1}}(\mathbf{R}^n)$.

ii) The kernel of $Q$ has dimension 1 and is spanned by $F$.

iii) There exists a constant $c>0$ such that

$$-\int_{\mathbf{R}^n} Q(f) f\frac{1}{F}\,dv \geq c\int_{\mathbf{R}^n} |f-\rho F|^2\frac{1}{F}\,dv.$$

iv) For all $h \in L^2_{F^{-1}}(\mathbf{R}^n)$, the equation $Q(g)=h$ has a solution if and only if $\int h(v)\,dv = 0$, and there is a unique such solution (denoted $Q^{-1}(h)$) satisfying $\int_{\mathbf{R}^n} g(v)\,dv = 0$.

v) The operator $Q^{-1}$ is bounded in $L^2_{F^{-1}}(\mathbf{R}^n)$ :

$$\|Q^{-1}(h)\|_{L^2_{F^{-1}}} \leq C\|h\|_{L^2_{F^{-1}}} \quad \text{for all } h \text{ such that } \int_{\mathbf{R}^n} h(v)\,dv = 0. \tag{10.24}$$

vi) $\int Q(f)\mathrm{sgn}(f)\,dv \leq 0$ for all $f$.

Equation (10.20) then yields (using (ii)) :

$$f_0^\varepsilon(x,v,t) = \rho^\varepsilon(x,t)F(v).$$

and so equation (10.21) becomes

$$\mathbf{T}g_1^\varepsilon = -\varepsilon v\cdot\nabla_x\rho^\varepsilon F$$

where we introduced the operator

$$\mathbf{T} = 1 + \varepsilon v \cdot \nabla_x.$$

Assuming (for the time being) that this operator can be inverted, we can then write

$$g_1^\varepsilon = -\mathbf{T}^{-1}(\varepsilon v \cdot \nabla_x \rho^\varepsilon) F. \tag{10.25}$$

Finally, the solvability condition for (10.22) (using (iv)) is

$$\int \left[\varepsilon^\alpha \partial_t \rho^\varepsilon F - Q^+(g_1^\varepsilon)\right] \, dv = 0$$

which reads

$$\varepsilon^\alpha \partial_t \rho^\varepsilon - \int g_1^\varepsilon \, dv = 0.$$

Using (10.25), we see that this is a condition on $\rho^\varepsilon$, which we can write as

$$\varepsilon^\alpha \partial_t \rho^\varepsilon + \int \mathbf{T}^{-1}(\varepsilon v \cdot \nabla_x \rho^\varepsilon) F \, dv = 0. \tag{10.26}$$

Because of the simple form of the operator $Q$, this condition also implies that the right hand side of (10.22) is actually 0 (rather than only having zero integral), and thus

$$g_2^\varepsilon = 0.$$

*Remark* 10.2.1. The usual Hilbert expansion consists in writing $f^\varepsilon = \rho F + f_1^\varepsilon + f_2^\varepsilon + r^\varepsilon$ where

$$\nu(x,v) f_1^\varepsilon = Q^+(f_1^\varepsilon) - \varepsilon v \cdot \nabla_x \rho F,$$

$$Q(f_2^\varepsilon) + \varepsilon v \cdot \nabla_x f_1^\varepsilon = \varepsilon^\alpha \partial_t \rho F.$$

We can see that the scheme that we use here consists in reshuffling the terms $Q^+(f_1^\varepsilon)$ and $\varepsilon v \cdot \nabla_x f_1^\varepsilon$ in the first order and second order equation.

Note also that while in the usual Hilbert expansion, the first term is solution of the asymptotic equation, here we have $\rho^\varepsilon$ which is solution of an approximation of the limiting model (10.26).

In order to prove Proposition 10.2.1, we thus have to show that $\rho^\varepsilon$, solution of (10.26), exists and that $g_1^\varepsilon$ and $r^\varepsilon$ go to zero in some appropriate norm.

This is best done using Fourier transform (with respect to $x$) : We denote by $\widehat{h}(\xi)$ the Fourier transform, with respect to $x$, of a function $h$. Taking the Fourier transform in (10.21), we get :

$$(1 + i\varepsilon v \cdot \xi)\widehat{g_1^\varepsilon} = -i\varepsilon v \cdot \xi \widehat{f_0}$$

and so

$$\widehat{g_1^\varepsilon} = -\frac{i\varepsilon v \cdot \xi}{1 + i\varepsilon v \cdot \xi} F(v) \, \widehat{\rho^\varepsilon}. \tag{10.27}$$

We can thus rewrite (10.26) as

$$\partial_t \widehat{\rho^\varepsilon} + a^\varepsilon(\xi) \, \widehat{\rho^\varepsilon} = 0. \tag{10.28}$$

with

$$a^\varepsilon(\xi) = \varepsilon^{-\alpha} \int_{\mathbf{R}^n} \frac{i\varepsilon v \cdot \xi}{1 + i\varepsilon v \cdot \xi} F(v) \, dv.$$

We now need the following lemma (see also [14]) :

*Lemma* 10.2.1. For all $\xi$, $a^\varepsilon(\xi) \geq 0$ and

$$|a^\varepsilon(\xi) - \kappa |\xi|^\alpha| \leq C\varepsilon^{2-\alpha}|\xi|^2 \longrightarrow 0, \tag{10.29}$$

with $\kappa$ given by (10.18). Furthermore, there exist positive constants $c_1$ and $c_2$ such that,

$$c_1 \min(\varepsilon^{-\alpha}, |\xi|^\alpha) \leq a^\varepsilon(\xi) \leq c_2(|\xi|^\alpha + \varepsilon^{(2-\alpha)}|\xi|^2) \tag{10.30}$$

*Proof of Lemma 10.2.1.* A simple computation (using the properties of $F$) yields

$$\begin{aligned}\int_{\mathbf{R}^n} \frac{i\varepsilon v\cdot\xi}{1+i\varepsilon v\cdot\xi}\,F(v)\,dv &= \int_{\mathbf{R}^n} \frac{(\varepsilon v\cdot\xi)^2}{1+(\varepsilon v\cdot\xi)^2}\,F(v)\,dv\\ &= \int_{|v|\le 1} \frac{(\varepsilon v\cdot\xi)^2}{1+(\varepsilon v\cdot\xi)^2}\,F(v)\,dv\\ &\quad+\int_{|v|\ge 1} \frac{(\varepsilon v\cdot\xi)^2}{1+(\varepsilon v\cdot\xi)^2}\,\frac{1}{|v|^{n+\alpha}}\,dv.\end{aligned}$$

The first integral is clearly bounded by $C\varepsilon^2|\xi|^2$, and using the change of variable $w=\varepsilon v|\xi|$ we can write the second integral as

$$\int_{|v|\ge 1} \frac{(\varepsilon v\cdot\xi)^2}{1+(\varepsilon v\cdot\xi)^2}\,\frac{1}{|v|^{n+\alpha}}\,dv = \varepsilon^\alpha|\xi|^\alpha\int_{|w|\ge\varepsilon|\xi|} \frac{(w\cdot e)^2}{1+(w\cdot e)^2}\,\frac{1}{|w|^{n+\alpha}}\,dw$$

We deduce

$$\begin{aligned}|\varepsilon^\alpha a^\varepsilon(\xi)-\varepsilon^\alpha\kappa|\xi|^\alpha| &\le C\varepsilon^2|\xi|^2+\varepsilon^\alpha|\xi|^\alpha\int_{|w|\le\varepsilon|\xi|} \frac{(w\cdot e)^2}{1+(w\cdot e)^2}\,\frac{1}{|w|^{n+\alpha}}\,dw\\ &\le C\varepsilon^2|\xi|^2\end{aligned}$$

This implies (10.29) and the second inequality in (10.30). The first inequality in (10.30) is obtained by distinguishing two cases :

If $\varepsilon|\xi|\le 1$, then the computation above yields $\int_{|v|\ge 1}\frac{(\varepsilon v\cdot\xi)^2}{1+(\varepsilon v\cdot\xi)^2}\,\frac{1}{|v|^{n+\alpha}}\,dv \ge c\varepsilon^\alpha|\xi|^\alpha$ and the result follows.

If $\varepsilon|\xi|\ge 1$, then we have

$$\begin{aligned}\int_{|v|\ge 1} \frac{(\varepsilon v\cdot\xi)^2}{1+(\varepsilon v\cdot\xi)^2}\,\frac{1}{|v|^{n+\alpha}}\,dv &\ge \varepsilon^\alpha|\xi|^\alpha\int_{|w|\ge\varepsilon|\xi|}\frac{1}{2}\frac{1}{|w|^{n+\alpha}}\,dw\\ &\ge 1\end{aligned}$$

and so

$$a^\varepsilon \ge \varepsilon^{-\alpha}.$$

□

We deduce

*Corollary* 10.2.2. For any $\rho_{in}$ and $\varepsilon>0$, there exists a unique $\rho^\varepsilon$ solution of (10.28), given by

$$\widehat{\rho^\varepsilon}(t,\xi)=e^{-ta^\varepsilon(\xi)}\widehat{\rho_{in}}(\xi)$$

Furthermore, $\rho^\varepsilon$ satisfies

$$||\,|\xi|^k\widehat{\rho^\varepsilon}(t)||_{L^\infty(\mathbf{R}^n)} \le ||\,|\xi|^k\widehat{\rho_{in}}||_{L^\infty(\mathbf{R}^n)}$$

and

$$||\,|\xi|^k\partial_t\widehat{\rho^\varepsilon}(t)||_{L^\infty(\mathbf{R}^n)} \le ||\,|\xi|^{k+2}\widehat{\rho_{in}}||_{L^\infty(\mathbf{R}^n)}$$

for all $k\ge 0$. Finally we have

$$||\rho^\varepsilon-\rho_0||_{L^\infty(0,T;L^1(\mathbf{R}^n))} \le CT||\rho_{in}||_{W^{2,1}(\mathbf{R}^n)}$$

and

$$||\rho^\varepsilon-\rho_0||_{L^\infty(0,T;L^1(\mathbf{R}^n))} \le CT||\rho_{in}||_{H^2(\mathbf{R}^n)}$$

where $\rho_0$ is the unique solution of (10.17).

*Démonstration.* We only prove the last assertion : We note that

$$\partial_t(\widehat{\rho^\varepsilon} - \widehat{\rho_0}) + a^\varepsilon(\xi)(\widehat{\rho^\varepsilon} - \widehat{\rho_0}) = (\kappa|\xi|^\alpha - a^\varepsilon(\xi))\widehat{\rho_0}$$

and so Lemma 10.2.1 yields

$$\frac{d}{dt}|\widehat{\rho^\varepsilon} - \widehat{\rho_0}| \leq C\varepsilon^{2-\alpha}|\xi|^2\widehat{\rho_0}.$$

We deduce

$$\frac{d}{dt}||\rho^\varepsilon - \rho_0||_{L^1(\mathbf{R}^n)} \leq C\varepsilon^{2-\alpha}||\rho_0||_{W^{2,1}(\mathbf{R}^n)}$$

which gives the $L^1(\mathbf{R}^n)$ convergence of $\rho^\varepsilon$. A similar computation (multiplying by $\widehat{\rho^\varepsilon} - \widehat{\rho_0}$) gives the $L^2(\mathbf{R}^n)$ convergence.
□

*Proof of Proposition 10.2.1.* We now notice that (10.27) yields

$$|\widehat{g_1^\varepsilon}| \leq \frac{|\varepsilon v \cdot \xi|}{\sqrt{1 + (\varepsilon v \cdot \xi)^2}} F(v)\widehat{\rho^\varepsilon}$$

and thus

$$\begin{aligned} ||\widehat{g_1^\varepsilon}||_{L^\infty_\xi(\mathbf{R}^n;L^1_v(\mathbf{R}^N)))} &\leq \sup_\xi \left\{ \int \frac{|\varepsilon v \cdot \xi|}{\sqrt{1 + (\varepsilon v \cdot \xi)^2}} F(v)\, dv \widehat{\rho^\varepsilon}(\xi) \right\} \\ &\leq \sup_\xi \left\{ C(\varepsilon^\alpha|\xi|^\alpha + \varepsilon|\xi|)\widehat{\rho^\varepsilon} \right\} \\ &\leq C\left(\varepsilon^\alpha + \varepsilon\right) ||(1 + |\xi|^2)\widehat{\rho^\varepsilon}||_{L^\infty(\mathbf{R}^n)}, \end{aligned}$$

and similarly

$$\begin{aligned} ||\partial_t\widehat{g_1^\varepsilon}||_{L^\infty_\xi(\mathbf{R}^n;L^1_v(\mathbf{R}^N)))} &\leq \sup_\xi \left\{ C(\varepsilon^\alpha|\xi|^\alpha + \varepsilon|\xi|)\partial_t\widehat{\rho^\varepsilon} \right\} \\ &\leq \sup_\xi \left\{ C(\varepsilon^\alpha|\xi|^\alpha + \varepsilon|\xi|)a^\varepsilon(\xi)\widehat{\rho^\varepsilon} \right\} \end{aligned}$$

so, using Lemma 10.2.1, we deduce

$$||\partial_t\widehat{g_1^\varepsilon}||_{L^\infty_\xi(\mathbf{R}^n;L^1_v(\mathbf{R}^N)))} \leq (\varepsilon^\alpha + \varepsilon)||(1 + |\xi|^3)\widehat{\rho^\varepsilon}||_{L^\infty(\mathbf{R}^n)}.$$

It follows that $g_1^\varepsilon$ and $\partial_t g_1^\varepsilon$ converge to 0 strongly in $L^\infty(0,\infty;L^1(\mathbf{R}^n \times \mathbf{R}^n))$. Finally, multiplying the equation for the remainder $r^\varepsilon$, (10.23), by $\mathrm{sign}(r^\varepsilon)$ (and using Proposition 10.2.1 (vi)), we get

$$\partial_t||r^\varepsilon||_{L^1(\mathbf{R}^n)} \leq ||\partial_t g_1^\varepsilon||_{L^1(\mathbf{R}^n)}.$$

Using the fact that $r^\varepsilon(t=0) = -g_1^\varepsilon(t=0)$, we deduce

$$||r^\varepsilon||_{L^\infty(0,T;L^1(\mathbf{R}^{2n}))} \longrightarrow 0$$

and so

$$||f^\varepsilon - \rho^\varepsilon F||_{L^\infty(0,T;L^1(\mathbf{R}^n \times \mathbf{R}^n))} \longrightarrow 0.$$

Similarly, we have (using the computations of Lemma 10.2.1) :

$$\begin{aligned} \int_{\mathbf{R}^n} |\widehat{g_1^\varepsilon}|^2 \frac{1}{F(v)}\, dv &\leq \int_{\mathbf{R}^n} \frac{|\varepsilon v \cdot \xi|^2}{1 + (\varepsilon v \cdot \xi)^2} F(v)\, dv \widehat{\rho^\varepsilon}^2 \\ &\leq C(\varepsilon^\alpha|\xi|^\alpha + \varepsilon^2|\xi|^2)\widehat{\rho^\varepsilon}^2 \end{aligned}$$

and so

$$||\widehat{g_1^\varepsilon}||_{L^2_{F^{-1}}(\mathbf{R}^n \times \mathbf{R}^n)} \leq C\varepsilon^\alpha||\rho^\varepsilon||_{H^1(\mathbf{R}^n)}$$

and in a same way

$$||\partial_t\widehat{g_1^\varepsilon}||_{L^2_{F^{-1}}(\mathbf{R}^n \times \mathbf{R}^n)} \leq C\varepsilon^\alpha||\rho^\varepsilon||_{H^3(\mathbf{R}^n)}.$$

A similar argument (multiplying (10.23) by $r^\varepsilon/F$) gives the strong convergence in $L^\infty(0,\infty;L^2_{F^{-1}}(\mathbf{R}^n \times \mathbf{R}^n))$. □

### 10.2.2 Case of space homogeneous cross section

We now show, in a still relatively simple case, why the general case is more complicated. In this section, we assume that $Q$ is given by :

$$Q(f) = \int_{\mathbf{R}^n} \sigma(v, v')[f(v')F(v) - f(v)F(v')]\, dv' \tag{10.31}$$

with $\sigma$ depending only on $v$ and $v'$ (and not on $x$), and satisfying

$$0 < \sigma_0 \leq \sigma(v, v') = \sigma(v', v) \leq \sigma_1.$$

We might try to proceed as in the previous section, writing

$$f^\varepsilon = f_0^\varepsilon + g_1^\varepsilon + g_2^\varepsilon + r^\varepsilon,$$

with

$$\begin{aligned} f_0^\varepsilon(x, v, t) &= \rho^\varepsilon(x, t)F(v), \\ g_1^\varepsilon &= -\mathbf{T}^{-1}(\varepsilon v \cdot \nabla_x \rho^\varepsilon), F \\ Q(g_2^\varepsilon) &= \varepsilon^\alpha \partial_t \rho^\varepsilon F - Q^+(g_1^\varepsilon). \end{aligned}$$

where the operator $\mathbf{T}$ is now defined by $\mathbf{T}(f) = \nu f + \varepsilon v \cdot \nabla_x f$.

The solvability condition for $g_2^\varepsilon$ (obtained by integrating this last equation with respect to $v$) reads

$$\begin{aligned} \partial_t \rho^\varepsilon &= \varepsilon^{-\alpha} \int_{\mathbf{R}^n} Q^+(g_1^\varepsilon)\, dv \\ &= \varepsilon^{-\alpha} \int_{\mathbf{R}^n} \nu g_1^\varepsilon\, dv \\ &= -\varepsilon^{-\alpha} \int_{\mathbf{R}^n} \nu(v)\mathbf{T}^{-1}(\varepsilon v \cdot \nabla_x \rho^\varepsilon)F\, dv. \end{aligned}$$

We thus obtain that $\rho^\varepsilon$ must solve

$$\partial_t \rho^\varepsilon + L^\varepsilon(\rho^\varepsilon) = 0$$

with

$$L^\varepsilon(\rho) = \varepsilon^{-\alpha} \int_{\mathbf{R}^n} \nu(v)\mathbf{T}^{-1}(\varepsilon v \cdot \nabla_x \rho^\varepsilon)F\, dv.$$

Once again, we can identify the nature of this operator using Fourier transform : We get

$$\begin{aligned} \widehat{L^\varepsilon(\rho)} &= \varepsilon^{-\alpha} \int_{\mathbf{R}^n} \frac{i\varepsilon v \cdot \xi}{\nu(v) + i\varepsilon v \cdot \xi} \nu(v)F(v)\, dv \widehat{\rho^\varepsilon} \\ &= a^\varepsilon(\xi)\widehat{\rho^\varepsilon}. \end{aligned}$$

However, unlike the previous section, we do not have $Q(g_2^\varepsilon) = 0$. Instead, we have

$$Q(g_2^\varepsilon) = -\varepsilon^\alpha L^\varepsilon(\rho^\varepsilon)F - Q^+(g_1^\varepsilon)$$

which, in Fourier, reads :

$$Q(\widehat{g_2^\varepsilon}) = -\varepsilon^\alpha \int_{\mathbf{R}^n} \frac{i\varepsilon v' \cdot \xi}{\nu(v') + i\varepsilon v' \cdot \xi} [\nu(v') - \sigma(v, v')]F(v')\, dv' \widehat{\rho^\varepsilon} F(v)$$

and so

$$\begin{aligned} |Q(\widehat{g_2^\varepsilon})| &\leq 2\sigma_1 \varepsilon^\alpha \int_{\mathbf{R}^n} \frac{|\varepsilon v' \cdot \xi|}{\sqrt{\sigma_0^2 + (\varepsilon v' \cdot \xi)^2}} F(v')\, dv' |\widehat{\rho^\varepsilon}| F(v) \\ &\leq C\varepsilon^\alpha(\varepsilon^\alpha |\xi|^\alpha + \varepsilon|\xi|)|\widehat{\rho^\varepsilon}|F(v). \end{aligned}$$

This clearly implies that $g_2^\varepsilon$ goes to zero as $\varepsilon$ goes to $0$, but it also means that it will be difficult to control the term $\varepsilon^{1-\alpha} v \cdot \nabla g_2^\varepsilon$ because of the lack of integrability of $vF$ (at least when $\alpha < 1$).

For this reason, we need to add a term in the expansion, as explained in the next section.

## 10.3 Proof of Theorem 10.1.1

### 10.3.1 A reshuffled Hilbert expansion

In order to prove Theorem 10.1.1, we use a Hilbert type expansion and some techniques similar to those first introduced in [13].

We recall that $f^\varepsilon$ is a solution of

$$\varepsilon^\alpha \partial_t f^\varepsilon + \varepsilon v \cdot \nabla_x f^\varepsilon = Q(f^\varepsilon) \tag{10.32}$$

where

$$Q(f) = Q^+(f) - \nu f \quad \text{with} \quad Q^+(f) = \int_{\mathbf{R}^n} \sigma(x,v,v') F(v) f(v') dv',$$

and we write the following expansion

$$f^\varepsilon = \rho^\varepsilon(t,x) F(v) + g_1^\varepsilon(t,x,v) + g_2^\varepsilon(t,x,v) + g_3^\varepsilon(t,x,v) + r^\varepsilon(t,x,v) \tag{10.33}$$

where the terms $g_1^\varepsilon$, $g_2^\varepsilon$, $g_3^\varepsilon$ satisfy

$$(\nu(x,v) + \varepsilon v \cdot \nabla_x) g_1^\varepsilon = -\varepsilon v \cdot \nabla_x (\rho^\varepsilon F) \tag{10.34}$$

$$Q(g_2^\varepsilon) = -Q^+(g_1^\varepsilon) - \varepsilon^\alpha \partial_t \rho^\varepsilon F \tag{10.35}$$

$$(\nu(x,v) + \varepsilon v \cdot \nabla_x) g_3^\varepsilon = -\varepsilon v \cdot \nabla_x g_2^\varepsilon. \tag{10.36}$$

The remainder solves

$$\partial_t r^\varepsilon + \varepsilon^{1-\alpha} v \cdot \nabla_x r^\varepsilon = \frac{Q(r^\varepsilon)}{\varepsilon^\alpha} - \partial_t g_1^\varepsilon - \partial_t g_2^\varepsilon - \partial_t g_3^\varepsilon + \frac{Q^+(g_3^\varepsilon)}{\varepsilon^\alpha}. \tag{10.37}$$

We recall that we defined the operator

$$\mathbf{T}(g) = \nu g + \varepsilon v \cdot \nabla_x g. \tag{10.38}$$

A simple computation (see [13]) shows that this operator is invertible with

$$\mathbf{T}^{-1}(f) = \int_0^{+\infty} e^{-\int_0^z \nu(x - \varepsilon v s, v)\, ds} f(x - \varepsilon v z, v)\, dz \tag{10.39}$$

for any function $f(x,v)$. In particular, equations (10.34) and (10.36) are solvable without additional conditions, but (10.35) requires the following solvability condition (obtained by integrating (10.35) with respect to $v$) :

$$\partial_t \rho^\varepsilon + L^\varepsilon(\rho^\varepsilon) = 0 \tag{10.40}$$

with

$$\begin{aligned} L^\varepsilon(\rho) &= -\varepsilon^{-\alpha} \int_{\mathbf{R}^n} Q^+(g_1^\varepsilon) dv \\ &= -\varepsilon^{-\alpha} \int_{\mathbf{R}^n} \nu(x,v) g_1^\varepsilon dv \\ &= \varepsilon^{-\alpha} \int_{\mathbf{R}^n} \nu(x,v)\, \mathbf{T}^{-1}(\varepsilon v \cdot \nabla_x \rho^\varepsilon)\, F\, dv. \end{aligned} \tag{10.41}$$

In view of (10.37), it is clear that in order to prove Theorem 10.1.1, we need to show that $g_1^\varepsilon$, $g_2^\varepsilon$ and $g_3^\varepsilon$ can be defined and that the terms $\partial_t g_1^\varepsilon$, $\partial_t g_2^\varepsilon$ $\partial_t g_3^\varepsilon$ and $\frac{Q^+(g_3^\varepsilon)}{\varepsilon^\alpha}$ in the right hand side of (10.37) go to 0 in $L^1(0,\infty; L^2_{F^{-1}}(\mathbf{R}^n \times \mathbf{R}^n))$.

The first step, detailed in the next section, will be to study the properties of this operator $L^\varepsilon$ and show that the equation (10.40) has a smooth solution $\rho^\varepsilon$. In section 10.3.3, we will derive some important estimate on $Q^{-1}(h)$ that will be needed to get some estimates on $g_2^\varepsilon$. In Section 10.3.4, we derive estimates on $g_1^\varepsilon$, $g_2^\varepsilon$ and $g_3^\varepsilon$ and their derivatives and show that $f^\varepsilon - \rho^\varepsilon F$ converges to zero strongly. The final step (Section 10.3.5) is to show that $\rho^\varepsilon$, solution of (10.40), converges strongly to $\rho_0$ solution of (10.13).

### 10.3.2 Properties of the operator $L^\varepsilon$ and the equation (10.40)

First, of all, we note that for any function $\rho(x)$, (10.39) and (10.41) imply

$$\begin{aligned} L^\varepsilon(\rho) &= \varepsilon^{-\alpha}\int_{\mathbf{R}^n} \nu(x,v)\,\mathbf{T}^{-1}(\varepsilon v\cdot\nabla_x\rho^\varepsilon)\,F\,dv \\ &= \varepsilon^{-\alpha}\int_{\mathbf{R}^n}\int_0^{+\infty} e^{-\int_0^z \nu(x-\varepsilon v s,v)\,ds}\,\varepsilon v\cdot\nabla_x\rho(x-\varepsilon v z)\,F(v)\,dz\,dv. \end{aligned}$$

Alternatively, we can also write

$$\begin{aligned} &\mathbf{T}^{-1}(\varepsilon v\cdot\nabla_x\rho) \\ &\quad= \int_0^{+\infty} e^{-\int_0^z \nu(x-\varepsilon v s,v)\,ds}\,\varepsilon v\cdot\nabla_x\rho(x-\varepsilon v z)\,dz \\ &\quad= -\int_0^{+\infty} e^{-\int_0^z \nu(x-\varepsilon v s,v)\,ds}\,\frac{d}{dz}\big[\rho(x-\varepsilon v z)\big]\,dz \\ &\quad= -\int_0^{+\infty} e^{-\int_0^z \nu(x-\varepsilon v s,v)\,ds}\,\frac{d}{dz}\big[\rho(x-\varepsilon v z)-\rho(x)\big]\,dz \\ &\quad= -\int_0^{+\infty} e^{-\int_0^z \nu(x-\varepsilon v s,v)\,ds}\nu(x-\varepsilon v z,v)\,\big[\rho(x-\varepsilon v z)-\rho(x)\big]\,dz \end{aligned}$$

which leads to

$$L^\varepsilon(\rho) = -\varepsilon^{-\alpha}\int_{\mathbf{R}^n}\int_0^{+\infty} e^{-\int_0^z \nu(x-\varepsilon v s,v)\,ds}\nu(x-\varepsilon v z,v)\,\big[\rho(x-\varepsilon v z)-\rho(x)\big]\,F(v)\,dz\,dv \tag{10.42}$$

which is the formula found in [13].

We can now prove the following lemma :

*Lemma* 10.3.1. For all $k$ integer, $k\geq 2$, the operator $L^\varepsilon$ is a bounded operator from $H^k(\mathbf{R}^n)$ to $H^{k-2}(\mathbf{R}^n)$ :

$$||L^\varepsilon(\rho)||_{H^{k-2}(\mathbf{R}^n)} \leq C||\rho||_{H^k(\mathbf{R}^n)} \quad \text{for all } \rho\in H^k(\mathbf{R}^n). \tag{10.43}$$

*Démonstration.* We first prove that

$$||L^\varepsilon(\rho)||_{L^2(\mathbf{R}^n)} \leq C||\rho||_{H^2(\mathbf{R}^n)} \qquad \text{for all } \rho\in H^2(\mathbf{R}^n). \tag{10.44}$$

A similar computation is performed in [13], and recalled here for the reader's convenience.

We write

$$\begin{aligned} L^\varepsilon(\rho) &= \varepsilon^{-\alpha}\int_{\mathbf{R}^n} \nu(x,v)\,\mathbf{T}^{-1}(\varepsilon v\cdot\nabla_x\rho)\,F\,dv \\ &= I_1^\varepsilon + I_2^\varepsilon + I_3^\varepsilon \end{aligned}$$

where

$$\begin{aligned} I_1^\varepsilon &= \varepsilon^{-\alpha}\int_{|v|\leq 1} \nu(x,v)\,\mathbf{T}^{-1}(\varepsilon v\cdot\nabla_x\rho)\,F\,dv, \\ I_2^\varepsilon &= \varepsilon^{-\alpha}\int_{1\leq|v|\leq 1/\varepsilon} \nu(x,v)\,\mathbf{T}^{-1}(\varepsilon v\cdot\nabla_x\rho)\,F\,dv, \\ I_3^\varepsilon &= \varepsilon^{-\alpha}\int_{|v|\geq 1/\varepsilon} \nu(x,v)\,\mathbf{T}^{-1}(\varepsilon v\cdot\nabla_x\rho)\,F\,dv. \end{aligned}$$

We have

$$\begin{aligned} \mathbf{T}^{-1}(\varepsilon v\cdot\nabla_x\rho) &= \int_0^{+\infty} e^{-\int_0^z \nu(x-\varepsilon v s,v)\,ds}\,\varepsilon v\cdot\nabla_x\rho(x-\varepsilon v z)\,dz \\ &= \int_0^{+\infty} \left[e^{-\int_0^z \nu(x-\varepsilon v s,v)\,ds} - e^{-\nu(x,v)z}\right]\,\varepsilon v\cdot\nabla_x\rho(x-\varepsilon v z)\,dz \\ &\quad + \int_0^{+\infty} e^{-\nu(x,v)z}\,\left[\varepsilon v\cdot\nabla_x\rho(x-\varepsilon v z)-\varepsilon v\cdot\nabla_x\rho(x)\right]\,dz \\ &\quad + \int_0^{+\infty} e^{-\nu(x,v)z}\,\varepsilon v\cdot\nabla_x\rho(x)\,dz \end{aligned}$$

and so an integration by parts yields

$$\begin{aligned}\mathbf{T}^{-1}(\varepsilon v\cdot\nabla_x\rho) &= \int_0^{+\infty}\left[e^{-\int_0^z\nu(x-\varepsilon vs,v)\,ds}-e^{-\nu(x,v)z}\right]\varepsilon v\cdot\nabla_x\rho(x-\varepsilon vz)\,dz\\ &+\int_0^{+\infty}e^{-\nu(x,v)z}\,\frac{1}{\nu(x,v)}\varepsilon^2 v\cdot D_x^2\rho(x-\varepsilon vz)\cdot v\,dz\\ &+\int_0^{+\infty}e^{-\nu(x,v)z}\,\varepsilon v\cdot\nabla_x\rho(x)\,dz.\end{aligned}$$

Using the symmetry assumption on $F$ and $\nu$, we deduce :

$$\begin{aligned}I_1^\varepsilon &= \varepsilon^{-\alpha}\int_{|v|\le1}\nu(x,v)\int_0^{+\infty}\left[e^{-\int_0^z\nu(x-\varepsilon vs,v)\,ds}-e^{-\nu(x,v)z}\right]\varepsilon v\cdot\nabla_x\rho(x-\varepsilon vz)\,F(v)\,dz\,dv\\ &+\varepsilon^{-\alpha}\int_{|v|\le1}\int_0^{+\infty}e^{-\nu(x,v)z}\varepsilon^2 v\cdot D_x^2\rho(x-\varepsilon vz)\cdot v\,dzF(v)\,dv\end{aligned}$$

and since $F$ is bounded, and

$$\left|e^{-\int_0^z\nu(x-s\varepsilon v,v)ds}-e^{-z\nu(x,v)}\right|\le Cz^2e^{-\sigma_1 z}\varepsilon|v|$$

we deduce

$$||I_1^\varepsilon||_{L^2(\mathbf{R}^n)}\le C\varepsilon^{2-\alpha}||\rho||_{H^2(\mathbf{R}^n)}$$

Next, Assumption (10.9) and the change of variable $w=\varepsilon v$ yields :

$$\begin{aligned}I_2^\varepsilon &= \varepsilon^{-\alpha}\int_{1\le|v|\le1/\varepsilon}\int_0^{+\infty}e^{-\int_0^z\nu(x-\varepsilon vs,v)\,ds}\nu(x,v)\;\varepsilon\cdot\nabla_x\rho(x-\varepsilon vz)\frac{1}{|v|^{n+\alpha}}\,dz\,dv\\ &= \int_{\varepsilon\le|w|\le1}\int_0^{+\infty}e^{-\int_0^z\nu(x-ws,w/\varepsilon)\,ds}\nu(x,w/\varepsilon)\;w\cdot\nabla_x\rho(x-wz)\frac{1}{|w|^{n+\alpha}}\,dz\,dw\end{aligned}$$

and proceeding as with $I_1^\varepsilon$, we deduce

$$\begin{aligned}I_2^\varepsilon &= \int_{\varepsilon\le|w|\le1}\nu(x,w/\varepsilon)\int_0^{+\infty}\left[e^{-\int_0^z\nu(x-ws,w/\varepsilon)\,ds}-e^{-\nu(x,w/\varepsilon)z}\right]w\cdot\nabla_x\rho(x-wz)\,\frac{1}{|w|^{n+\alpha}}\,dz\,dw\\ &+\int_{\varepsilon\le|w|\le1}\int_0^{+\infty}e^{-\nu(x,w/\varepsilon)z}w\cdot D_x^2\rho(x-wz)\cdot w\,dz\frac{1}{|w|^{n+\alpha}}\,dw\end{aligned}$$

and so

$$||I_2^\varepsilon||_{L^2(\mathbf{R}^n)}\le C||\rho||_{H^2(\mathbf{R}^n)}.$$

Finally, to estimate $I_3^\varepsilon$, we use the formula

$$\mathbf{T}^{-1}(\varepsilon v\cdot\nabla_x\rho)=-\int_0^{+\infty}e^{-\int_0^z\nu(x-\varepsilon vs,v)\,ds}\nu(x-\varepsilon vz,v)\left[\rho(x-\varepsilon vz)-\rho(x)\right]dz.$$

Assumption (10.9) and the change of variable $w=\varepsilon v$ then yields :

$$I_3^\varepsilon=-\int_{|w|\ge1}\int_0^{+\infty}e^{-\int_0^z\nu(x-ws,w/\varepsilon)\,ds}\nu(x,w/\varepsilon)\,\nu(x-wz,v)\left[\rho(x-wz)-\rho(x)\right]\frac{1}{|w|^{n+\alpha}}\,dz\,dw$$

and so

$$||I_3^\varepsilon||_{L^2(\mathbf{R}^n)}\le C||\rho||_{L^2(\mathbf{R}^n)}.$$

We now show how to obtain higher order estimates : We have

$$\begin{aligned}\partial_{x_i} L^\varepsilon(\rho^\varepsilon) &= \varepsilon^{-\alpha}\int_{\mathbf{R}^n} \partial_{x_i}\nu(x,v)\mathbf{T}^{-1}(\varepsilon v\cdot\nabla_x\rho)F(v)dv \\ &\quad -\varepsilon^{-\alpha}\int_{\mathbf{R}^n} \nu(x,v)\mathbf{T}^{-1}(\partial_{x_i}\nu\mathbf{T}^{-1}(\varepsilon v\cdot\nabla_x\rho))F(v)\,dv \\ &\quad +\varepsilon^{-\alpha}\int_{\mathbf{R}^n} \nu(x,v)\mathbf{T}^{-1}(\varepsilon v\cdot\nabla_x\partial_{x_i}\rho)F(v)dv.\end{aligned}$$

The last term is just $L^\varepsilon(\partial_{x_i}\rho)$ which can thus be bounded by the previous step. The first term is treated in a same way as $L^\varepsilon(\rho)$ except that we need to use the fact that $\frac{\partial_{x_i}\nu}{\nu}$ is even and bounded.

Therefore, we will only detail the computations concerning the second term. We split it into three integrals corresponding respectively to $|v|<1$, $1<|v|<\frac{1}{\varepsilon}$ and $|v|>\frac{1}{\varepsilon}$.

First of all, we handle the $|v|>\frac{1}{\varepsilon}$ part. For that purpose, we use (10.9) and the usual change of variable. It leads to

$$\begin{aligned}I_1^\varepsilon &= -\varepsilon^{-\alpha}\int_{|v|>\frac{1}{\varepsilon}} \nu(x,v)\mathbf{T}^{-1}(\partial_{x_i}\nu\mathbf{T}^{-1}(\varepsilon v\cdot\nabla_x\rho))F(v)\,dv \\ &= -\int_{|w|>1}\nu(x,\frac{w}{\varepsilon})\int_0^{+\infty} e^{-\int_0^z \nu(x-ws,\frac{w}{\varepsilon})\,ds}\,\partial_{x_i}\nu(x-wz,\frac{w}{\varepsilon})\int_0^{+\infty} e^{-\int_0^{\overline{z}}\nu(x-w(s+z),\frac{w}{\varepsilon})\,ds} \\ &\qquad \nu(x-w(z+\overline{z}),\frac{w}{\varepsilon})[\rho(x-w(z+\overline{z}))-\rho(x-\varepsilon vz)]\frac{1}{|w|^{n+\alpha}}dvdzd\overline{z}.\end{aligned}$$

Thus,

$$||I_1^\varepsilon||_{L^2(\mathbf{R}^n)}\le C||\rho||_{L^2(\mathbf{R}^n)}.$$

Next, for $|v|<1$, we write

$$\begin{aligned}I_2^\varepsilon &= -\varepsilon^{-\alpha}\int_{|v|<1}\nu(x,v)\mathbf{T}^{-1}(\partial_{x_i}\nu\mathbf{T}^{-1}(\varepsilon v\cdot\nabla_x\rho))F(v)\,dv \\ &= -\varepsilon^{-\alpha}\int_{|v|<1}\nu(x,v)\int_0^{+\infty} e^{-\int_0^z \nu(x-\varepsilon vs,v)\,ds}\,\partial_{x_i}\nu(x-\varepsilon vz,v)\int_0^{+\infty} e^{-\int_0^{\overline{z}}\nu(x-\varepsilon v(s+z),v)\,ds} \\ &\qquad \varepsilon v\cdot\nabla_x\rho(x-\varepsilon v(z+\overline{z}))F(v)dvdzd\overline{z}\end{aligned}$$

which can be written

$$\begin{aligned}I_2^\varepsilon &= -\varepsilon^{-\alpha}\int_{|v|<1}\nu(x,v)\int_0^{+\infty} e^{-\int_0^z \nu(x-\varepsilon vs,v)\,ds}\,\partial_{x_i}\nu(x-\varepsilon vz,v) \\ &\qquad \int_0^{+\infty}[e^{-\int_0^{\overline{z}}\nu(x-\varepsilon v(s+z),v)\,ds}-e^{-\overline{z}\nu(x,v)}]\varepsilon v\cdot\nabla_x\rho(x-\varepsilon v(z+\overline{z}))F(v)dvdzd\overline{z} \\ &\quad -\varepsilon^{-\alpha}\int_{|v|<1}\nu(x,v)\int_0^{+\infty} e^{-\int_0^z \nu(x-\varepsilon vs,v)\,ds}\,\partial_{x_i}\nu(x-\varepsilon vz,v)\int_0^{+\infty} e^{-\overline{z}\nu(x,v)} \\ &\qquad \varepsilon v\cdot\nabla_x\rho(x-\varepsilon v(z+\overline{z}))F(v)dvdzd\overline{z}.\end{aligned}$$

Since

$$[e^{-\int_0^{\overline{z}}\nu(x-\varepsilon v(s+z),v)\,ds}-e^{-\overline{z}\nu(x,v)}]\le\varepsilon|v|(z+\overline{z})^2e^{-\sigma_1\overline{z}},$$

we get

$$||I_2^\varepsilon||_{L^2(\mathbf{R}^n)} \leq ||I_{21}^\varepsilon||_{L^2(\mathbf{R}^n)} + \varepsilon^{2-\alpha}||\nabla\rho||_{L^2(\mathbf{R}^n)}$$

where

$$I_{21}^\varepsilon = -\varepsilon^{-\alpha}\int_{|v|<1}\nu(x,v)\int_0^{+\infty} e^{-\int_0^z \nu(x-\varepsilon vs,v)\,ds}\,\partial_{x_i}\nu(x-\varepsilon vz,v)\int_0^{+\infty} e^{-\overline{z}\nu(x,v)}\,\varepsilon v\cdot\nabla_x\rho(x-\varepsilon v(z+\overline{z}))F(v)dvdzd\overline{z}.$$

On the other hand

$$\begin{aligned}
I_{21}^\varepsilon &= -\varepsilon^{-\alpha}\int_{|v|<1}\nu(x,v)\int_0^{+\infty} e^{-\int_0^z \nu(x-\varepsilon vs,v)\,ds}\,\partial_{x_i}\nu(x-\varepsilon vz,v)\int_0^{+\infty} e^{-\overline{z}\nu(x,v)}\,\varepsilon v\cdot\nabla_x\rho(x-\varepsilon v(z+\overline{z}))F(v)dvdzd\overline{z}\\
&= -\varepsilon^{-\alpha}\int_{|v|<1}\nu(x,v)\int_0^{+\infty} e^{-\int_0^z \nu(x-\varepsilon vs,v)\,ds}\,[\partial_{x_i}\nu(x-\varepsilon vz,v)-\partial_{x_i}\nu(x,v)]\int_0^{+\infty} e^{-\overline{z}\nu(x,v)}\,\varepsilon v\cdot\nabla_x\rho(x-\varepsilon v(z+\overline{z}))F(v)dvdzd\overline{z}\\
&\quad -\varepsilon^{-\alpha}\int_{|v|<1}\nu(x,v)\int_0^{+\infty} e^{-\int_0^z \nu(x-\varepsilon vs,v)\,ds}\,\partial_{x_i}\nu(x,v)\int_0^{+\infty} e^{-\overline{z}\nu(x,v)}\,\varepsilon v\cdot\nabla_x\rho(x-\varepsilon v(z+\overline{z}))F(v)dvdzd\overline{z}
\end{aligned}$$

and then, since

$$[\partial_{x_i}\nu(x-\varepsilon vz,v)-\partial_{x_i}\nu(x,v)] \leq \varepsilon|v|||\nu||_{W^{2,\infty}},$$

we have

$$||I_{21}^\varepsilon||_{L^2(\mathbf{R}^n)} \leq ||I_{22}||_{L^2(\mathbf{R}^n)} + \varepsilon^{2-\alpha}||\nabla\rho||_{L^2(\mathbf{R}^n)}$$

where

$$I_{22}^\varepsilon = \varepsilon^{-\alpha}\int_{|v|<1}\nu(x,v)\int_0^{+\infty} e^{-\int_0^z \nu(x-\varepsilon vs,v)\,ds}\,\partial_{x_i}\nu(x,v)\int_0^{+\infty} e^{-\overline{z}\nu(x,v)}\,\varepsilon v\cdot\nabla_x\rho(x-\varepsilon v(z+\overline{z}))F(v)dvdzd\overline{z}.$$

In a same way

$$||I_{22}^\varepsilon||_{L^2(\mathbf{R}^n)} \leq ||I_{23}^\varepsilon||_{L^2} + \varepsilon^{2-\alpha}||\nabla\rho||_{L^2(\mathbf{R}^n)}$$

with

$$I_{23}^\varepsilon = -\varepsilon^{-\alpha}\int_{|v|<1}\nu(x,v)\int_0^{+\infty} e^{-z\nu(x,v)}\,\partial_{x_i}\nu(x,v)\int_0^{+\infty} e^{-\overline{z}\nu(x,v)}\,\varepsilon v\cdot\nabla_x\rho(x-\varepsilon v(z+\overline{z}))F(v)dvdzd\overline{z}$$

An integration by part in $z$ gives

$$\begin{aligned}
I_{23}^\varepsilon &= -\varepsilon^{-\alpha}\int_{|v|<1}\int_0^{+\infty} e^{-z\nu(x,v)}\,\partial_{x_i}\nu(x,v)\int_0^{+\infty} e^{-\overline{z}\nu(x,v)}\,\varepsilon^2 v\cdot D^2\rho(x-\varepsilon v(z+\overline{z}))vF(v)dvdzd\overline{z}\\
&\quad -\varepsilon^{-\alpha}\int_{|v|<1}\partial_{x_i}\nu(x,v)\int_0^{+\infty} e^{-\overline{z}\nu(x,v)}\,\varepsilon v\cdot\nabla_x\rho(x-\varepsilon v(\overline{z}))F(v)dvdzd\overline{z}.
\end{aligned}$$

Another integration by parts gives

$$\begin{aligned}
&-\varepsilon^{-\alpha}\int_{|v|<1}\partial_{x_i}\nu(x,v)\int_0^{+\infty} e^{-\overline{z}\nu(x,v)}\,\varepsilon v\cdot\nabla_x\rho(x-\varepsilon v(\overline{z}))F(v)dvdzd\overline{z}\\
= \;& -\varepsilon^{-\alpha}\int_{|v|<1}\frac{\partial_{x_i}\nu(x,v)}{\nu(x,v)}\int_0^{+\infty} e^{-\overline{z}\nu(x,v)}\,\varepsilon^2 v\cdot D^2\rho(x-\varepsilon v(\overline{z}))vF(v)dvdzd\overline{z}\\
&-\varepsilon^{-\alpha}\int_{|v|<1}\frac{\partial_{x_i}\nu(x,v)}{\nu(x,v)}\varepsilon v\cdot\nabla_x\rho(x)F(v)dv
\end{aligned}$$

in which the last term vanishes (by symmetries).

Finally, we get

$$||I_2^\varepsilon||_{L^2(\mathbf{R}^n)} \leq C||\rho||_{H^2(\mathbf{R}^n)}$$

and the same computations for $\varepsilon < |w| < 1$ where $w$ is the change of variable $w = \dfrac{v}{\varepsilon}$ reads

$$||I_3^\varepsilon||_{L^2(\mathbf{R}^n)} \leq C||\rho||_{H^2(\mathbf{R}^n)}$$

where

$$I_3^\varepsilon = -\varepsilon^{-\alpha} \int_{1<|v|<\frac{1}{\varepsilon}} \nu(x,v)\mathbf{T}^{-1}(\partial_{x_i}\nu\mathbf{T}^{-1}(\varepsilon v\cdot\nabla_x\rho))F(v)\,dv.$$

We obtain finally,

$$||\partial_{x_i}L^\varepsilon(\rho)||_{L^2(\mathbf{R}^n)} \leq C(||L^\varepsilon(\rho)||_{L^2(\mathbf{R}^n)} + C(\sum_{|k|=2}||\partial_x^k\rho||_{L^2(\mathbf{R}^n)} + \sum_{|k|=3}||\partial^k\rho||_{L^2(\mathbf{R}^n)}).$$

Proceeding similarly, we can show (10.43) for all $k \geq 2$. □

*Proposition.* For all $\rho_{in} \in H^k(\mathbf{R}^n)$ (for some $k \geq 0$), the equation

$$\begin{cases} \partial_t\rho + L^\varepsilon(\rho) = 0 & \text{in } \mathbf{R}_+ \times \mathbf{R}^n \\ \rho(0,x) = \rho_{in} & \text{in } \mathbf{R}^n \end{cases} \tag{10.45}$$

has a unique solution solution $\rho^\varepsilon \in L^\infty(0,\infty;H^k(\mathbf{R}^n))$ for all $\varepsilon > 0$. Furthermore, if $k \geq 2$, then

$$||\rho^\varepsilon||_{L^\infty([0,\infty),H^k(\mathbf{R}^n))} \leq C \quad \text{and}$$

$$||L^\varepsilon(\rho^\varepsilon)||_{L^\infty([0,\infty),H^{k-2}(\mathbf{R}^n))} = ||\partial_t\rho^\varepsilon||_{L^\infty([0,\infty),H^{k-2}(\mathbf{R}^n))} \leq C,$$

with $C$ independent of $\varepsilon$.

*Démonstration.* First, we have to prove that $\rho^\varepsilon$ exists for $\varepsilon > 0$. For a fixed $\varepsilon > 0$, the operator $L^\varepsilon$ (defined by (10.42)) is clearly a bounded operator on $L^2(\mathbf{R}^n)$. Furthermore, we have :

$$\begin{aligned}
\int_{\mathbf{R}^n} L^\varepsilon(\rho)\rho dx &= \varepsilon^{-\alpha}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} \rho\,\nu\mathbf{T}^{-1}(\varepsilon v\cdot\nabla_x\rho)F\,dv\,dx \\
&= -\varepsilon^{-\alpha}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} \rho\,\varepsilon v\cdot\nabla_x\big(\mathbf{T}^{-1}(\varepsilon v\cdot\nabla_x\rho)\big)F\,dv\,dx \\
&\quad +\varepsilon^{-\alpha}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} \rho\,\mathbf{T}\mathbf{T}^{-1}(\varepsilon v\cdot\nabla_x\rho)F\,dv\,dx \\
&= \varepsilon^{-\alpha}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} \varepsilon v\cdot\nabla_x\rho\,\mathbf{T}^{-1}(\varepsilon v\cdot\nabla_x\rho)F\,dv\,dx \\
&= \varepsilon^{-\alpha}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} \mathbf{T}(g_1)\,g_1\frac{1}{F}\,dv\,dx
\end{aligned}$$

where $g_1 = -\mathbf{T}^{-1}(\varepsilon v\cdot\nabla_x\rho)F$. Hence

$$\begin{aligned}
\int_{\mathbf{R}^n} L^\varepsilon(\rho)\,\rho\,dx &= \varepsilon^{-\alpha}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} |g_1|^2\frac{\nu}{F}\,dv\,dx + \varepsilon^{-\alpha}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}(\varepsilon v\cdot\nabla g_1)\,g_1\frac{1}{F}\,dv\,dx \\
&= \varepsilon^{-\alpha}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} |g_1|^2\frac{\nu}{F}\,dv\,dx \\
&\geq 0.
\end{aligned} \tag{10.46}$$

In particular, the operator $L^\varepsilon$ is monotone and Lax Milgram's theorem implies that for any $f \in L^2(\mathbf{R}^n)$, the equation

$$\rho + L^\varepsilon(\rho) = f \tag{10.47}$$

has a solution in $L^2(\mathbf{R}^n)$. We deduce that $L^\varepsilon$ is maximal monotone, and thus Hille-Yosida's theorem implies that there exists a unique $\rho^\varepsilon$ in $C^0(\mathbf{R}^+, L^2(\mathbf{R}^n)) \cap C^1(\mathbf{R}^{+*}, L^2(\mathbf{R}^n))$ solution of (10.45).

We now prove the uniform estimates on $||\rho^\varepsilon||_{L^\infty([0,\infty],H^k(\mathbf{R}^n))}$. Note that since $\partial_t\rho^\varepsilon = -L^\varepsilon(\rho^\varepsilon)$ satisfy the same equation as $\rho^\varepsilon$ with initial condition $\partial_t\rho^\varepsilon(0,\cdot) = -L^\varepsilon(\rho_0) \in H^{k-2}(\mathbf{R}^n)$, the other estimates will then follow.

Multiplying equation Equation (10.41) by $\rho^\varepsilon$ and using (10.46), we now get :

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}||\rho^\varepsilon||^2_{L^2(\mathbf{R}^n)} &= \int_{\mathbf{R}^n} \partial_t\rho^\varepsilon\,\rho^\varepsilon\,dx \\ &= -\int_{\mathbf{R}^n} L^\varepsilon(\rho^\varepsilon)\,\rho^\varepsilon\,dx \\ &= -\varepsilon^{-\alpha}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} |g_1^\varepsilon|^2\,\frac{\nu}{F}\,dv\,dx.\end{aligned}$$

Since $\rho_0^\varepsilon$ is bounded in $L^2(\mathbf{R}^n)$, this implies in particular

$$||\rho^\varepsilon||_{L^\infty(0,\infty;L^2(\mathbf{R}^n))} \leq C$$

and

$$||g_1^\varepsilon||_{L^2(0,\infty;L^2_{F^{-1}}(\mathbf{R}^n\times\mathbf{R}^n))} \leq C\varepsilon^{\frac{\alpha}{2}}. \tag{10.48}$$

Next, we show how one gets a similar estimate on $\nabla_x\rho^\varepsilon$ : We write

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}||\nabla_x\rho^\varepsilon||^2_{L^2(\mathbf{R}^n)} &= \int_{\mathbf{R}^n}\partial_t\nabla_x\rho^\varepsilon\cdot\nabla_x\rho^\varepsilon dx = -\int_{\mathbf{R}^n}\nabla_x\rho^\varepsilon\cdot\nabla_x L^\varepsilon(\rho^\varepsilon)dx \\ &= -\varepsilon^{-\alpha}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}\nabla_x\rho^\varepsilon\cdot\nabla_x\left(\nu\mathbf{T}^{-1}(\varepsilon v\cdot\nabla_x\rho^\varepsilon)\right)F\,dv\,dx \\ &= -\varepsilon^{-\alpha}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}\nabla_x\rho^\varepsilon\cdot\nabla_x\left(\mathbf{T}\mathbf{T}^{-1}(\varepsilon v\cdot\nabla_x\rho^\varepsilon)\right)F\,dv\,dx \\ &\quad +\varepsilon^{-\alpha}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}\nabla_x\rho^\varepsilon\cdot\nabla_x\left(\varepsilon v\cdot\nabla_x(\mathbf{T}^{-1}(\varepsilon v\cdot\nabla_x\rho^\varepsilon))\right)F\,dv\,dx \\ &= -\varepsilon^{-\alpha}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}\nabla_x(\varepsilon v\cdot\nabla_x\rho^\varepsilon)\cdot\nabla_x\left(\mathbf{T}^{-1}(\varepsilon v\cdot\nabla_x\rho^\varepsilon)\right)F\,dv\,dx.\end{aligned}$$

Which gives (since $g_1^\varepsilon = \mathbf{T}^{-1}(\varepsilon v\cdot\nabla_x\rho^\varepsilon)F$) :

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}||\nabla_x\rho^\varepsilon||^2_{L^2(\mathbf{R}^n)} &= -\varepsilon^{-\alpha}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}\nabla_x((\nu+\varepsilon v\cdot\nabla_x)g_1^\varepsilon)\cdot\nabla_x g_1^\varepsilon\frac{1}{F}\,dv\,dx \\ &= -\varepsilon^{-\alpha}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}\nabla_x(\nu g_1^\varepsilon)\cdot\nabla_x g_1^\varepsilon\frac{1}{F}\,dv\,dx \\ &= -\varepsilon^{-\alpha}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}(\nu|\nabla_x g_1^\varepsilon|^2+\nabla_x\nu\cdot\nabla_x g_1^\varepsilon g_1^\varepsilon)\frac{1}{F}\,dv\,dx \\ &\leq -\frac{\sigma_0}{2}\varepsilon^{-\alpha}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}|\nabla_x g_1^\varepsilon|^2\frac{1}{F}\,dv\,dx + C\varepsilon^{-\alpha}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}{g_1^\varepsilon}^2\frac{1}{F}\,dv\,dx\end{aligned}$$

where we used (10.7) and (10.8). Using (10.48), we deduce :

$$||\nabla_x\rho^\varepsilon||^2_{L^\infty(0,\infty;L^2(\mathbf{R}^n))} \leq C,$$

and

$$||\nabla_x g_1^\varepsilon||_{L^2(0,\infty;L^2_{F^{-1}}(\mathbf{R}^n\times\mathbf{R}^n))} \leq C\varepsilon^{\frac{\alpha}{2}}.$$

Proceeding similarly, we can show that for any multi-indice $k = (k_1, k_2, ..., k_n)$, we have :

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}||\partial_x^k \rho^\varepsilon||^2_{L^2(\mathbf{R}^n)} &\leq -\varepsilon^{-\alpha}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} \nu|\partial_x^k g_1^\varepsilon|^2 \frac{1}{F}\,dv\,dx + C\sum_{1\leq|i|\leq|k|}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}|\partial_x^i\nu||\partial_x^{k-i}g_1^\varepsilon||\partial_x^k g_1^\varepsilon)|\,\frac{1}{F}\,dv\,dx \\ &\leq -\frac{\sigma_0}{2}\varepsilon^{-\alpha}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}|\partial_x^k g_1^\varepsilon|^2\,\frac{1}{F}\,dv\,dx + C\sum_{1\leq|i|\leq|k|}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}|\partial_x^{k-i}g_1^\varepsilon|^2\,\frac{1}{F}\,dv\,dx.\end{aligned}$$

By induction on $|k|$, we deduce that

$$||\partial_x^k\rho^\varepsilon||^2_{L^\infty(0,\infty;L^2(\mathbf{R}^n))} \leq C,$$

and

$$||\partial_x^k g_1^\varepsilon||_{L^2(0,\infty;L^2_{F^{-1}}(\mathbf{R}^n\times\mathbf{R}^n))} \leq C\varepsilon^{\frac{\alpha}{2}}.$$

□

*Remark* 10.3.1. Note that as usual with the heat equation, it does not seem possible to derive estimates in $L^1(\mathbf{R}^n)$ (and such estimate do not follow from $L^2$ estimates in unbounded domain). This is the reason why in our main result, Theorem 10.1.1, we obtain convergence in $L^2_{F^{-1}}$ rather than $L^1$.

### 10.3.3 Some estimates for the collision operator

In this section, we establish a couple of auxiliary results concerning the operator $Q$ that will be needed in the next section. First, we prove the following consequence of Proposition 10.2.1 :

*Corollary* 10.3.2. Let $h(v)$ be such that $\int_{\mathbf{R}^n} h(v)dv = 0$ and $h/F \in L^\infty(\mathbf{R}^n)$, and let $g = Q^{-1}(h)$. Then $g/F \in L^\infty(\mathbf{R}^n)$ and there exists a constant $K$ such that

$$\|g/F\|_{L^\infty(\mathbf{R}^n)} \leq K\|h/F\|_{L^\infty(\mathbf{R}^n)}.$$

*Démonstration.* We have

$$\|h\|_{L^2_{F^{-1}}(\mathbf{R}^n)} \leq \|h/F\|_{L^\infty(\mathbf{R}^n)},$$

and so Proposition 10.2.1 implies that $g = Q^{-1}(h)$ exists and satisfies

$$\|g\|_{L^2_{F^{-1}}(\mathbf{R}^n)} \leq C\|h\|_{L^2_{F^{-1}}(\mathbf{R}^n)} \leq C\|h/F\|_{L^\infty(\mathbf{R}^n)}.$$

Next, we can write

$$g = \frac{1}{\nu}\left[\int_{\mathbf{R}^n}\sigma(x,v,v')g(v')\,dv'\,F(v) + h(v)\right].$$

We then deduce that

$$\begin{aligned}|g(v)| &\leq C(\|g\|_{L^1(\mathbf{R}^n)}F(v) + |h(v)|) \\ &\leq C(\|g\|_{L^2_{F^{-1}}(\mathbf{R}^n)}F(v) + |h(v)|) \\ &\leq C\|h/F\|_{L^\infty(\mathbf{R}^n)}F(v),\end{aligned}$$

which completes the proof. □

Next, we prove :

*Proposition.* There exists $C$ such that for all function $h(x,v)$ we have

$$\left\|\frac{Q^+(\mathbf{T}^{-1}(\varepsilon v\cdot\nabla_x h)F)}{F}\right\|_{L^2_x(\mathbf{R}^n;L^\infty_v(\mathbf{R}^n))}$$
$$\leq C\varepsilon^\alpha||h||_{L^2_x(\mathbf{R}^n;L^\infty_v(\mathbf{R}^n))} + C(\varepsilon^\alpha+\varepsilon)||\nabla_x h||_{L^2_x(\mathbf{R}^n;L^\infty_v(\mathbf{R}^n))}$$

for all $\varepsilon > 0$.

*Démonstration.* We start by noticing that

$$Q^+(\mathbf{T}^{-1}(\varepsilon v\cdot\nabla_x h)F)(x,v)$$
$$=\int_{\mathbf{R}^n}\sigma(x,v,v')\int_0^{+\infty}e^{-\int_0^z\nu(x-\varepsilon sv',v')\,ds}\varepsilon v'\cdot\nabla_x h(x-\varepsilon zv',v')F(v')dzdv'F(v)$$

is very similar to $\varepsilon^\alpha L^\varepsilon(h)$ (which is of order $\varepsilon^\alpha$), though the lack of symmetry with respect to $v$ prevents us from using the result directly (and explain why this term is of smaller order).

In particular, it is still natural to split this integral into three parts :

$$Q^+(\mathbf{T}^{-1}(\varepsilon v\cdot\nabla_x h)F)(x,v)=(I_1^\varepsilon+I_2^\varepsilon+I_3^\varepsilon)F(v)$$

where

$$I_1^\varepsilon=\int_{|v'|\le 1}\sigma(x,v,v')\,\mathbf{T}^{-1}(\varepsilon v'\cdot\nabla_x h)\,F(v')\,dv',$$
$$I_2^\varepsilon=\int_{1\le|v'|\le 1/\varepsilon}\sigma(x,v,v')\,\mathbf{T}^{-1}(\varepsilon v'\cdot\nabla_x h)\,F(v')\,dv',$$
$$I_3^\varepsilon=\int_{|v'|\ge 1/\varepsilon}\sigma(x,v,v')\,\mathbf{T}^{-1}(\varepsilon v'\cdot\nabla_x h)\,F(v')\,dv'.$$

We then note that

$$I_1^\varepsilon=\int_{|v'|\le 1}\sigma(x,v,v')\int_0^{+\infty}e^{-\int_0^z\nu(x-\varepsilon v's,v')\,ds}\,\varepsilon v'\cdot\nabla_x h(x-\varepsilon v'z,v)\,F(v')\,dz\,dv'$$

and so

$$|I_1^\varepsilon(v)|\le C\varepsilon\int_{|v'|\le 1}\int_0^{+\infty}e^{-\sigma_0 z}\,||\nabla_x h(x-\varepsilon v'z,\cdot)||_{L_v^\infty}\,dz\,dv'.$$

We deduce

$$||I_1^\varepsilon||_{L_x^2(\mathbf{R}^n;L_v^\infty(\mathbf{R}^n)))}\le C\varepsilon||\nabla_x h||_{L_x^2(\mathbf{R}^n;L_v^\infty(\mathbf{R}^n))}.$$

Next, Assumption (10.9) and the change of variable $w'=\varepsilon v'$ yields :

$$\begin{aligned}I_2^\varepsilon &= \int_{1\le|v'|\le 1/\varepsilon}\int_0^{+\infty}\sigma(x,v,v')e^{-\int_0^z\nu(x-\varepsilon v's,v')\,ds}\,\varepsilon v'\cdot\nabla_x h(x-\varepsilon v'z,v')\frac{1}{|v'|^{n+\alpha}}\,dz\,dv'\\ &= \varepsilon^\alpha\int_{\varepsilon\le|w'|\le 1}\sigma(x,v,\frac{w'}{\varepsilon})\int_0^{+\infty}e^{-\int_0^z\nu(x-w's,\frac{w'}{\varepsilon})\,ds}\,w'\cdot\nabla_x h(x-w'z,\frac{w'}{\varepsilon})\frac{1}{|w'|^{n+\alpha}}\,dz\,dw',\end{aligned}$$

and so

$$|I_2^\varepsilon(v)| = C\varepsilon^\alpha\int_{\varepsilon\le|w'|\le 1}\int_0^{+\infty}e^{-\sigma_0 z}||\nabla_x h(x-w'z,\cdot)||_{L_v^\infty}\frac{|w'|}{|w'|^{n+\alpha}}\,dz\,dw'$$

Using the fact that $\int_{\varepsilon\le|w'|\le 1}\frac{|w'|}{|w'|^{n+\alpha}}\,dw'\le C(1+\varepsilon^{1-\alpha})$, we deduce

$$\begin{aligned}||I_2^\varepsilon||_{L_x^2(\mathbf{R}^n;L_v^\infty(\mathbf{R}^n))} &\le C\varepsilon^\alpha\left(\int_{\varepsilon\le|w'|\le 1}\frac{|w'|}{|w'|^{n+\alpha}}\,||\nabla_x h||^2_{L_x^2(\mathbf{R}^n;L_v^\infty(\mathbf{R}^n))}\,dw'\right)^{1/2}\\ &\qquad\times\left(\int_{\varepsilon\le|w'|\le 1}\frac{|w'|}{|w'|^{n+\alpha}}\,dw'\right)^{1/2}\\ &\le C(\varepsilon+\varepsilon^\alpha)||\nabla_x h||_{L_x^2(\mathbf{R}^n;L_v^\infty(\mathbf{R}^n))}.\end{aligned}$$

Finally, to estimate $I_3^\varepsilon$, we use the formula

$$\mathbf{T}^{-1}(\varepsilon v\cdot\nabla_x h)=-\int_0^{+\infty}e^{-\int_0^z\nu(x-\varepsilon vs,v)\,ds}\nu(x-\varepsilon vz,v)\left[h(x-\varepsilon vz,v)-h(x)\right]dz.$$

Assumption (10.9) and the change of variable $w = \varepsilon v$ then yields :

$$I_3^\varepsilon = -\varepsilon^\alpha \int_{|w'|\geq 1} \int_0^{+\infty} e^{-\int_0^z \nu(x-w's,\frac{w'}{\varepsilon})\,ds} \nu(x,\frac{w'}{\varepsilon})\,\nu(x-w'z,v)\left[h(x-w'z,\frac{w'}{\varepsilon})-h(x,\frac{w'}{\varepsilon})\right]\frac{1}{|w'|^{n+\alpha}}\,dz\,dw'$$

and so

$$||I_3^\varepsilon||_{L^2_x(\mathbf{R}^n;L^\infty_v(\mathbf{R}^n))} \leq C\varepsilon^\alpha ||h||_{L^2_x(\mathbf{R}^n;L^\infty_v(\mathbf{R}^n))}.$$

□

### 10.3.4 Convergence of $f^\varepsilon - \rho^\varepsilon F$

We are now ready to prove that $f^\varepsilon - \rho^\varepsilon F$ converges to zero. More precisely, we are going to show the following proposition :

*Proposition.* Let $f^\varepsilon$ be a solution of (10.32) with $f^\varepsilon(0,x,v) = \rho_{in}(x)F(v)$ and $\rho^\varepsilon$ be the solution of (10.45). Then

$$||f^\varepsilon - \rho^\varepsilon F(v)||_{L^\infty(0,\infty;L^2_{F^{-1}}(\mathbf{R}^n\times\mathbf{R}^n))} \leq C(\varepsilon^{\alpha/2}+\varepsilon^{1-\alpha/2})||\rho||_{L^\infty(0,\infty;H^4(\mathbf{R}^n))}.$$

*Démonstration.* The expansion (10.33) gives

$$||f^\varepsilon - \rho^\varepsilon F(v)||_{L^\infty(0,\infty;L^2_{F^{-1}}(\mathbf{R}^{2n}))} \leq ||g_1^\varepsilon||_{L^\infty L^2_{F^{-1}}} + ||g_2^\varepsilon||_{L^\infty L^2_{F^{-1}}} + ||g_3^\varepsilon||_{L^\infty L^2_{F^{-1}}} + ||r^\varepsilon||_{L^\infty L^2_{F^{-1}}} \quad (10.49)$$

Furthermore, the remainder $r^\varepsilon$ can be estimated by multiplying the equation (10.37) by $r^\varepsilon/F$ and integrating with respect to $x$ and $v$. Proposition 10.2.1 (iii) then yields

$$\begin{aligned} ||r^\varepsilon||_{L^\infty L^2_{F^{-1}}} \leq\ & ||\partial_t g_1^\varepsilon||_{L^1(0,\infty;L^2_{F^{-1}}(\mathbf{R}^{2n}))} + ||\partial_t g_2^\varepsilon||_{L^1(0,\infty;L^2_{F^{-1}}(\mathbf{R}^{2n}))} \\ & + ||\partial_t g_3^\varepsilon||_{L^1(0,\infty;L^2_{F^{-1}}(\mathbf{R}^{2n}))} + \varepsilon^{-\alpha}||Q^+(g_3^\varepsilon)||_{L^1(0,\infty;L^2_{F^{-1}}(\mathbf{R}^{2n}))} \quad (10.50) \\ & + ||g_1^\varepsilon(0,\cdot)||_{L^2_{F^{-1}}(\mathbf{R}^{2n})} + ||g_2^\varepsilon(0,\cdot)||_{L^2_{F^{-1}}(\mathbf{R}^{2n})} + ||g_3^\varepsilon(0,\cdot)||_{L^2_{F^{-1}}(\mathbf{R}^{2n})}. \end{aligned}$$

We now see that Proposition 10.3.4 will be proved if we can show that all the norms arising in the right hand side of (10.49) and (10.50) go to zero.

All the necessary estimates are collected in a sequence of Lemmas below, which altogether complete the proof of Proposition 10.3.4. □

*Lemma* 10.3.3. There exists a constant $C$ such that

$$||g_1^\varepsilon||_{L^\infty(0,\infty;L^2_{F^{-1}}(\mathbf{R}^{2n}))} \leq C\varepsilon^{\frac{\alpha}{2}}||\rho^\varepsilon||_{L^\infty(0,\infty;H^1(\mathbf{R}^n))}$$

and

$$||\partial_t g_1^\varepsilon||_{L^\infty(0,\infty;L^2_{F^{-1}}(\mathbf{R}^{2n}))} \leq C\varepsilon^{\frac{\alpha}{2}}||\rho^\varepsilon||_{L^\infty(0,\infty;H^3(\mathbf{R}^n))}.$$

*Lemma* 10.3.4. There exists a constant $C$ such that

$$\begin{aligned} ||g_2^\varepsilon||_{L^\infty(0,\infty;L^2_{F^{-1}}(\mathbf{R}^{2n}))} &\leq ||g_2^\varepsilon/F||_{L^\infty(0,\infty;L^2_x(\mathbf{R}^n;L^\infty_v(\mathbf{R}^n)))} \\ &\leq C(\varepsilon^\alpha+\varepsilon)||\rho^\varepsilon||_{L^\infty(0,\infty;H^2(\mathbf{R}^n))} \end{aligned}$$

and

$$||\partial_t g_2^\varepsilon||_{L^\infty(0,\infty;L^2_{F^{-1}}(\mathbf{R}^{2n}))} \leq C(\varepsilon^\alpha+\varepsilon)||\rho^\varepsilon||_{L^\infty(0,\infty;H^4(\mathbf{R}^n))}.$$

*Lemma* 10.3.5. There exists a constant $C$ such that

$$||g_3^\varepsilon||_{L^\infty(0,\infty;L^2_{F^{-1}}(\mathbf{R}^{2n}))} \leq C(\varepsilon^\alpha+\varepsilon)||\rho^\varepsilon||_{L^\infty(0,\infty;H^2(\mathbf{R}^n))}$$

$$||\partial_t g_3^\varepsilon||_{L^\infty(0,\infty;L^2_{F^{-1}}(\mathbf{R}^{2n}))} \leq C(\varepsilon^\alpha+\varepsilon)||\rho^\varepsilon||_{L^\infty(0,\infty;H^4(\mathbf{R}^n))}$$

and

$$||Q^+(g_3^\varepsilon)||_{L^\infty(0,\infty;L^2_{F^{-1}}(\mathbf{R}^{2n}))} \leq C\varepsilon^\alpha[\varepsilon^{\alpha/2}+\varepsilon^{1-\alpha/2}]||\rho^\varepsilon||_{L^\infty(0,\infty;H^2(\mathbf{R}^n))}.$$

The proof of this last lemma will require the following estimate :

*Lemma* 10.3.6. There exists a constant $C$ such that

$$\left\| \frac{\nabla_x g_2^\varepsilon}{F} \right\|_{L^\infty(0,\infty;L^2_x(\mathbf{R}^n;L^\infty_v(\mathbf{R}^n))} \leq C(\varepsilon^\alpha + \varepsilon)||\rho^\varepsilon||_{L^\infty(0,\infty;H^2(\mathbf{R}^n))} + C\varepsilon^{\alpha/2}||\rho^\varepsilon||_{L^\infty(0,\infty;H^1(\mathbf{R}^n))}.$$

The rest of this section is devoted to the proof of these lemmas.

*Proof of Lemma 10.3.3.* We recall that (10.48) yields

$$||g_1^\varepsilon||_{L^2(0,\infty;L^2_{F^{-1}}(\mathbf{R}^{2n}))} \leq C\varepsilon^{\frac{\alpha}{2}}.$$

However, we have to work a bit more to get an estimate uniformly in time.

We recall that

$$\begin{aligned} g_1^\varepsilon(t,x,v) &= \mathbf{T}^{-1}(\varepsilon v \cdot \nabla_x \rho^\varepsilon)F \\ &= -\varepsilon \int_0^{+\infty} e^{-\int_0^z (\nu(x-\varepsilon s v,v)\, ds} v \cdot \nabla_x \rho^\varepsilon(x-\varepsilon z v)F(v)dz. \end{aligned} \tag{10.51}$$

Therefore

$$\begin{aligned} &\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} \frac{g_1^\varepsilon(t,x,v)^2}{F}dvdx \\ &\leq C\varepsilon^\alpha \int_{\mathbf{R}^n}\int_{|w|>\varepsilon} \left( \int_0^{+\infty} e^{-\int_0^z (\nu(x-sw,\frac{w}{\varepsilon})\, ds} w \cdot \nabla_x \rho^\varepsilon(x-zw)dz \right)^2 F\left(\frac{w}{\varepsilon}\right) \frac{dw\,dx}{\varepsilon^{n+\alpha}} \\ &\quad + \varepsilon^2 \int_{\mathbf{R}^n}\int_{|v|\leq 1} \left( \int_0^{+\infty} e^{-\int_0^z (\nu(x-\varepsilon s v,v)\, ds} v \cdot \nabla_x \rho^\varepsilon(x-\varepsilon z v)dz \right)^2 F(v)dv\,dx. \end{aligned}$$

Splitting the first integral and integrating by parts, we get :

$$\begin{aligned} &\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} \frac{g_1^\varepsilon(t,x,v)^2}{F}dvdx \\ &\leq \varepsilon^\alpha \int_{\mathbf{R}^n}\int_{|w|>1} \left( \int_0^{+\infty} \nu(x-zw,\frac{w}{\varepsilon}) e^{-\int_0^z (\nu(x-sw,\frac{w}{\varepsilon})\, ds} [\rho^\varepsilon(x-zw)-\rho^\varepsilon(x)]dz \right)^2 \frac{1}{|w|^{n+\alpha}} dw\,dx \\ &\quad + \varepsilon^\alpha \int_{\mathbf{R}^n}\int_{1>|w|>\varepsilon} \left( \int_0^{+\infty} e^{-\int_0^z (\nu(x-sw,\frac{w}{\varepsilon})\, ds} w \cdot \nabla_x \rho^\varepsilon(x-zw)dz \right)^2 \frac{1}{|w|^{n+\alpha}} dw\,dx \\ &\quad + \varepsilon^2 C \int_{\mathbf{R}^n}\int_{|v|\leq 1} \left( \int_0^{+\infty} e^{-\sigma_0 z} |\nabla_x \rho^\varepsilon(x-\varepsilon z v)|dz \right)^2 dvdx \end{aligned}$$

and so

$$||g_1^\varepsilon||_{L^2_{F^{-1}}(\mathbf{R}^{2n})} \leq C(\varepsilon^{\frac{\alpha}{2}}||\rho^\varepsilon||_{L^2(\mathbf{R}^n)} + (\varepsilon^{\frac{\alpha}{2}} + \varepsilon)||\nabla_x \rho^\varepsilon||_{L^2(\mathbf{R}^n)})$$

which gives the first inequality in Lemma 10.3.3.

Differentiating (10.51) with respect to $t$ (note that only $\rho^\varepsilon$ depends on $t$) and proceeding similarly, we deduce :

$$\begin{aligned} ||\partial_t g_1^\varepsilon||_{L^2F^{-1}(\mathbf{R}^{2n})} &\leq C(\varepsilon^{\frac{\alpha}{2}}||\partial_t \rho^\varepsilon||_{L^2(\mathbf{R}^n)} + (\varepsilon^{\frac{\alpha}{2}} + \varepsilon)||\partial_t \nabla_x \rho^\varepsilon||_{L^2(\mathbf{R}^n)}) \\ &\leq C(\varepsilon^{\frac{\alpha}{2}}||\rho^\varepsilon||_{H^2(\mathbf{R}^n)} + (\varepsilon^{\frac{\alpha}{2}} + \varepsilon)||\nabla_x \rho^\varepsilon||_{H^2(\mathbf{R}^n)}). \end{aligned}$$

□

*Proof of Lemma 10.3.4.* We recall that (10.35) gives

$$\begin{aligned} g_2^\varepsilon &= Q^{-1}\big( - Q^+(g_1^\varepsilon) - \varepsilon^\alpha \partial_t \rho^\varepsilon F\big) \\ &= Q^{-1}\big( - Q^+(\mathbf{T}^{-1}(\varepsilon v\cdot\nabla_x\rho^\varepsilon)F) + \varepsilon^\alpha L^\varepsilon(\rho^\varepsilon)F\big) \end{aligned}$$

and so Corollary 10.3.2 yields

$$||g_2^\varepsilon/F||_{L^\infty_v(\mathbf{R}^n)} \leq C|| - Q^+(\mathbf{T}^{-1}(\varepsilon v\cdot\nabla_x\rho^\varepsilon)F)/F + \varepsilon^\alpha L^\varepsilon(\rho^\varepsilon)||_{L^\infty_v(\mathbf{R}^n)}$$

and so

$$\begin{aligned} ||g_2^\varepsilon/F||_{L^2_x(\mathbf{R}^n;L^\infty_v(\mathbf{R}^n))} &\leq C||Q^+(\mathbf{T}^{-1}(\varepsilon v\cdot\nabla_x\rho^\varepsilon)F)/F||_{L^2_x(\mathbf{R}^n;L^\infty_v(\mathbf{R}^n))} + \varepsilon^\alpha||L^\varepsilon(\rho^\varepsilon)||_{L^2(\mathbf{R}^n)} \\ &\leq C||Q^+(\mathbf{T}^1(\varepsilon v\cdot\nabla_x\rho^\varepsilon)F)/F||_{L^2_x(\mathbf{R}^n;L^\infty_v(\mathbf{R}^n))} + \varepsilon^\alpha||\rho^\varepsilon||_{H^2(\mathbf{R}^n)} \end{aligned}$$

Furthermore, Proposition 10.3.3 implies

$$||Q^+(\mathbf{T}^{-1}(\varepsilon v\cdot\nabla_x\rho^\varepsilon)F)/F||_{L^2_x(\mathbf{R}^n;L^\infty_v(\mathbf{R}^n))} \leq C(\varepsilon+\varepsilon^\alpha)||\rho^\varepsilon||_{H^1(\mathbf{R}^n)}$$

which yields the first estimate.

Next, we write

$$\partial_t g_2^\varepsilon = Q^{-1}\big( - Q^+(\mathbf{T}^{-1}(\varepsilon v\cdot\nabla_x\partial_t\rho^\varepsilon)F) + \varepsilon^\alpha L^\varepsilon(\partial_t\rho^\varepsilon)F\big)$$

and so the same argument as above yields

$$||\partial_t g_2^\varepsilon||_{L^\infty(0,\infty;L^2_{F^{-1}}(\mathbf{R}^{2n}))} \leq C(\varepsilon^\alpha+\varepsilon)||\partial_t\rho^\varepsilon||_{L^\infty(0,\infty;H^2(\mathbf{R}^n))}.$$

Since $\partial_t\rho^\varepsilon = -L^\varepsilon(\rho^\varepsilon)$, we have

$$||\partial_t\rho^\varepsilon||_{L^\infty(0,\infty;H^2(\mathbf{R}^n))} = ||\rho^\varepsilon||_{L^\infty(0,\infty;H^4(\mathbf{R}^n))}$$

and the second estimate follows. □

*Proof of Lemma 10.3.5.* We note that

$$||\mathbf{T}^{-1}h||_{L^2_{F^{-1}}(\mathbf{R}^{2n})} \leq C||h||_{L^2_{F^{-1}}(\mathbf{R}^{2n})}$$

for any $h$ in $L^2_{F^{-1}}(\mathbf{R}^{2n})$. Thus, noticing that $g_3^\varepsilon$, satisfies

$$(\nu+\varepsilon v\cdot\nabla_x)g_3^\varepsilon = \varepsilon v\cdot\nabla_x g_2^\varepsilon \quad \text{or} \quad (\nu+\varepsilon v\cdot\nabla_x)(g_3^\varepsilon - g_2^\varepsilon) = \nu g_2^\varepsilon$$

we get :

$$||g_3^\varepsilon||_{L^2_{F^{-1}}(\mathbf{R}^{2n})} \leq C||g_2^\varepsilon||_{L^2_{F^{-1}}(\mathbf{R}^{2n})} \leq C(\varepsilon^\alpha+\varepsilon)||\rho^\varepsilon||_{H^2(\mathbf{R}^n)}$$

and a similar argument gives the bound on $\partial_t g_3^\varepsilon$.

It remains to estimate $Q^+(g_3^\varepsilon)$. Proposition 10.3.3 implies

$$\left\|\frac{Q^+(g_3^\varepsilon)}{F}\right\|_{L^2_x(\mathbf{R}^n;L^\infty_v(\mathbf{R}^n))} \leq C\varepsilon^\alpha \left\|\frac{g_2^\varepsilon}{F}\right\|_{L^2_x(\mathbf{R}^n;L^\infty_v(\mathbf{R}^n))} + C(\varepsilon^\alpha+\varepsilon)\left\|\frac{\nabla_x g_2^\varepsilon}{F}\right\|_{L^2_x(\mathbf{R}^n;L^\infty_v(\mathbf{R}^n))}, \qquad (10.52)$$

and Lemma 10.3.4 yields

$$||g_2^\varepsilon/F||_{L^2_x(\mathbf{R}^n;L^\infty_v(\mathbf{R}^n))} \leq C(\varepsilon^\alpha+\varepsilon)||\rho^\varepsilon||_{H^2(\mathbf{R}^n)}.$$

Furthermore, Lemma 10.3.6 completes the proof of Lemma 10.3.5, since it implies

$$\left\|\frac{Q^+(g_3^\varepsilon)}{F}\right\|_{L^2_x(\mathbf{R}^n;L^\infty_v(\mathbf{R}^n))} \leq C[\varepsilon^\alpha(\varepsilon^\alpha+\varepsilon) + (\varepsilon^\alpha+\varepsilon)^2 + \varepsilon^{\alpha/2}(\varepsilon^\alpha+\varepsilon)]||\rho^\varepsilon||_{H^2(\mathbf{R}^n)}.$$

□

*Proof of Lemma 10.3.6.* First, taking the derivative in the equation satisfied by $g_2^\varepsilon$, we obtain :

$$Q(\nabla_x g_2^\varepsilon) = Q_x(g_2^\varepsilon) - Q_x^+(g_1^\varepsilon) - Q^+(\nabla_x g_1^\varepsilon) - \varepsilon^\alpha \nabla_x(L^\varepsilon(\rho^\varepsilon))F. \tag{10.53}$$

where $Q_x$ denotes the operator defined as $Q$ with $\nabla_x\sigma$ instead of $\sigma$. We need to derive some bound on all the terms in the right hand side of (10.53).

First, it is easy to check that

$$\left\| \frac{Q_x(g_2^\varepsilon)}{F} \right\|_{L^2_x(\mathbf{R}^n;L^\infty_v(\mathbf{R}^n))} \leq C||g_2^\varepsilon/F||_{L^2_x(\mathbf{R}^n;L^\infty_v(\mathbf{R}^n))}$$

and

$$\left\| \frac{Q_x^+(g_1^\varepsilon)}{F} \right\|_{L^2_x(\mathbf{R}^n;L^\infty_v(\mathbf{R}^n))} \leq C||g_1^\varepsilon||_{L^2_{F^{-1}}(\mathbf{R}^{2n})}.$$

Furthermore, $\nabla_x g_1^\varepsilon$ satisfies

$$(\nu(x,v) + \varepsilon v\cdot\nabla_x)\nabla_x g_1^\varepsilon = -\nabla_x\nu g_1^\varepsilon - \varepsilon v\cdot\nabla_x(\nabla_x\rho^\varepsilon F)$$

and so

$$\begin{aligned} Q^+(\nabla_x g_1^\varepsilon) &= Q^+(\mathbf{T}^{-1}(-\varepsilon v\cdot\nabla_x\nabla_x\rho^\varepsilon F)) \\ &\quad -Q^+(\mathbf{T}^{-1}(\nabla_x\nu g_1^\varepsilon)). \end{aligned}$$

Proposition 10.3.3 implies

$$||Q^+(\mathbf{T}^{-1}(-\varepsilon v\cdot\nabla_x\nabla_x\rho^\varepsilon F))/F||_{L^2_x(\mathbf{R}^n;L^\infty_v(\mathbf{R}^n))} \leq C(\varepsilon^\alpha+\varepsilon)||\rho^\varepsilon||_{H^2(\mathbf{R}^n)}$$

while we clearly have

$$\begin{aligned} ||Q^+(\mathbf{T}^{-1}(\nabla_x\nu g_1^\varepsilon))/F||_{L^2_x(\mathbf{R}^n;L^\infty_v(\mathbf{R}^n))} &\leq C||\mathbf{T}^{-1}(\nabla_x\nu g_1^\varepsilon)||_{L^2_{F^{-1}}(\mathbf{R}^{2n})} \\ &\leq C||\nabla_x\nu g_1^\varepsilon||_{L^2_{F^{-1}}(\mathbf{R}^{2n})} \\ &\leq C||g_1^\varepsilon||_{L^2_{F^{-1}}(\mathbf{R}^{2n})} \\ &\leq C\varepsilon^{\alpha/2}||\rho^\varepsilon||_{H^1(\mathbf{R}^n)}. \end{aligned}$$

We deduce

$$\left\| \frac{Q^+(\nabla_x g_1^\varepsilon)}{F} \right\|_{L^2_x(\mathbf{R}^n;L^\infty(\mathbf{R}^n))} \leq C(\varepsilon^\alpha+\varepsilon)||\rho^\varepsilon||_{H^2(\mathbf{R}^n)} + C\varepsilon^{\alpha/2}||\rho^\varepsilon||_{H^1(\mathbf{R}^n)}.$$

Finally,

$$\begin{aligned} \varepsilon^\alpha\nabla_x(L^\varepsilon(\rho^\varepsilon)) &= -\int_{\mathbf{R}^n}\nabla_x(Q^+(g_1^\varepsilon))\,dv \\ &= -\int_{\mathbf{R}^n} Q^+(\nabla_x g_1^\varepsilon)\,dv - \int_{\mathbf{R}^n} Q_x^+(g_1^\varepsilon)\,dv \end{aligned}$$

and so

$$\begin{aligned} ||\varepsilon^\alpha\nabla_x(L^\varepsilon(\rho^\varepsilon))||_{L^2(\mathbf{R}^n)} &\leq \left\| \frac{Q^+(\nabla_x g_1^\varepsilon)}{F} \right\|_{L^2_x(\mathbf{R}^n;L^\infty_v(\mathbf{R}^n))} + \left\| \frac{Q_x^+(g_1^\varepsilon)}{F} \right\|_{L^2_x(\mathbf{R}^n;L^\infty_v(\mathbf{R}^n))} \\ &\leq C(\varepsilon^\alpha+\varepsilon)||\rho^\varepsilon||_{H^2(\mathbf{R}^n)} + C\varepsilon^{\alpha/2}||\rho^\varepsilon||_{H^1(\mathbf{R}^n)}. \end{aligned}$$

Putting the pieces together, (10.53) implies (together with Corollary 10.3.2) that

$$\left\| \frac{\nabla_x g_2^\varepsilon}{F} \right\|_{L^2_x L^\infty_v} \leq C(\varepsilon^\alpha+\varepsilon)||\rho^\varepsilon||_{H^2} + C\varepsilon^{\alpha/2}||\rho^\varepsilon||_{H^1(\mathbf{R}^n)}.$$

□

### 10.3.5 Passage to the limit in the fractional diffusion equation

Proposition 10.3.4 implies

$$||f^\varepsilon - \rho^\varepsilon F(v)||_{L^\infty(0,\infty;L^2_{F^{-1}}(\mathbf{R}^n\times\mathbf{R}^n))} \longrightarrow 0.$$

To conclude the proof of Theorem 10.1.1, it thus only remains to show the following proposition :

*Proposition.* The solution $\rho^\varepsilon$ of (10.45) converges strongly in $L^\infty(0,T;L^2(\mathbf{R}^n))$ to $\rho_0(t,x)$ solution of (10.13). More precisely :

$$||\rho^\varepsilon - \rho_0||_{L^\infty(0,T;L^2(\mathbf{R}^n))} \leq CT\varepsilon^{1-\alpha/2}||\rho_0||_{L^\infty(0,\infty;H^2(\mathbf{R}^n))}.$$

First, we recall the following simpler result (which is proved in [13]) :

*Proposition.* The solution $\rho^\varepsilon$ of (10.45) converges weak$^*$ in $L^\infty(0,\infty;H^k(\mathbf{R}^n))$ to $\rho_0$ solution of (10.13).

*Démonstration.* Let $\phi(t,x)$ be a test function in $\mathcal{D}([0,\infty),\mathbf{R}^n)$, the weak formulation of the equation (10.45) gives

$$\int_0^\infty \int_{\mathbf{R}^n} \rho^\varepsilon \left(\partial_t\phi - L^\varepsilon(\phi)\right) dtdx = \int_{\mathbf{R}^n} \phi(0,x)\rho_{in}(0,x)dx.$$

Furthermore, it is proved in [13] that for any smooth function $\phi$, $L^\varepsilon(\phi)$ converges to $L(\phi)$ uniformly in $x$ and $t$. Together with the bounds on $\rho^\varepsilon$ given by Proposition 10.3.2, this implies Proposition 10.3.5. Note also that by lower semicontinuity of the norm with respect to the weak convergence, we get :

$$||\rho_0||_{L^\infty(0,\infty;H^k(B_R))} \leq ||\rho^\varepsilon||_{L^\infty(0,\infty;H^k(\mathbf{R}^n))}.$$

□

*Proof of Proposition 10.3.5.* In order to show the strong convergence, we note that

$$\partial_t(\rho^\varepsilon - \rho_0) + L^\varepsilon(\rho^\varepsilon - \rho_0) = L^\varepsilon(\rho_0) - L^0(\rho_0)$$

and so

$$\frac{d}{dt}\int |\rho^\varepsilon - \rho_0|^2\, dx \leq \left(\int |L^\varepsilon(\rho_0) - L^0(\rho_0)|^2\, dx\right)^{1/2} \left(\int |\rho^\varepsilon - \rho_0|^2\, dx\right)^{1/2}.$$

We deduce

$$\begin{aligned}\left(\int |\rho^\varepsilon(t) - \rho_0(t)|^2\, dx\right)^{1/2} &\leq \int_0^T \left(\int |L^\varepsilon(\rho_0) - L^0(\rho_0)|^2\, dx\right)^{1/2} dt \\ &\leq T||L^\varepsilon(\rho_0) - L^0(\rho_0)||_{L^\infty(0,\infty;L^2(\mathbf{R}^n))}\end{aligned}$$

A computation similar to that of the proof of Lemma 10.3.1 now gives

$$\int_{\mathbf{R}^n} |L^\varepsilon(\rho_0) - L(\rho_0)|^2 dx \leq \int_{\mathbf{R}^n} |I_2^\varepsilon(\rho_0) + I_3^\varepsilon(\rho_0) - L(\rho_0)|^2 dx + C\varepsilon^{2-\alpha}||\rho_0||^2_{L^\infty(0,\infty;H^2(\mathbf{R}^n))}.$$

where

$$\begin{aligned}I_2^\varepsilon(\rho_0) &= \int_{\varepsilon\leq|w|\leq 1} \nu(x,w/\varepsilon) \int_0^{+\infty} \left[e^{-\int_0^z \nu(x-ws,w/\varepsilon)\,ds} - e^{-\nu(x,w/\varepsilon)z}\right] w\cdot\nabla_x\rho_0(x-wz)\,\frac{1}{|w|^{n+\alpha}}\,dz\,dw \\ &\quad + \int_{\varepsilon\leq|w|\leq 1}\int_0^{+\infty} e^{-\nu(x,w/\varepsilon)z} w\cdot D_x^2\rho_0(x-wz)\cdot w\,dz\frac{1}{|w|^{n+\alpha}}\,dw\end{aligned}$$

and

$$I_3^\varepsilon(\rho_0) = -\int_{|w|\geq 1}\int_0^{+\infty} e^{-\int_0^z \nu(x-ws,w/\varepsilon)\,ds}\nu(x,w/\varepsilon)\,\nu(x-wz,v)\left[\rho_0(x-wz)-\rho_0(x)\right]\frac{1}{|w|^{n+\alpha}}\,dz\,dw.$$

But

$$\int_{\mathbf{R}^n} |I_2^\varepsilon(\rho_0) + I_3^\varepsilon(\rho_0) - L(\rho_0)|^2 dx$$

$$\leq \int_{\mathbf{R}^n} \left( \int_0^\infty \int_{|w|<\varepsilon} \nu(x, w/\varepsilon) \frac{[e^{-\int_0^z \nu(x-ws, w/\varepsilon)\, ds} - e^{-\nu(x,w/\varepsilon)z}]}{|w|^{n+\alpha}} \, w \cdot \nabla_x \rho_0(x - wz) dz \, dw \right)^2 dx$$

$$+ \int_{\mathbf{R}^n} \left( \int_0^\infty \int_{|w|<\varepsilon} e^{-\nu(x,w/\varepsilon)z} w \cdot D_x^2 \rho_0(x - wz) \cdot w \, dz \frac{1}{|w|^{n+\alpha}} \, dw \right)^2 dx.$$

and so

$$\begin{aligned} \int_{\mathbf{R}^n} |I_2^\varepsilon(\rho_0) + I_3^\varepsilon(\rho_0) - L(\rho_0)|^2 dx &\leq C \int_{|w|<\varepsilon} |w|^{2-\alpha-n} dw ||\rho_0||^2_{L^\infty(0,\infty;H^2(\mathbf{R}^n))} \\ &\leq C\varepsilon^{2-\alpha} ||\rho_0||^2_{L^\infty(0,\infty;H^2(\mathbf{R}^n))} \end{aligned}$$

which allows to conclude.
□

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# Chapitre 11

# On the slowing down of charged particles in a binary stochastic mixture

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**abstract**

A kinetic equation is addressed for the straight line slowing-down of charged particles, the geometrical domain consists of randomly distributed spherical grains of dense material imbedded in a light material. The dense material is assumed to be a Boolean medium (the sphere centers are sampled according to a Poisson random field). We focus on the fraction of particles $P$ which stop in the light medium. After setting some properties of the Boolean medium, we perform an asymptotic analysis in two extreme cases corresponding to grain radius very small and very large with respect to the stopping distance of the dense material. A fitted analytic formula is proposed for the quantity $P$ and results of numerical simulations are presented in order to validate the proposed formula.

## 11.1 Introduction

The motivation of this study is the following. In the framework of Inertial Confinement Fusion experiments with Deuterium ($D$) targets, the thermonuclear reaction

$$D + D \Longrightarrow T + \text{proton}$$

creates Tritium ($T$) ions with a very high velocity - so they are called suprathermal. These tritium ions travel in the plasma and after slowing down they stop and may undergo new thermonuclear reactions with the Deuterium ions creating neutrons (the corresponding cross section is much larger than the one of $D + D$ reactions). So it is of primal interest to know where the Tritium ions stop. Here we are concerned with this problem in the case of a heterogeneous plasma. This plasma consists of a mixture of two materials : a light one which is Deuterium and a dense one whose presence is due to the hydrodynamic instabilities during the implosion (see Lindl [12] for physical background). Therefore,

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one has to estimate the fraction of suprathermal ions which stops in each material in order to estimate accurately the number of $D+T$ reactions.

For the modeling of this problem, one has to describe the motion of particles in the plasma and to know the mixture characteristic. Slowing down of suprathermal particles is classically described by a linear Vlasov equation (see eq. (11.1) below) ; the slowing down is assumed to be on straight line, that is to say the scattering phenomena are neglected. But, the structure of heterogeneous medium is not well known because it arises from complex hydrodynamic flow and we have to adopt a statistical description. This problem has been addressed in numerous articles [14], [15], [9], [11] ; but in these papers, the heterogeneous medium is always assumed to be Markovian with the same law in all directions. Although it enables to compute explicitly the fraction of particles lost in the dense material and leads to simple formulas, this model should be questioned because it seems not to be realistic (it is not clear that such a medium may be built easily in the three-dimensional case, see the end of Section 5 for this point).

In this paper, we use a different approach. The heterogeneous medium is assumed to consist of randomly distributed spherical grains of dense material embedded in the light material. With classical tools of stochastic geometry (see [21], [18]), the dense material is built as a Boolean medium $\mathcal{G}$ which is statistically homogeneous and isotropic. The problem may be studied by an asymptotic analysis in two extreme cases corresponding to grain radius which are very small and very large with respect to the stopping distance in the dense material $\mathcal{G}$. So, for these two extreme cases, we may evaluate the fraction $P$ of particles which stop in the light material $\mathcal{G}^c$, by an analytical way. In the sequel this fraction $P$ is expressed as a function of the relevant parameters of the model and mainly the volume fraction of $\mathcal{G}$, so we often write $P = P(f)$. For the general cases, we have to fit a formula which interpolate the ones obtained by asymptotic analysis ; it is done by performing a set of numerical simulations corresponding to different physical coefficients and to various sampling of the Boolean medium $\mathcal{G}$.

The paper is organized as follows. In section 2, we describe the physical model and we build the stochastic Boolean medium $\mathcal{G}$. The relevant parameters of the model are exhibited and some properties of the Boolean medium are also given, especially the estimation of the mean chord lengths, using the so-called capacity functional [18]. Section 3 is devoted to the asymptotic analysis of the problem in the cases where the grain radius is very small or very large (compared to the grain stopping distance) and we write the corresponding analytic formulas for the fraction of particles $P(f)$ which stop in the light material. We state also a fitted formula for $P(f)$ which may be applied to general cases. In section 4, we present the results of the numerical simulations in various situations for the validation of the previous formula. Our results are also compared with those of Levermore et al. [9].

## 11.2 The physical model and the geometry

In this paper, we assume that the slowing down of the suprathermal particles occurs only on straight lines (the scattering phenomena are neglected ; for a more realistic model, see [2]). Let us denote $\mathbf{v} = \Omega w$ the particle velocity of the particles, with $w \in \mathbf{R}^+$ and $\Omega \in \mathbf{S}^2$, the unit sphere. To model the behavior of the density of the charged particles $u = u(t, \mathbf{x}, w, \Omega)$ at time $t$, position $\mathbf{x} \in \mathbf{R}^3$ and velocity $\Omega w$, we consider the transport equation

$$\frac{\partial u}{\partial t} + w\Omega.\frac{\partial u}{\partial \mathbf{x}} = k(\mathbf{x})\frac{\partial}{\partial w}(S(w)u). \tag{11.1}$$

Here the slowing down function is assumed to be factorized by a product of an universal function $S(w) = s_0(w^2/\beta^3 + 1/w)$ with a slowing coefficient $k(x)$ depending only on the space variable. One assumes that the heterogeneous medium consists of two materials : the sub-domain $\mathcal{G}$ is the union of spherical granules of a "dense" plasma (the polluant) and in the other one $\mathcal{G}^c$ there is the reacting matter, called "light". So one states $k(\mathbf{x}) = k_g$ in $\mathcal{G}$ and $k(\mathbf{x}) = k_l$ in $\mathcal{G}^c$. In the function $S$, the $w^2$ term models the interactions with the electrons while the $1/w$ term models the interactions with ions (the coefficient $\beta$ depends mainly on the plasma temperature and is assumed to be constant in this paper).

The particles are assumed to be created with an uniform angular direction and a fixed energy, corresponding to the velocity $w_{\max}$, that is to say the initial velocity distribution is

$$d\Omega\delta_{w_{\max}}(dw) \qquad (\delta_{w_{\max}}, \text{ the Dirac mass in } w_{\max}).$$

We assume also that all the particles are created uniformly in the sub-domain $\mathcal{G}^c$, then the initial value of $u$ may read as

$$u(0,\mathbf{x},w,\Omega)dwd\Omega = u^0(\mathbf{x},w,\Omega)dwd\Omega \equiv \mathbf{1}_{\mathcal{G}^c}(\mathbf{x})\delta_{w_{\max}}(dw)d\Omega.$$

It is easy to show that for $t$ large enough, $u(t,x,w,\Omega)dw$ converges to a Dirac mass at $w=0$ (the particle stops after a finite time). We want to determine the probability that the particles stop in the light medium $\mathcal{G}^c$, that is to say we wish to get an evaluation of the following quantity

$$P = \lim_{t\to\infty} \frac{\int\int\int \mathbf{1}_{\mathcal{G}^c}(\mathbf{x})u(t,x,\Omega,w)\frac{d\Omega}{4\pi}d\mathbf{x}dw}{\int \mathbf{1}_{\mathcal{G}^c}(\mathbf{x})d\mathbf{x}} \tag{11.2}$$

As a matter of fact the above limit is reached for $t$ large enough (see the remark below).

### 11.2.1 Stopping distances

We address first the case where the slowing down coefficient $k$ is constant. For a fixed value of the variable $\Omega$, the solution $u$ of equation (11.1) may be replaced by $u = u(t,y,w)$ (where $y \in \mathbf{R}^+$ corresponds to an abscissa along a trajectory) which is solution to

$$\frac{\partial u}{\partial t} + w\frac{\partial}{\partial y}u = k\frac{\partial}{\partial w}(S(w)u).$$

The characteristic curves $(Y(t),W(t))$ for this equation are given by

$$dY(t) = W(t)dt, \qquad dW(t) = -kS(W(t))dt$$

thus we get

$$\frac{W(t)}{S(W(t))}dW(t) = -kdY(t).$$

Defining the function $H$ by $H(W) = \int_0^W \frac{v}{S(v)}dv = \frac{\beta^3}{3s_0}\log(1+\frac{W^3}{\beta^3})$, for $W>0$, we have for all $t^1, t^2$ (with $t^2 > t^1$)

$$k(Y(t^2) - Y(t^1)) = H(W(t^1)) - H(W(t^2)).$$

Recall that the initial velocity is $W(0) = w_{\max}$ , then the velocity becomes zero after a finite distance which is equal to $k^{-1}H(w_{\max})$. This distance is called the stopping distance. The coefficient $s_0$ is chosen such that $H(w_{\max}) = 1$ (that is $s_0 = \frac{\beta^3}{3}\log(1+w_{\max}^3/\beta^3)$ ). Then the stopping distance is equal to $k^{-1}$.

In our model, two stopping distances are to be considered : $k_l^{-1}$ in the light medium $\mathcal{G}^c$ and $k_g^{-1}$ in $\mathcal{G}$. We always have $k_l^{-1} > k_g^{-1}$.

*Remark* 11.2.1. Since $k(Y)dt = -S(W(t))^{-1}dW(t)$, the stopping time is bounded from above by $t_{\max}$ which satisfies

$$k_l t_{\max} = \int_0^{w_{\max}} \frac{1}{S(v)}dv \le \frac{\beta^2}{2s_0}\int_0^\infty \frac{dz}{1+z^{3/2}} \le \frac{3\beta^2}{2s_0} = \frac{9}{2\beta\log(1+w_{\max}^3\beta^{-3})}$$

Thus by setting $t_{\max} = (k_l\frac{2}{9}\beta\log(1+w_{\max}^3\beta^{-3}))^{-1}$, formula (11.2) may be simplified and reads as

$$P = \frac{\int\int\int 1_{\mathcal{G}^c}(\mathbf{x})u(t_{\max},\mathbf{x},\Omega,w)\frac{d\Omega}{4\pi}d\mathbf{x}dw}{\int \mathbf{1}_{\mathcal{G}^c}(\mathbf{x})d\mathbf{x}}$$

### 11.2.2 Stochastic geometry

In the sequel, for any compact set $A$, one denotes by $|A|$ its volume, by $A_{(\mathbf{z})}$ the compact set translated by a vector $\mathbf{z}$ and by $S_{\mathbf{o}}$ the sphere of radius $r$ (which is a deterministic value) whose center is located at a fixed point $\mathbf{o}$ . Moreover let us set $A \oplus A' = \{\mathbf{x} + \mathbf{x}' / \mathbf{x} \in A,\ \mathbf{x}' \in A'\}$ and by $\sharp\{.\}$ the cardinal on a discrete set.

Let us now define the random closed set $\mathcal{G}$. For the sake of simplicity, it is done in the whole space $\mathbf{R}^3$, but from the numerical point of view, we shall consider the problem on a bounded set, more precisely on a torus of $\mathbf{R}^3$ (the numerical simulations based on the equation (11.1) will be performed on such a torus). We assume that

$$\mathcal{G} = \cup_{\mathbf{q}} S_{\mathbf{q}}$$

where $S_{\mathbf{q}}$ is the sphere of radius $r$ whose center is the random point $\mathbf{q}$ and the set of points $\{\mathbf{q}\}$ is a stationary Poisson point process whose intensity is equal to a constant $\lambda$ times the Lebesgue measure. Such a medium is called in the literature a *Poisson germ-grain random closed set* or shortly a *Boolean medium* (see for instance [18]). Recall that the Poisson point process satisfies

$$\mathbb{P}(\sharp\{\text{points } \mathbf{q} \text{ in } A\} = n) = \frac{(\lambda|A|)^n}{n!} \exp(-\lambda|A|), \qquad \text{for all compact set } A$$

and for two disjoint compact sets $A$ and $A'$, the random variables $\sharp\{\text{points } \mathbf{q} \text{ in } A\}$ and $\sharp\{\text{points } \mathbf{q} \text{ in } A'\}$ are independent ; moreover $\lambda|A|$ is the mean number of points $\mathbf{q}$ in $A$. Of course, we allow overlapping of the spheres $S_{\mathbf{q}}$. Notice that the construction of a random closed set without overlapping of spheres is more delicate, see [1] for instance.

*Remark* 11.2.2. The hypothesis that the granules are spherical may be weakened from a theoretical point of view (the sphere may be replaced by a very smooth convex body) but in this paper we wish to perform numerical calculus so it is better to deal with spheres in order to have fast computations. To assume that the radius of the sphere is equal to a fixed deterministic value is clearly not realistic ; but in practice it may be assumed that in a small regium (at macroscopic level), the typical diameter of the heterogeneities is a random variable whose standard deviation is small. On the other hand, we will see that the final formulas for $P$ are smooth functions of the parameter $r$, so if the probability distribution function is centered around a mean value $\overline{r}$, we may think that the expected value of $P$ for a random distribution of $r$ may be roughly approximated by the value of $P$ evaluated with the mean value $\overline{r}$.

The Boolean medium $\mathcal{G}$ is statistically homogeneous and isotropic. According to the stationarity of $\mathcal{G}$, we have $\mathbb{P}(A \cap \mathcal{G} \neq \emptyset) = \mathbb{P}(A_{(\mathbf{z})} \cap \mathcal{G} \neq \emptyset)$ for all compact set $A$. Moreover, $\mathbb{P}(\mathbf{x} \in \mathcal{G})$ does not depend on the deterministic point $\mathbf{x}$; this justifies to set

$$f = \mathbb{P}(\mathbf{x} \in \mathcal{G}) \qquad \text{for all fixed point } \mathbf{x}$$

This quantity is called the mean volume fraction $f$ of the Boolean medium $\mathcal{G}$; it corresponds to the intuitive definition of the volume fraction, according to the following simple but fundamental result.

*Lemma* 11.2.1. For any deterministic compact set $A$, we get

$$f = \frac{1}{|A|} \mathbb{E}\left(\left|A \bigcap \mathcal{G}\right|\right)$$

Moreover, it satisfies

$$f = 1 - \exp(-\lambda|S_{\mathbf{o}}|) \tag{11.3}$$

For any deterministic compact set $B$, we get

$$\mathbb{P}(B \cap \mathcal{G} \neq \emptyset) = 1 - \exp(-\lambda|S_{\mathbf{o}} \oplus B|)\,. \tag{11.4}$$

*Démonstration.* According to the definition of $f$, we get

$$f = \frac{1}{|A|} \int_A \mathbb{P}(\mathbf{x} \in \mathcal{G})\, d\mathbf{x} = \frac{1}{|A|} \mathbb{E}\left(\int_A 1_{\mathcal{G}}(\mathbf{x}) d\mathbf{x}\right) = \frac{1}{|A|} \mathbb{E}\left(\left|A \bigcap \mathcal{G}\right|\right)$$

We now prove (11.4) [notice that (13.26) follows from that relation whith $B = \{\mathbf{o}\}$]. Notice that $1 - \mathbb{P}(B \cap \mathcal{G} \neq \emptyset) = \mathbb{P}\left(\mathbf{x} \notin (\cup_{\mathbf{q}} S_{\mathbf{q}}), \forall \mathbf{x} \in B\right)$. But $\mathbf{x} \notin S_{\mathbf{q}}$ means that $\mathbf{q} \notin (S_{\mathbf{o}})_{(\mathbf{x})}$, thus

$$\mathbb{P}\left(\mathbf{x} \notin \cup_{\mathbf{q}} S_{\mathbf{q}}, \forall \mathbf{x} \in B\right) = \mathbb{P}[\text{no point } \mathbf{q}, \text{ s.t. } \mathbf{q} \in S_{\mathbf{o}} \oplus \{\mathbf{x}\}, \forall \mathbf{x} \in B] = \exp(-\lambda |S_{\mathbf{o}} \oplus B|),$$

according to the Poisson law. □

*Remark* 11.2.3. As it is claimed in [18], it may be checked that the Boolean medium $\mathcal{G}$ is mixing, that is to say if $\mathbf{z} \to \infty$, we get for all compact sets $A, B$

$$\mathbb{P}(A \cap \mathcal{G} \neq \emptyset, B_{(\mathbf{z})} \cap \mathcal{G} \neq \emptyset) \to \mathbb{P}(A \cap \mathcal{G} \neq \emptyset)\mathbb{P}(B \cap \mathcal{G} \neq \emptyset)$$

(indeed $A$ and $B_{(\mathbf{z})}$ are disjoint for $\mathbf{z}$ large enough). Therefore $\mathcal{G}$ is ergodic. As a consequence, according to the ergodic property, one has $\lim_{\alpha \to \infty} \frac{|\mathcal{G} \cap (\alpha B)|}{\alpha |B|} = f$, almost surely ( $\alpha B$ denotes the set $\{\alpha \mathbf{x} / \mathbf{x} \in B\}$).

The functional defined for all compact set $A$ by $\mathbb{P}(A \cap \mathcal{G} \neq \emptyset)$ is called the Choquet's capacity functional related to the Boolean medium $\mathcal{G}$.

As it is explained above, we shall consider the problem on a torus for numerical purpose. So, we introduce a characteristic length $\mathcal{L}$ which is somehow the observation length of the phenomena (this length is larger than $k_l^{-1}$) and one considers a Poisson point process $\{\mathbf{q}\}$ and a Boolean medium $\mathcal{G}$ which are defined on the torus $[0, \mathcal{L}[^3$. Since $\lambda \mathcal{L}^3$ is the mean number of points $\mathbf{q}$ in the torus, we may state

$$\lambda = N \mathcal{L}^{-3}$$

where $N$ is the mean number of sphere centers. Notice that if the volume fraction is small, there is few overlapping and $f$ is close to $\frac{N}{\mathcal{L}^3} |S_{\mathbf{o}}|$.

Lastly, it may be also interesting to address two-dimensional problems; then of course $\mathcal{L}^3$ has to be replaced by $\mathcal{L}^2$ in the previous formulas.

### 11.2.3 Trajectories and chord lengths.

The trajectory of a given particle belongs to a half line $D$ (corresponding to an angular direction $\Omega$); besides its initial position $Y^0$ (in $\mathcal{G}^c$) it is characterized by the intersections of $D$ with the Boolean medium $\mathcal{G}$. So denote by $Y^i$ the $i-$th entrance intersection of $D$ with $\mathcal{G}$ and by $Y^{i,s}$ the corresponding exit intersection; $Y^e$ is the final point of the particle trajectory, that to say the point where its velocity vanishes.

Let $l_g^i = |Y^{i,s} - Y^i|$ and $l_l^i = |Y^i - Y^{i-1,s}|$ (for $i \geq 2$) be the length of the $i$-th section of the trajectory respectively in $\mathcal{G}$ and in the light medium $\mathcal{G}^c$.

According to the calculus made above, there are two formulas satisfied by $Y^e$ according to the medium in which the particle stops : either there exists $i$ such that

$$1 = k_l |Y^1 - Y^0| + k_g l_g^1 + k_l l_l^2 ... + k_l l_l^i + k_g l_g^i + k_l |Y^e - Y^{i,s}| \tag{11.5}$$

then it stops in the light medium, or

$$1 = k_l |Y^1 - Y^0| + k_g l_g^1 + k_l l_l^2 ... + k_l l_l^i + k_g |Y^e - Y^i| \tag{11.6}$$

then it stops in a grain (by convention in these formulas, if $i = 1$,then $k_g l_g^1 + ... + k_l l_l^i$ is withdrawn).

Thus the probability $P$ to stop outside the Boolean medium $\mathcal{G}$ is given by

$$P = \mathbb{P}(k_l |Y^1 - Y^0| > 1) + \sum_{i \geq 1} \mathbb{P}\left(k_l |Y^1 - Y^0| + k_g l_g^1 + ... + k_g l_g^i \leq 1 \right. \tag{11.7}$$
$$\left. \text{and } k_l |Y^1 - Y^0| + k_g l_g^1 + ... + k_g l_g^i + k_l l_l^{i+1} > 1\right) \quad .$$

**Mean chord lengths.**

Since the Boolean medium is statistically homogeneous and isotropic, the chords $l_g^i$ are independent and equidistributed and the chords $l_l^i$ are independent and equidistributed. The following result related to the chord length in a Boolean medium is usefull for the sequel.

*Proposition.* **Underwood's theorem** [22]. Let $S(\mathcal{G})$, the expected value of the interface area $\partial\mathcal{G}$ divided by $\mathcal{L}^3$, then we get

$$\begin{aligned}\mathbb{E}(l_g) &= 4f/S(\mathcal{G}),\\ \mathbb{E}(l_l) &= 4(1-f)/S(\mathcal{G}).\end{aligned}$$

It is proved in appendix where the expression of $S(\mathcal{G})$ is also given. In 2D the coefficient 4 has to be replaced by $\pi$.

Therefore, we get

$$\frac{\mathbb{E}(l_l)}{\mathbb{E}(l_g)} = \frac{1-f}{f} \tag{11.8}$$

The problem of finding the chord-length distribution of a binary stochastic mixture is an important one, which has been dealt in numerous studies since Debye et al. [3], see [17], [16], [20], [7], [21].

## 11.3 An analytic formula for the probability $P(f)$

There are four characteristic lengths

$$r, \quad k_g^{-1}, \quad k_l^{-1}, \quad \mathcal{L},$$

and one parameter $f$ which is called the mean volume fraction of the granules. We always have

$$k_g^{-1} \leq k_l^{-1} \leq \mathcal{L} \qquad \text{and} \qquad r \leq \mathcal{L}$$

$$\text{Moreover} \quad k_l^{-1} \sim \mathcal{L}$$

We present in the two first paragraphs some mathematical arguments to compute the probability $P = P(f)$ to stop in the light medium when the sphere radius $r$ is respectively much smaller or much larger than to the stopping distance $k_g^{-1}$ corresponding to the medium $\mathcal{G}$. In the third paragraph, we propose an analytic formula to model this probability $P(f)$.

### 11.3.1 Case of homogenization.

We address the case where
- the stopping distances $k_g^{-1} \sim k_l^{-1}$ are of order $\mathcal{L}$,
- the radius of the sphere is small, i.e. $r = \varepsilon \ll \mathcal{L}$.

Notice that, since $f$ is fixed, the mean number of granules $N$ in the torus $[0, \mathcal{L}[^3$ is of order of $\varepsilon^{-3}$. We now introduce a different scaling in the space variable and we denote

$$L_g^k = l_g^k/\varepsilon, \quad L_l^k = l_l^k/\varepsilon, \quad \widetilde{L}_l^1 = |Y^1 - Y^0|/\varepsilon, \qquad \widetilde{Y}^e = Y^e/\varepsilon.$$

So we get

$$\varepsilon^{-1} = k_l\widetilde{L}_l^1 + k_g L_g^1 + ... + k_g L_g^{i-1} + k_l|\widetilde{Y}^e - \widetilde{Y}^{i-1,s}|, \tag{11.9}$$

if the particle stops in the light medium and

$$\varepsilon^{-1} = k_l\widetilde{L}_l^1 + k_g L_g^1 + ... + k_l L_l^i + k_g|\widetilde{Y}^e - \widetilde{Y}^i|$$

if the particle stops in a granule.

We want to estimate the following quantity

$$P_\varepsilon = \mathbb{P}\left(k_l\widetilde{L}_l^1 > \frac{1}{\varepsilon}\right) + \sum_{i\geq 1} \tag{11.10}$$

$$\mathbb{P}\left(k_l\widetilde{L}_l^1 + k_g L_g^1 + ... + k_g L_g^i < \frac{1}{\varepsilon} \text{ and } k_l\widetilde{L}^1 + k_g L_g^1 + ... + k_g L_g^i + k_l L_l^{i+1} \geq \frac{1}{\varepsilon}\right).$$

*Proposition.* Assume that the random variables $(L_l^i)$ are identically distributed, the random variables $(L_g^i)$ are identically distributed and that all the random variables $(L_g^i)$ and $(L_l^i)$ are independent. Then, when $\varepsilon$ goes to 0, we get

$$P_\varepsilon \to \frac{k_l \mathbb{E}(L_l)}{k_g \mathbb{E}(L_g) + k_l \mathbb{E}(L_l)}$$

*Démonstration.* The probability distribution of $\widetilde{L}_l^1$ is not the same as for the $L_l^i$ , but since we are concerned with the formula (11.10) with small $\varepsilon$ , one sees that $P_\varepsilon$ has the same limit that $P^\varepsilon$ which is obtained by replacing $\widetilde{L}_l^1$ by a random variable $L_l^1$ which is distributed as the other $L_l^i$ ; that is $P^\varepsilon$ reads as

$$P^\varepsilon = \sum_{i=1}^{\infty} \mathbb{P}\left( \sum_{j=1}^{i} (k_l L_l^j + k_g L_g^j) < \frac{1}{\varepsilon} < \sum_{j=1}^{i} (k_l L_l^j + k_g L_g^j) + k_l L_l^{i+1} \right)$$

Then, using the independence between $k_l L_l^{i+1}$ and $\sum_{j=1}^{i} (k_l L_l^j + k_g L_g^i)$, we get

$$\begin{aligned} P^\varepsilon &= \sum_{i=1}^{\infty} \int_0^{1/\varepsilon} \mathbb{P}\left( \sum_{j=1}^{i} (k_l L_l^j + k_g L_g^j) \in [x, x+dx] \right) \mathbb{P}\left( \frac{1}{\varepsilon} < x + k_l L_l^{i+1} \right) \\ &= \int_0^{\infty} \sum_{i=1}^{\infty} \mathbb{P}\left( \sum_{j=1}^{i} (k_l L_l^j + k_g L_g^j) \in [\frac{1}{\varepsilon} - y, \frac{1}{\varepsilon} - y + dy] \right) \mathbb{P}(k_l L_l > y) \mathbf{1}_{y<1/\varepsilon} \end{aligned}$$

Since the random variables $k_l L_l^j + k_g L_g^j$ are identically distributed and independent, we can build the renewal process associated to these variables. Let us recall the renewal theorem (cf. [5], chap. 11). If $Z^j$ denote independent identically distributed random variables and $U(x, dx) = \mathbb{P}\left( \cup_i (\sum_{j=1}^{i} Z^j \in [x, x+dx]) \right)$ the renewal measure, then the theorem claims that $U(x, dx) \to dx/\mathbb{E}(Z^1)$ when $x \to \infty$. Notice that $U(x, dx)$ is the expected number of points of the renewal process $\sum_{j=1}^{i} Z^j$ in the intervall $[x, x+dx]$ .

We now apply this theorem to the random variables $Z^j = k_l L_l^j + k_g L_g^j$. So, when $x \to \infty$, we have

$$\sum_{i=1}^{\infty} \mathbb{P}\left( \sum_{j=1}^{i} (k_l L_l^j + k_g L_g^j) \in [x, x+dx] \right) \to \frac{dx}{\mathbb{E}(k_g L_g + k_l L_l)} \tag{11.11}$$

Let now $\varepsilon \to 0$. Since the function $y \to P(k_l L_l > y) \mathbf{1}_{y<1/\varepsilon}$ is bounded by $P(k_l L_l > y)$ which is integrable, according to Lebesgue theorem, the property (11.11) implies that

$$P^\varepsilon = \int_0^{\infty} U\left( \frac{1}{\varepsilon} - y, dy \right) \mathbb{P}(k_l L_l > y) \mathbf{1}_{y<\frac{1}{\varepsilon}} \to \frac{1}{\mathbb{E}(k_g L_g + k_l L_l)} \int_0^{\infty} \mathbb{P}(k_l L_l > y) dy.$$

Since $\int_0^{\infty} \mathbb{P}(L_l > y/k_l) dy = \int_0^{\infty} k_l z \mathbb{P}(L_l \in [z, z+dz]) = k_l \mathbb{E}(L_l)$, we get the result. □

According to this proposition and relation (11.8), we may claim that, when $k_g r \ll 1$, we get

$$P_\varepsilon \simeq P_{\hom}(f) \equiv \frac{k_l(1-f)}{k_l(1-f) + k_g f} \tag{11.12}$$

*Remark* 11.3.1. Since $f k_g + (1-f) k_l$ is the averaged value $\overline{k}$ of $k$, the coefficient $k = k_\varepsilon$ whose values in $\mathcal{G}$ and $\mathcal{G}^c$ are $k_g$ and $k_l$, converges towards $\overline{k}$ in $L^2$ weakly. Then the justification of (11.12) can be reinforced by the result obtained in [6] which claims that the solution $u_\varepsilon$ of the initial problem $\frac{\partial}{\partial t} u_\varepsilon + w\Omega . \frac{\partial}{\partial \mathbf{x}} u_\varepsilon = k_\varepsilon \frac{\partial}{\partial w}(S(w) u_\varepsilon)$ may be approximated (with some technical assumptions on the random heterogeneous medium) by the solution $U$ to the homogenized problem

$$\frac{\partial U}{\partial t} + w\Omega . \frac{\partial}{\partial \mathbf{x}} U = \overline{k} \frac{\partial}{\partial w}(S(w) U),$$

The technical arguments of the proof are analogous to the ones used in [4].

### 11.3.2 Case of small stopping distances

We address now the case where
- the radius $r$ of the spheres is of order $\mathcal{L}$.
- the stopping distance $k_g^{-1}$ is very small with respect to $k_l^{-1}$ and $r$,that is

$$k_g^{-1} \ll k_l^{-1} \qquad \text{and} \qquad k_g^{-1} \ll r$$

In these asymptotic conditions, we want to express the fact that if a particle crosses a sphere, it stops in this sphere since the slowing power is very large compared to the radius of the spheres.

Then, if one assumes that a particle is initialy located at point $\mathbf{y}$ chosen uniformly in $\mathcal{G}^c$ and that its direction is equal to a fixed vector $\Omega$, its probabilty $P(f)$ to stop in $\mathcal{G}^c$ is nearly equal to the probabilty $P_{ssd}(f)$ that the travel distance of this particle up to the set $\mathcal{G}$ is smaller than $k_l^{-1}$. So, introducing this travel distance $d(\mathbf{y}) = \inf(s/\mathbf{y} + s\Omega \in \mathcal{G})$, we have to evaluate the quantity $\mathbb{P}\left(d(\mathbf{y}) < k_l^{-1}\right)$.

*Proposition.* For any point $\mathbf{y}$ in $\mathcal{G}^c$, we get

$$P_{ssd} = \mathbb{P}\left(d(\mathbf{y}) < k_l^{-1}\right) = \exp\left(\frac{3}{4k_l r}\log(1-f)\right) = (1-f)^{3/4k_l r}$$

*Démonstration.* Consider a random point $\mathbf{y}$ uniformly distributed in $[0,\mathcal{L}[^3$ independently from the points $\mathbf{q}$. According to the definition of $f$, we have

$$\begin{aligned} 1 - P_{ssd} &= \mathbb{P}\left(d(\mathbf{y}) > k_l^{-1} \ / \ \mathbf{y} \in \mathcal{G}^c\right) \\ &= \mathbb{P}\left([d(\mathbf{y}) > k_l^{-1}] \cap [\mathbf{y} \in \mathcal{G}^c]\right)/\mathbb{P}(\mathbf{y} \in \mathcal{G}^c) \\ &= \mathbb{P}\left([d(\mathbf{y}) > k_l^{-1}] \cap [\mathbf{y} \in \mathcal{G}^c]\right)/(1-f) \end{aligned}$$

Denote by $\Gamma$ the cylinder whose base is the half sphere $S_{\mathbf{o}}$ and whose length is equal to $k_l^{-1}$, we get

$$\mathbb{P}\left([d(\mathbf{y}) > k_l^{-1}] \cap [\mathbf{y} \in \mathcal{G}^c]\right) = \mathbb{P}\left(\mathbf{y} \notin \mathcal{G} \oplus \Gamma\right) = \mathbb{P}\left(\mathbf{o} \notin \mathcal{G} \oplus \Gamma\right)$$

indeed these quantities do not depend on the point $\mathbf{y}$ . But $\mathbf{o} \in \mathcal{G} \oplus \Gamma$ means that $\mathbf{o} - \mathbf{z} \in \mathcal{G}$ for one point $\mathbf{z} \in \Gamma$. Then $\mathbb{P}\left(\mathbf{o} \in \mathcal{G} \oplus \Gamma\right) = \mathbb{P}((-\Gamma) \cap \mathcal{G} \neq \varnothing)$. According to lemma 1, we see that

$$\mathbb{P}\left(\mathbf{o} \notin \mathcal{G} \oplus \Gamma\right) = 1 - \mathbb{P}(\Gamma \cap \mathcal{G} \neq \emptyset) = \exp\left(-\lambda|S_{\mathbf{o}} \oplus \Gamma|\right) = \exp\left(-\lambda|S_{\mathbf{o}}|\right)\exp\left(-\lambda|\Gamma|\right).$$

Since $|\Gamma| = \pi r^2 k_l^{-1} = \frac{3}{4k_l r}|S_{\mathbf{o}}|$, the result comes from relation (13.26). □

In two-dimension case, the volume of the cylinder is equal to $2rk_l^{-1}$, then the same argument leads to $P_{ssd}(f) = \exp(\frac{2}{\pi k_l r}\log(1-f))$. So one can state, for whatever dimension

$$P_{\mathrm{ssd}}(f) = \exp\left(\frac{\alpha}{k_l r}\log(1-f)\right),$$

with $\alpha = 2/\pi$ in the two-dimension case and $\alpha = 3/4$ in the three-dimension case.

### 11.3.3 Fitting by an analytic formula

We wish to find a fitting formula which would be valid when the parameters range over a wide domain, for instance

$$k_l r \in [10^{-1}, 10], \quad k_g r \in [10^{-1}, 10^2], \quad k_g/k_l \in [1, 10].$$

For the volume fraction $f$, it suffices for a practical point of view to assume $f \in [0, 0.6]$. For this purpose, to approximate $P(f)$ we propose a combination of the two formulas $P_{\mathrm{ssd}}(f)$ and $P_{\mathrm{hom}}(f)$ found above

$$P_{\mathrm{hom}} = \frac{k_l(1-f)}{k_l(1-f) + k_g f}, \qquad P_{\mathrm{ssd}}(f) = \exp\left(\frac{\alpha}{k_l r}\log(1-f)\right)$$

In other words, we look for a fitting function $Z$ such that :

$$P(f) = P_{\rm ssd}(f)Z + P_{\rm hom}(f)(1-Z), \quad \text{with } 0 \leq Z \leq 1.$$

When $r$ becomes very small compared to $k_l^{-1}$ and $k_g^{-1}$, one must have $Z \simeq 0$, and when $k_l r$ and $k_g r$ becomes large one must have $Z \simeq 1$. It is natural to search a fitting function $Z$ depending only on the quantity $k_g r$ that is to say the ratio between the granule radius and the slowing distance in the granules. The simplest choice is $Z = \exp(-bk_g r)$ with $b$ constant. With the help of the numerical results presented below, we may adjust the constant $b$ and it turns out that a good choice is $b = 1$.

Finally, we propose the following analytic formula

$$P_{\rm fit}(f) = \exp\left(\frac{\alpha}{k_l r}\log(1-f)\right)(1-e^{-k_g r}) + e^{-k_g r}\frac{k_l(1-f)}{k_l(1-f)+k_g f} \tag{11.13}$$

## 11.4 Numerical results

In this section, numerical simulations are presented for estimating the probability $P(f)$ as a function of $f$. The numerical algorithm is based on the solution of the initial kinetic equation (11.1) by a Monte-Carlo method ; its description is sketched in appendix 1.

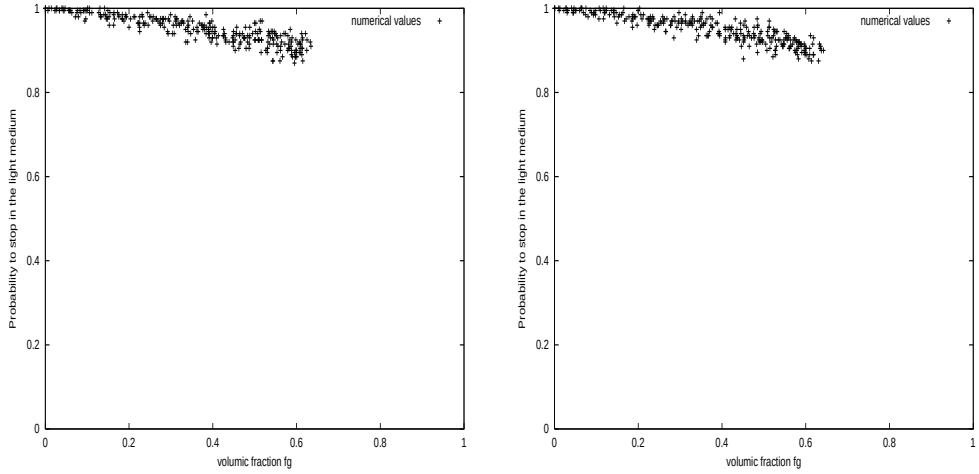


FIGURE 11.1 – $P(f)$ versus the volume fraction $f$. Case of large sphere radius : $rk_l = 6$. Left : $k_g = k_l$. Right : $k_g = 9k_l$, Two-dimension computations

We first consider the case where the radius of the spheres is very large with respect to the stopping distance in the light medium, here $rk_l = 6$. We check on Fig. 11.1 that the probability $P(f)$ depends weakly on the volume fraction $f$ and we recover – what we mathematically expected – that $P(f)$ does not depend any more on the ratio between $k_g$ and $k_l$ since the particle will stop in the first sphere encountered whatever $k_g$ is.

We now compare the numerical results with the analytic formula $P_{\rm fit}$ given by (11.13) for a representative panel of parameters. We observe that the analytic formula is a good approximation in the two extremal cases of characteristic length and for any ratio of the slowing distances.

The first considered cases correspond to a sphere radius which is small compared with the stopping distance in $\mathcal{G}^c$ : $rk_l = 0.1$. In these cases, the radius of the spheres is small enough so that the particles go through a large number of spheres before stopping.

On Fig. 11.2 computations performed in the two-dimension and the three-dimension framework are plotted in the case $rk_g = 0.9$ . On Fig. 11.3, the computations are performed for other values of $rk_g$ (the computation are made in three-dimension as in the sequel). We check on Fig. 11.2 and

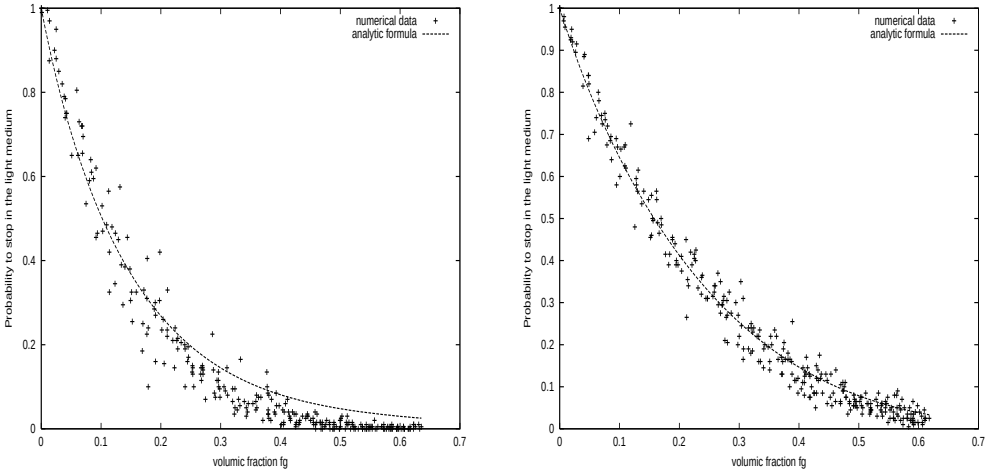


FIGURE 11.2 – $P(f)$ versus the volume fraction $f$. Case of small sphere radius : $rk_l = 0.1$ and $rk_g = 0.9$ . Left : 2-dim. ; Right : 3-dim. computations

Fig. 11.3 that the numerical simulations are in agreement with the mathematical estimation of $P(f)$ which is here very close to $P_{\text{hom}}(f)$. On Fig. 11.4, we have plotted results obtained in cases where the characteristic lengths $r$ and $k_l^{-1}$ are such that $rk_l = 1$; they are performed for two different values of $rk_g$. On Fig 11.5, we have plotted results obtained for a sphere radius which is large compared with the stopping distance in $\mathcal{G}^c$, here we choose $rk_l = 6$. They are also performed for two different values of $rk_g$. One can check that in all the cases there is a good agreement between the numerical results and our analytic formula $P_{\text{fit}}$.

## 11.5 Comparison with a Markovian medium

In this section, we would like to compare our results based on a representation of the heterogeneous binary mixture by a Boolean medium with the ones obtained in [9] by another representation of the mixture, see also [11] [14], [15]. In these papers, the main hypothesis is that the length of both media are distributed according Markovian laws (so the approach is specially one-dimensional). For this model, one has to introduce the characteristic stopping distance of each material which are equal to $k_l^{-1}$ and $k_g^{-1}$ and some characteristic lengths of each medium $\lambda_l$ , $\lambda_g$ .

Then the proposed analytical approximation reads as follows

$$P(f) = P_{\text{lev}}(f) = \frac{k_l\lambda_l}{k_l\lambda_l + k_g\lambda_g} + \frac{k_g\lambda_g}{k_l\lambda_l + k_g\lambda_g} \exp\left(-\frac{1}{k_l\lambda_l} - \frac{1}{k_g\lambda_g}\right).$$

In order to make comparison, we have to define precisely the two quantities $\lambda_l$ , $\lambda_g$ in the framework of the Boolean medium ; it is natural to claim that these quantities are the mean chord length for $\mathcal{G}^c$ and for $\mathcal{G}$. So, according to (11.8), we get

$$\lambda_l = \lambda_g(1-f)/f$$

Underwood's theorem yields a value for $\lambda_g$ which is not very handy in the general case ; so we choose for $\lambda_g$ the mean chord length of an isolated sphere, that is to say $\lambda_g = r/\alpha$, where $\alpha$ is a constant depending on the dimension ($\alpha = 2/\pi$ or $\alpha = 3/4$ as above). Therefore, denoting $f_l = 1 - f$, we get $\lambda_l = rf_l/(\alpha f)$ and the approximation proposed in [9] reads as

$$\begin{aligned} P_{\text{lev}}(f) &= \frac{k_l f_l}{k_l f_l + k_g f} + \frac{k_g f}{k_l f_l + k_g f} \exp\left(-\frac{\alpha}{r}(\frac{1}{k_g} + \frac{f}{f_l k_l})\right), \\ &= \frac{k_l f_l}{k_l f_l + k_g f}\left(1 - \exp(-\frac{\alpha}{k_g r})\exp(-\frac{\alpha f}{f_l k_l r})\right) + \exp(-\frac{\alpha}{k_g r})\exp(-\frac{\alpha f}{f_l k_l r}), \end{aligned}$$

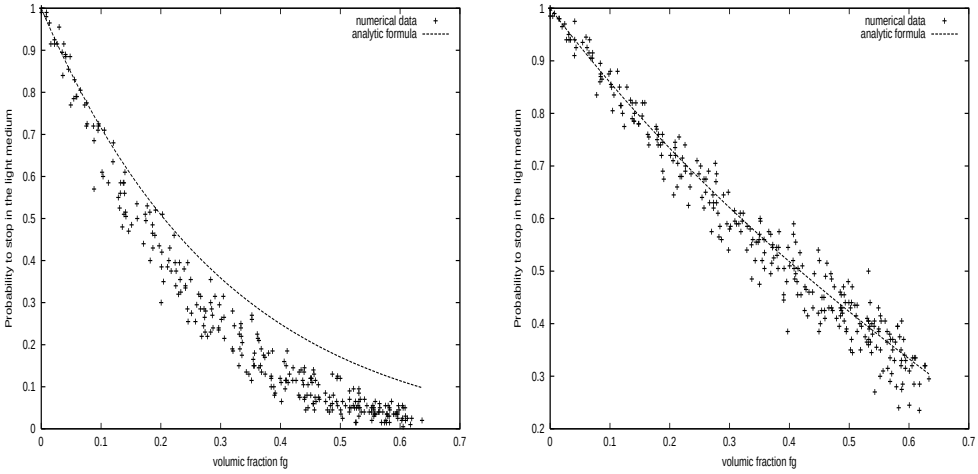


FIGURE 11.3 – Case of small sphere radius : $rk_l = 0.1$. Left : $rk_g = 0.3$; Right : $rk_g = 0.1$ (3-dim. computations)

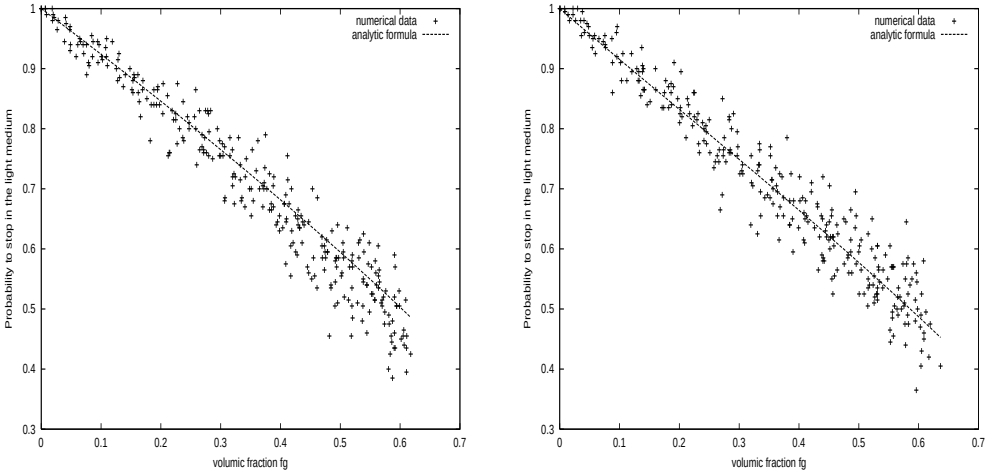


FIGURE 11.4 – Case $rk_l = 1$. Left : $rk_g = 9$ ; Right : $rk_g = 3$,

We can now compare this formula with our formula

$$P_{\text{fit}}(f) = \exp\left(\frac{\alpha}{k_l r}\log(1-f)\right)(1-Z) + Z\frac{k_l f_l}{k_l f_l + k_g f}, \quad \text{with } Z = \exp(-k_g r).$$

Notice that in the case where $k_g r \ll 1$ that is to say $Z \simeq 1$, we have $P_{\text{fit}} = P_{\text{lev}} = P_{\text{hom}}$. Now in the case where $k_g r \gg 1$, and $k_g/k_l \gg 1$, that is to say $Z \simeq 0$, we have $P_{\text{fit}}(f) \simeq \exp(\frac{\alpha}{k_l r}\log(1-f))$, but $P_{\text{lev}}(f) \simeq \exp\left(-\frac{\alpha}{k_l r}\frac{f}{1-f}\right)$. Therefore one checks that for very small values of $f$, the two quantities are very close, but at second order with respect to $f$, there is a difference between these two quantities ($\log(1-f)$ is not equal to $-f/(1-f)$ at second order with respect to $f$).

Let us now perform some numerics to compare the two approaches. On Fig. 11.6 and Fig. 11.7, the results given by the numerical simulations are compared with the two analytic formulas $P_{\text{lev}}$ and $P_{\text{fit}}$ in two cases : firstly when the sphere radius is small and the ratio between the two stopping distances is large ; secondly when the sphere radius is of order 1 and the stopping distances are equal.

The comparison between the two analytic formula suggests that the two models do not differ a lot when the stopping distance are of the same order of magnitude but in the other case our formula $P_{\text{fit}}$ is a better one.

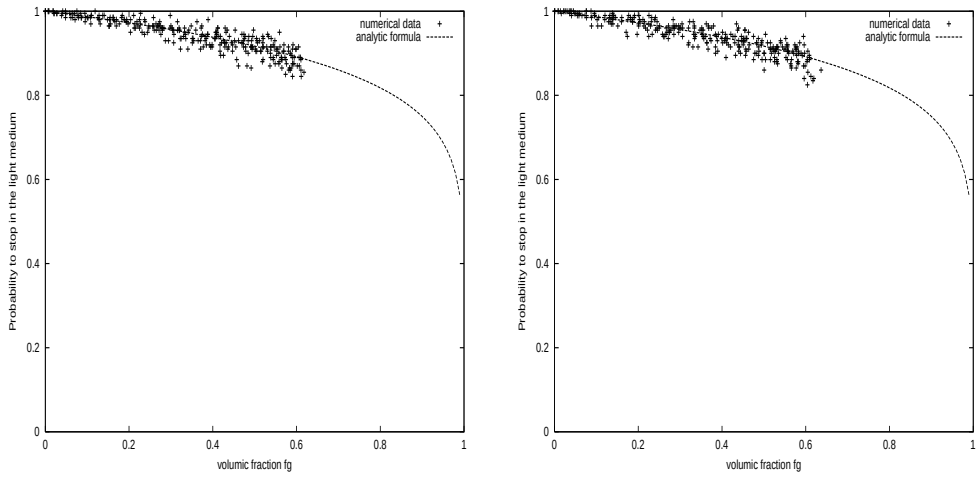


FIGURE 11.5 – case of large sphere radius : $rk_l = 6$. Left : $rk_g = 54$ ; Right : $rk_g = 18$,

For understanding the difference between the two approaches, it must be noticed that the building of the random medium considered in [9], [10], [11] is basically a one dimensional construction and the way to extend their results to a fully three dimensional random medium is not clear. Particularly, it is easy to find examples where the chord distribution is exponential but depends on the direction (simply think to a stratified medium), but the assumption that the distribution of chords is markovian with the **same** law in each direction (which is the case if the medium is isotropic) is a very strong condition. In [13], it is claimed that for a binary random mixture with non-overlapping disks or spheres, the chord-length distribution in the background material is approximatively of exponential type (and thus Markovian) if the volume fraction $f$ is not too large ; but the chord-length distribution for the spheres is not of the same type. For the sake of completeness, let us mention [19] for an example in 2D of isotropic binary mixture where the distribution of chords is Markovian for both materials (but this building is quite complicated).

Nevertheless, although the mixture we consider in our paper does not fullfill the assumptions of a Markovian medium, the two approaches give results which are not dramatically different (provided that the mean chord lengths are properly defined).

## 11.6 Conclusion

In the framework of a stochastic binary mixture with spherical heterogeneities, we have proposed an analytical formula for the fraction $P(f)$ of suprathermal particles stopped in the light material as a function of the size of heterogeneities, the stopping distances in each material and the volume fraction $f$. We addressed the issue of the influence of the details of the distribution of heterogeneities : the main conclusion is that it is not of primal importance to know these details since we are only interested in results on the mean fraction of particles stopped in $\mathcal{G}^c$. For this purpose, the analytical formula is obtained on one hand by direct numerical simulations on a large number of realistic mixtures and on the other hand by interpolation between asymptotic results obtained in idealized situations.

## Appendix 1 : Description of the algorithm for the numerical simulations.

In practice, we dont start from a given volume fraction to construct a heterogeneous medium. In fact, we first construct the heterogeneous medium and afterwards we evaluate the volume fraction and the probability to stop in the light medium for this given configuration. Thus to build the binary mixture, we fix a value for the sphere radius and we simply sample the location of the centers of the spheres according to the above Poisson law. Once, the geometry is given, a classical Monte-Carlo

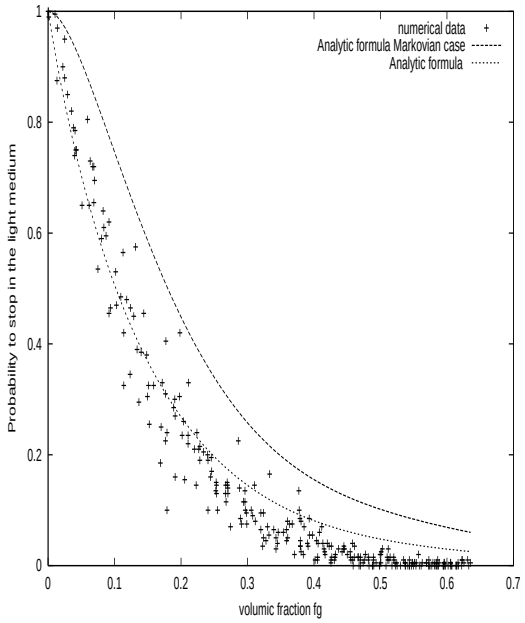


FIGURE 11.6 – Comparison between numerical simulations and the formulas $P_{\text{fit}}$ (doted line) and $P_{\text{lev}}$ (dashed line) in the case $rk_g = 1$ and $rk_l = 0.1$

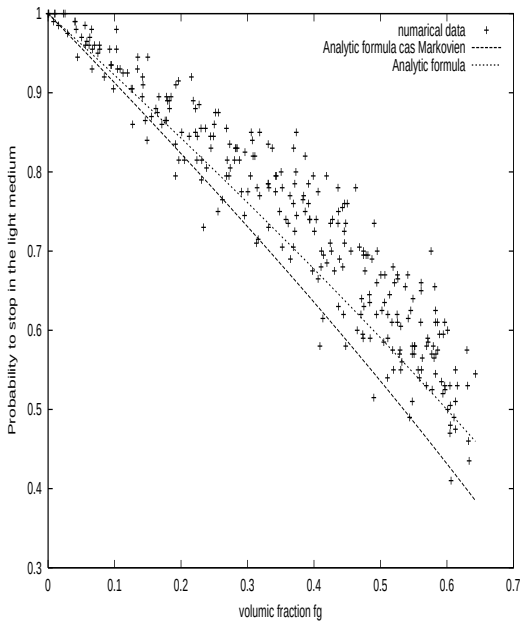


FIGURE 11.7 – Idem, in the case $rk_g = 1$ and $rk_l = 1$.

method is used to solve the transport equation (11.1) (see for example [8]). So, we have to perform two steps.

**Step 1 : Initialization.** We sample the random variables

– the initial position of the particles

– the direction of the velocity of the particles.

The initial position must be in $\mathcal{G}^c$. So we sample it uniformly in the torus and if it is not in $\mathcal{G}^c$, we have to sample out another point until we find a point in the light medium $\mathcal{G}^c$. Notice that since we apply this to a large number of points, this step allows us to estimate numerically the volume fraction $f$ of the heavy medium.

**Step 2 : Iterations to compute the stopping point.**

Recall that we use formula (11.5) and (11.6) to compute the trajectory of the particle. The numerical computation can be reduced to compute the position of $(Y^i; Y^{i,s})$.

At each iteration $i$ corresponding to $(Y^{i-1}; Y^{i,s})$, we compute the remaining potential $H^i$ by substracting to $H^{i-1}$ the distance needed to reach $Y^{i,s}$ multiplied by the slowing coefficient corresponding to the medium where the particle moves. We iterate the computation as long as the remaining potential is strictly positive. When the potential becomes negative, it means that the particle stopped before reaching $Y^{i,s}$. Therefore, for any configuration of the binary mixture, we may estimate the probability $P$ to stop in the light medium by using the formula (11.2).

# Appendix 2 : Proof of proposition 1.

**Step i).** Let $\mathbf{y}$ be an arbitrary fixed point and $\Omega$ an arbitrary unit vector. Since the Boolean medium $\mathcal{G}$ is isotropic, one may define a function $M$ on $\mathbf{R}^+$ by the following relation

$$M(z) = \mathbb{P}[\mathbf{y} \in \mathcal{G},\, \mathbf{y} + z\Omega \in \mathcal{G}^c]$$

($-M(z)$ is the correlation function of the Boolean medium up to an additive constant). We have $(\mathbf{o} \in \mathcal{G}) \cap (\mathbf{o} + z\Omega \in \mathcal{G}^c) = [(\mathbf{o} \in \mathcal{G}) \cup (\mathbf{o} + z\Omega \in \mathcal{G})] - [\mathbf{o} + z\Omega \in \mathcal{G}]$. Then, according to lemma 1, we get

$$\begin{aligned} f + M(z) = \mathbb{P}[(\mathbf{o} \in \mathcal{G}) \cup (\mathbf{o} + z\Omega \in \mathcal{G})] &= \mathbb{P}[\mathcal{G} \cap \{\mathbf{o}, \mathbf{o} + z\Omega\} \neq \emptyset] \\ &= 1 - \exp\left(-\lambda |S_{\mathbf{o}} \oplus \{\mathbf{o}, \mathbf{o} + z\Omega\}|\right) \end{aligned}$$

But $|S_{\mathbf{o}} \oplus \{\mathbf{o}, \mathbf{o} + z\Omega\}| = 2|S_{\mathbf{o}}| - |S_{\mathbf{o}} \cap S_{\mathbf{o}+z\Omega}|$ and after some tedious calculus it may be found that $|S_{\mathbf{o}} \cap S_{\mathbf{o}+z\Omega}| = |S_{\mathbf{o}}| - \pi r^2 (z - z^3 r^{-2}/12)$ (for $z$ smaller than $2r$). Therefore, we get

$$f + M(z) = 1 - (1 - f) \exp\left(-\lambda \pi r^2 (z - \frac{z^3}{12r^2})\right),$$

$$\frac{\partial M}{\partial z}(0) = (1 - f)\lambda \pi r^2. \tag{11.14}$$

**Step ii).** Following [21], introduce also the lineal-path function $L_c(z)$ which is the probability that a line segment $[\mathbf{y}, \mathbf{y}+z\Omega]$ lies in $\mathcal{G}$ (it is independent from the initial point $\mathbf{y}$). For small values of $z$, we can easily see that $L_c(z) = \mathbb{P}[\mathbf{y} \in \mathcal{G}, \mathbf{y} + z\Omega \in \mathcal{G}] + o(z)$. Then we have

$$\mathbb{P}[\mathbf{y} \in \mathcal{G}] = f = M(z) + L_c(z) + o(z)$$

Therefore, by deriving this relation with respect to $z$, we get

$$\frac{\partial L_*}{\partial z}(0) + \frac{\partial M}{\partial z}(0) = 0. \tag{11.15}$$

**Step iii).** Let $p_c(z)$ the probability law density of the chord length $l_g$, then

$$\mathbb{E}(l_g) = \int z p_c(z) dz.$$

Let us now show that the probability that the point $\mathbf{y}$ belongs to $\mathcal{G}$ and that the line segment $[\mathbf{y}, \mathbf{y}+z\Omega]$ is included in a chord of $\mathcal{G}$ is equal to

$$L_c(z) = f\frac{\int_z^\infty (x-z)p_c(x)dx}{\mathbb{E}(l_g)}. \tag{11.16}$$

see [21]. Indeed, on a arbitrary line, the probability that a fixed point $\mathbf{y}$ on this line belongs to a chord $l_g$ of length in $[x, x+dx]$ is equal to $xp_c(x)dx/\mathbb{E}(l_g)$. Moreover, the probability that a fixed line segment $[\mathbf{y}, \mathbf{y}+z\Omega]$ is included in a chord of length $x$, knowing that $\mathbf{y}$ belongs to this chord is equal to $\frac{x-z}{x}\mathbf{1}_{z<x}$. Then, the probability that the line segment $[\mathbf{y}, \mathbf{y}+z\Omega]$ is included in a chord $l_g$ knowing that $\mathbf{y}$ belongs to this chord is $\int_z^\infty (x-z)p_c(x)dx/\mathbb{E}(l_g)$ and (11.16) follows.

Since we have $\frac{\partial L_*}{\partial z}(0) = -f/\mathbb{E}(l_g)$,according to (11.15) and (11.14) we get

$$\frac{f}{\mathbb{E}(l_g)} = (1-f)\lambda\pi r^2. \tag{11.17}$$

So, denoting $S(\mathcal{G}) = (1-f)\lambda 4\pi r^2$, the first part of the theorem follows. It suffices to reverse the part of $\mathcal{G}$ and $\mathcal{G}^c$ in steps ii) and iii) above in order to end the proof of the theorem.

Notice that one may check that the quantity $S(\mathcal{G})\mathcal{L}^3$ is the expected value of the interface area $\partial\mathcal{G}$. Moreover

$$\mathbb{E}(l_g) = f\left((1-f)\lambda|S_{\mathbf{o}}|\right)^{-1}\frac{4r}{3} = \frac{4r}{3}\frac{f}{(1-f)|\log(1-f)|} \simeq \frac{4r}{3}$$

for $f$ small. Therefore we find the classical result that for a isolated sphere of radius $r$, the mean value of the chord length is equal to $r\frac{4}{3}$.

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# Chapitre 12

# Homogenization of a One-Dimensional Spectral Problem for a Singularly Perturbed Elliptic Operator with Neumann Boundary Conditions

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**abstract**

We study the asymptotic behavior of the first eigenvalue and eigenfunction of a one-dimensional periodic elliptic operator with Neumann boundary conditions. The second order elliptic equation is not self-adjoint and is singularly perturbed since, denoting by $\varepsilon$ the period, each derivative is scaled by an $\varepsilon$ factor. The main difficulty is that the domain size is not an integer multiple of the period. According to the size of the fractional part of the periodic cell cut by the domain boundary, several asymptotic behaviors are obtained, including an homogenization limit or an exponential localization on one of the extreme points of the domain.

## 12.1 Introduction

This paper is devoted to the homogenization of a spectral problem for a singularly perturbed elliptic equation in a one-dimensional periodic medium with Neumann boundary conditions. Without loss of generality we consider a bounded domain $\Omega = (0,1)$ and we denote by $\varepsilon > 0$ its period, or rather the period of the coefficients of the equation posed in $\Omega$. Although we shall sometime use the notations $\nabla$ and div for the gradient and the divergence operators, they simply mean derivation with respect to the single spatial variable. We study the following eigenvalue problem

$$\begin{cases} -\varepsilon^2 \mathrm{div}\left(a\left(\frac{x}{\varepsilon}\right)\nabla u^\varepsilon\right) + \varepsilon b\left(\frac{x}{\varepsilon}\right)\nabla u^\varepsilon + c\left(\frac{x}{\varepsilon}\right)u^\varepsilon = \lambda^\varepsilon \rho\left(\frac{x}{\varepsilon}\right)u^\varepsilon \text{ in } \Omega, \\ a\left(\frac{x}{\varepsilon}\right)\nabla u^\varepsilon = 0 \text{ on } \partial\Omega. \end{cases} \tag{12.1}$$

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We assume that $a, b, c$ and $\rho$ are continuous periodic functions of period one, defined in the unit cell $Y = [0,1]$. As usual $x$ denotes the macroscopic variable in $\Omega$, while $y$ is the microscopic variable in $Y$, and they are related by the scaling $y = x/\varepsilon$. We further assume that $a$ and $\rho$ are strictly positive, more precisely there exists a positive constant $C$ such that

$$\forall y \in Y, \quad 0 < C < a(y) < C^{-1}, \quad 0 < C < \rho(y) < C^{-1}.$$

By the Krein-Rutman theorem there exists, at least, a first eigenvalue and eigenvector of (12.1) that we shall denote by $\lambda^\varepsilon$ and $u^\varepsilon$. Furthermore, $\lambda^\varepsilon$ is real, simple and the smallest in modulus of all other eigenvalues, and $u^\varepsilon$ can be chosen to be positive in $\Omega$ and is thus unique if it is normalized, say by the choice of $u^\varepsilon(0)$. Since (12.1) is actually an ordinary differential equation in one space dimension, the eigenfunction $u^\varepsilon$ belongs at least to $C^1(\overline{\Omega})$.

We study the asymptotic behavior of the smallest eigenpair $(\lambda^\varepsilon, u^\varepsilon)$, when $\varepsilon$ tends to zero. In contrast to the case of Dirichlet boundary conditions, studied in [7], the behavior of the first eigencouple depends on the fractional part of $1/\varepsilon$. Furthermore, new asymptotic regimes, corresponding to an exponential localization of the first eigenfunction at one of the extreme points of the domain, are obtained for some values of this fractional part. Nevertheless, for other values of the fractional part we still obtain an homogenized limit as was always the case for Dirichlet boundary conditions. Our main results are Theorems 12.2.1 and 12.2.2 below. We therefore choose the sequence $\varepsilon \equiv \varepsilon_n$ to be of the form

$$\varepsilon = \frac{1}{n+\delta}, \tag{12.2}$$

where $n$ is an integer and $0 \leq \delta < 1$ is a constant which is the rescaled size of the fractional part of the extremal periodic cell cut by the right domain boundary. In the sequel, when $\varepsilon \equiv \varepsilon_n$ is said to go to 0, we mean that $n$ goes to infinity with $\delta$ fixed.

The special case $\delta = 0$, corresponding to an entire number of cells in the domain, is already known. It already appears in [14] for a similar system of two elliptic equations. In this later case, the proof is a little more involved and uses an exponential change of unknowns together with a viscosity solution approach to the resulting Hamilton-Jacobi equation. In the case of (12.1) a simpler proof is available for the following proposition.

*Proposition* 12.1.1. Assume that $\delta = 0$ in (12.2). Let $(\lambda_N, u_N)$ be the first eigenpair of the following Neumann cell problem

$$\begin{cases} -\mathrm{div}_y\left(a(y)\nabla_y u_N\right) + b(y)\nabla_y u_N + c(y)u_N = \lambda_N \rho(y) u_N \text{ in } Y, \\ \qquad\qquad a(0)\nabla_y u_N(0) = a(1)\nabla_y u_N(1) = 0 \\ \qquad\qquad u_N(0) = 1. \end{cases} \tag{12.3}$$

Define $\theta_N = \log\left(u_N(1)\right)$. Then, the function $w_N(y) = e^{-\theta_N y} u_N(y)$ is 1-periodic and the first eigenpair of (12.1) is exactly given by

$$\lambda^\varepsilon = \lambda_N, \qquad u_\varepsilon(x) = e^{\frac{\theta_N x}{\varepsilon}} w_N\left(\frac{x}{\varepsilon}\right).$$

*Démonstration.* By the Krein-Rutman theorem $u_N$ is positive, therefore $\theta_N$ is well defined, and thus we can define a 1-periodic function $w_N = e^{-\theta_N y} u_N(y)$ on each period. Clearly, $e^{\frac{\theta_N x}{\varepsilon}} w_N\left(\frac{x}{\varepsilon}\right)$ is a positive $C^1$ solution of (12.1) for the eigenvalue $\lambda_N$. Another application of the Krein-Rutman theorem, which implies that a positive eigenfunction can happen only for the first eigenvalue, yields that $\lambda_N$ is indeed the smallest eigenvalue $\lambda^\varepsilon$ and then $u_\varepsilon(x) = e^{\frac{\theta_N x}{\varepsilon}} w_N\left(\frac{x}{\varepsilon}\right)$. □

The fact that we can get an explicit and exact formula (in terms of $\varepsilon$) for the solution of (12.1) is quite special to this case (even though it sometimes happens when $\delta \neq 0$). Nevertheless this example shows that Neumann cell eigenvalue problems are key to the problem, and that the solutions could be of exponential-periodic type.

## 12.2 Main results

Before we can state our main results, Theorems 12.2.1 and 12.2.2, we need to introduce some notations and auxiliary problems. Since the case $\delta = 0$ is already covered by Proposition 12.1.1, we assume from now on that $0 < \delta < 1$ in (12.2). Instead of the single Neumann cell problem (12.3) there are now two such cell problems to consider, each of them corresponding to one endpoint of the domain $\Omega$. For $t \in [0,1]$ let us introduce the following Neumann cell problem on the shifted cell $(t-1,t)$ : we call $(u_N^t, \lambda_N^t)$, the first eigenpair of

$$\begin{cases} -\mathrm{div}_y\left(a(y)\nabla_y u_N^t\right) + b(y)\nabla_y u_N^t + c(y)u_N^t = \lambda_N^t \rho(y) u_N^t \text{ in } (t-1,t), \\ \qquad\qquad a(t)\nabla_y u_N^t(t) = a(t-1)\nabla_y u_N^t(t-1) = 0, \end{cases} \tag{12.4}$$

normalized by $u_N^t(t-1) = 1$. Another application of the Krein-Rutman theorem shows that there exists a first eigenvalue $\lambda_N^t$ (which is real, simple and the smallest in modulus of all other eigenvalues) and a corresponding eigenvector $u_N^t$ which can be chosen to be positive in $Y$. Only two values of the parameter $t$ matter : $t = 0$ for the left end point $x = 0$ and $t = \delta$ for the right end point $x = 1$ of $\Omega$.

### 12.2.1 Exponential-periodic cell problems

We shall recognize (see Lemma 12.2.1 below) that the auxiliary problem (12.4) is actually equivalent to the well-known exponential-periodic cell problem (or shifted cell problem) introduced in [2, 6, 7, 14]. These spectral cell problems are key ingredients in the homogenization of (12.1). Following the lead of [2, 6, 7, 14], for each $\theta \in \mathbb{R}$ we introduce an exponential-periodic cell problem which reads

$$\begin{cases} -\mathrm{div}_y(a(y)\nabla_y\psi_\theta) + b(y)\nabla_y\psi_\theta + c(y)\psi_\theta = \lambda_\theta \rho(y)\psi_\theta \quad \text{in } Y, \\ y \to e^{-\theta y}\psi_\theta(y) \quad Y\text{-periodic}, \end{cases} \tag{12.5}$$

together with its associated adjoint problem, with respect to the $L^2(Y)$ scalar product,

$$\begin{cases} -\mathrm{div}_y(a(y)\nabla_y\psi_\theta^*) - b(y)\nabla_y\psi_\theta^* + (c(y) - \mathrm{div}_y b(y))\psi_\theta^* = \lambda_\theta \rho(y)\psi_\theta^* \quad \text{in } Y, \\ y \to e^{\theta y}\psi_\theta^*(y) \quad Y\text{-periodic}. \end{cases} \tag{12.6}$$

In the above equations (12.5) and (12.6) $\lambda_\theta$ stands for the first eigenvalue and $\psi_\theta, \psi_\theta^*$ for the first eigenfunctions, which exist and are real-valued by virtue, once again, of the Krein-Rutman theorem. It also implies that $\lambda_\theta$ is of algebraic and geometric multiplicity one, that we can impose $\psi > 0$, $\psi^* > 0$ in $Y$ and that there are the only eigenfunctions which are positive. Of course, since (12.5), (12.6) and also (12.4) are just ordinary differential equations, their solutions belong at least to $C^1(Y)$. We choose the following normalization : $\psi_\theta(0) = 1 = \psi_\theta^*(0)$. We recall some properties of these problems, established in [2, 6, 7, 14].

*Proposition* 12.2.1. The following properties hold true.

- The map $\theta \to \lambda_\theta$ is strictly concave, and $\lim_{\theta\to\pm\infty} \lambda_\theta = -\infty$.
- At the unique $\theta_\infty$ such that $\lambda_\theta$ is maximal, the normalized eigenvectors $\psi_\infty \equiv \psi_{\theta_\infty}$ and $\psi_\infty^* \equiv \psi_{\theta_\infty}^*$ satisfy

$$a(y)\left(\psi_\infty \nabla_y \psi_\infty^*(y) - \psi_\infty^* \nabla_y \psi_\infty(y)\right) + b(y)\psi_\infty^*(y)\psi_\infty(y) \equiv 0 \text{ in } Y. \tag{12.7}$$

- For each $y \in Y$, the map $\theta \to \frac{1}{\psi_\theta(y)}\nabla_y\psi_\theta(y)$ is strictly increasing and one-to-one from $\mathbb{R}$ to $\mathbb{R}$.
- The maximizer $\theta_\infty$ satisfies

$$\theta_\infty = \int_0^1 \frac{b(y)}{2a(y)} dy. \tag{12.8}$$

*Démonstration.* We only prove the last point whose proof is not included in references [2, 6, 7, 14]. By dividing (12.7) by $\psi_\infty\psi_\infty^*$, we obtain

$$-\theta_\infty + \nabla\log(\tilde{\psi}_\infty^*) - \theta_\infty - \nabla\log(\tilde{\psi}_\infty) + \frac{b}{a} = 0$$

where $\tilde{\psi}_\infty(y) = e^{-\theta_\infty y}\psi_\infty(y)$ and $\tilde{\psi}^*_\infty(y) = e^{\theta_\infty y}\psi^*_\infty(y)$ are $Y$-periodic functions. Integrating with respect to $y$, we obtain (12.8). □

Actually the solution $u^t_N$ of (12.4) is an exponential periodic function as shown by the following result.

*Lemme* 12.2.1. For each $t \in [0,1]$ there exists $\theta^t_N \in \mathbf{R}$ such that the solution $u^t_N$ of (12.4) satisfies $u^t_N(y) = e^{\theta^t_N y} w^t_N(y)$ where $w^t_N$ is a 1-periodic function.

*Démonstration.* We define the constant $\theta^t_N = \log\left(u^t_N(t)\right)$. It is then easy to check that the function $w^t_N(y) = e^{-\theta^t_N y} u^t_N(y)$ is 1-periodic. □

Lemma 12.2.1 shows that the solution $u^t_N(y)$ of (12.4) coincides with that of (12.5), $\psi_{\theta^t_N}(y)/\psi_{\theta^t_N}(t-1)$, with the same eigenvalue $\lambda^t_N = \lambda_{\theta^t_N}$. In particular, it allows us to extend the function $u^t_N$ to the whole $\mathbf{R}$ although it is originally defined only in $(t-1,t)$. Depending on the respective positions of $\theta^0_N$ and $\theta^\delta_N$ with respect to $\theta_\infty$, we will exhibit the different behaviors of the sequence $u_\varepsilon$ when $\varepsilon$ goes to zero.

### 12.2.2 Convergence

In this subsection, Theorems 12.2.1 and 12.2.2 describe completely all possible asymptotic regimes of the spectral problem (12.1) using the auxiliary spectral problems (12.4) and (12.5). However we start with a special case, similar to Proposition 12.1.1, which is simpler than the general case that will follow. This special case occurs when the solutions $u^0_N$ and $u^\delta_N$ of (12.4), for $t = 0$ and $t = \delta$ respectively, are equal (up to a multiplicative factor).

*Proposition* 12.2.2. If the solutions $u^0_N$ and $u^\delta_N$ of (12.4) satisfies $u^0_N(y) = \frac{u^\delta_N(y)}{u^\delta_N(-1)}$, then the first eigenpair of (12.1) is exactly given by

$$\lambda^\varepsilon = \lambda^0_N, \qquad u_\varepsilon(x) = e^{\frac{\theta^0_N x}{\varepsilon}} \frac{w^0_N\left(\frac{x}{\varepsilon}\right)}{w^0_N(0)},$$

where the function $w^0_N(y) = e^{-\theta^0_N y} u^0_N(y)$, with $\theta^0_N = \log\left(u^0_N(0)\right)$, is the 1-periodic function defined in Lemma 12.2.1.

The proof of Proposition 12.2.2 is given in Proposition 12.6.1.

When Proposition 12.2.2 does not apply, i.e., when $u^0_N(y) \neq \frac{u^\delta_N(y)}{u^\delta_N(-1)}$, the asymptotic behavior of $u_\varepsilon$ can be of different nature. In some cases, described in Proposition 12.2.2, the solution of (12.1) concentrates on the boundaries of the domain.

**Theorem 12.2.1.** *The first eigenpair of (12.1) is localized on one of the end points of $\Omega$ in the following two cases.*

- *For $\left(\theta^0_N < \theta_\infty \text{ and } \theta^\delta_N \leq \theta_\infty\right)$ or for $\left(\theta^0_N < \theta_\infty < \theta^\delta_N \text{ and } \lambda^0_N < \lambda^\delta_N\right)$, then*

$$\left|\lambda^\varepsilon - \lambda^0_N\right| = \gamma_0 e^{2(\theta^0_N - \theta_\infty)/\varepsilon}(1 + o(1)),$$

*and*

$$\left\| u^\varepsilon(x) - u^\varepsilon(0) e^{\frac{\theta^0_N x}{\varepsilon}} \frac{w^0_N\left(\frac{x}{\varepsilon}\right)}{w^0_N(0)} \right\|_{L^\infty(\Omega)} \leq \frac{C}{\varepsilon} e^{(\theta^0_N - \theta_\infty)/\varepsilon} \|u^\varepsilon\|_{L^1(\Omega)},$$

*where $\gamma_0$ is a positive constant defined in Proposition 12.6.2, independent of $\varepsilon$.*

- *For $\left(\theta^\delta_N > \theta_\infty \text{ and } \theta^0_N \geq \theta_\infty\right)$ or for $\left(\theta^0_N < \theta_\infty < \theta^\delta_N \text{ and } \lambda^0_N > \lambda^\delta_N\right)$, then*

$$\left|\lambda^\varepsilon - \lambda^\delta_N\right| = \gamma_1 e^{2(\theta_\infty - \theta^\delta_N)/\varepsilon}(1 + o(1)),$$

*and*

$$\left\| u^\varepsilon(x) - u^\varepsilon(1) \frac{e^{\frac{\theta^\delta_N x}{\varepsilon}} w^\delta_N\left(\frac{x}{\varepsilon}\right)}{e^{\frac{\theta^\delta_N}{\varepsilon}} w^\delta_N(\delta)} \right\|_{L^\infty(\Omega)} \leq \frac{C}{\varepsilon} e^{(\theta_\infty - \theta^\delta_N)/\varepsilon} \|u^\varepsilon\|_{L^1(\Omega)},$$

*where $\gamma_1$ is a positive constant defined in Proposition 12.6.2, independent of $\varepsilon$.*
*The first eigenpair of (12.1) localizes at one or two end points of $\Omega$ in the following third case.*

– *For $\left(\theta_N^0 < \theta_\infty < \theta_N^\delta \text{ and } \lambda_N^\delta = \lambda_N^0\right)$, that is $\theta_N^\delta - \theta_\infty = \theta_\infty - \theta_N^0 > 0$, then*

$$\lambda^\varepsilon - \lambda_N^0 = -\gamma_\delta e^{(\theta_N^0 - \theta_\infty)/\varepsilon}(1 + o(1)),$$

*and*

$$\left\| u^\varepsilon(x) - u^\varepsilon(0) e^{\frac{\theta_N^0 x}{\varepsilon}} \frac{w_N^0\left(\frac{x}{\varepsilon}\right)}{w_N^0(0)} - u^\varepsilon(0) c_\delta e^{\frac{\theta_N^0 - \theta_\infty}{\varepsilon}} \frac{\psi_\infty(\delta-1) e^{\frac{\theta_N^\delta x}{\varepsilon}} w_N^\delta\left(\frac{x}{\varepsilon}\right)}{\psi_\infty(-1) w_N^0(0)} \right\|_{L^\infty(\Omega)} \leq \frac{C}{\varepsilon} e^{(\theta_N^0 - \theta_\infty)/\varepsilon} \|u^\varepsilon\|_{L^1(\Omega)},$$

*where $\gamma_\delta > 0$ and $c_\delta$ are constants defined in Proposition 12.6.2, independent of $\varepsilon$.*

*Remarque* 12.2.1. Throughout this paper, $C$ denotes a positive constant independent of $\varepsilon$.

*Remarque* 12.2.2. The right hand sides of all estimates in Theorem 12.2.1 are exponentially small with respect to $\varepsilon$. In the two first cases, the eigenfunction $u_\varepsilon$ is approximately the product of a periodic function and a scaled exponential, which clearly exhibits a localization effect on one and only one end point of $\Omega$ (at least when $\theta_N^0$ and $\theta_N^\delta$, respectively, are not equal to zero). The precise end point of $\Omega$ where localization occurs is deduced from the sign of $\theta_N^0$ or $\theta_N^\delta$, respectively. In the third case, the eigenfunction $u_\varepsilon$ localizes on one endpoint of $\Omega$ if $\theta_\infty \neq 0$ and on the two end points in the special case $\theta_\infty = 0$. Indeed, around $x = 0$, the ansatz says

$$u^\varepsilon(x) \approx u^\varepsilon(0) e^{\frac{\theta_N^0 x}{\varepsilon}} \frac{w_N^0\left(\frac{x}{\varepsilon}\right)}{w_N^0(0)},$$

whereas around $x = 1$, we use the following equivalent form of the ansatz

$$\begin{aligned} & u^\varepsilon(x) - u^\varepsilon(0) e^{\frac{\theta_N^0 x}{\varepsilon}} \frac{w_N^0\left(\frac{x}{\varepsilon}\right)}{w_N^0(0)} - u^\varepsilon(0) c_\delta e^{\frac{\theta_N^0 - \theta_\infty}{\varepsilon}} \frac{\psi_\infty(\delta-1) e^{\frac{\theta_N^\delta x}{\varepsilon}} w_N^\delta\left(\frac{x}{\varepsilon}\right)}{\psi_\infty(-1) w_N^0(0)} \\ = \quad & u^\varepsilon(x) - u^\varepsilon(0) e^{\frac{\theta_N^0 x}{\varepsilon}} \frac{w_N^0\left(\frac{x}{\varepsilon}\right)}{w_N^0(0)} - u^\varepsilon(0) c_\delta e^{\frac{\theta_\infty}{\varepsilon}} e^{\frac{\theta_N^\delta (x-1)}{\varepsilon}} \frac{\psi_\infty(\delta-1) w_N^\delta\left(\frac{x}{\varepsilon}\right)}{\psi_\infty(-1) w_N^0(0)}, \end{aligned}$$

which implies

$$u^\varepsilon(x) \approx u^\varepsilon(0) c_\delta e^{\frac{\theta_\infty}{\varepsilon}} e^{\frac{\theta_N^\delta (x-1)}{\varepsilon}} \frac{\psi_\infty(\delta-1) w_N^\delta\left(\frac{x}{\varepsilon}\right)}{\psi_\infty(-1) w_N^0(0)}.$$

Therefore, the localisation is determined by the drift factor $\theta_\infty$. If $\theta_\infty < 0$, the localization is in $x = 0$, and if $\theta_\infty > 0$ the localization occurs in $x = 1$. In the special case where $\theta_\infty = 0$ which includes the self adjoint case (see Proposition 12.2.1), a double localization occurs, as the solution localizes at both endpoints.

*Démonstration.* It is a consequence of Corollary 12.6.1 which is expressed in terms of $\phi^\varepsilon(x)$, a factorized solution defined by the relation $u^\varepsilon(x) = \psi_\infty\left(\frac{x}{\varepsilon}\right)\phi^\varepsilon(x)$, of the factorized cell eigenfunctions $\tilde{\varphi}_\theta^t(y) = e^{-\theta y}\varphi_\theta^t(y)$ where $\varphi_\theta^t$ is the first eigenfunction of (12.16) and of the factorized Neumann solutions $\phi_t(y)$ given by (12.17). Introducing the correspondences that, on one hand,

$$\begin{gathered} \frac{\phi_0\left(\frac{x}{\varepsilon}\right)}{\phi_0(0)} = e^{\frac{\theta_0 x}{\varepsilon}} \frac{\tilde{\varphi}_{\theta_0}^0\left(\frac{x}{\varepsilon}\right)}{\tilde{\varphi}_{\theta_0}^0(0)} = \frac{u_N^0\left(\frac{x}{\varepsilon}\right)}{\psi_\infty\left(\frac{x}{\varepsilon}\right) u_N^0(0)} = \frac{e^{\frac{\theta_N^0 x}{\varepsilon}} w_N^0\left(\frac{x}{\varepsilon}\right)}{\psi_\infty\left(\frac{x}{\varepsilon}\right) w_N^0(0)}, \\ \theta_0 = \theta_N^0 - \theta_\infty, \quad u^\varepsilon(0) = \phi^\varepsilon(0), \quad \mu_0 = \lambda_N^0 - \lambda_\infty, \end{gathered} \tag{12.9}$$

and on the other hand

$$\begin{gathered} \frac{\phi_\delta\left(\frac{x}{\varepsilon}\right)}{\phi_\delta\left(\frac{1}{\varepsilon}\right)} = \frac{e^{\frac{\theta_\delta x}{\varepsilon}} \tilde{\varphi}_{\theta_\delta}^\delta\left(\frac{x}{\varepsilon}\right)}{e^{\frac{\theta_\delta(1)}{\varepsilon}} \tilde{\varphi}_{\theta_\delta}^\delta(\delta)} = \frac{\psi_\infty\left(\frac{1}{\varepsilon}\right) u_N^\delta\left(\frac{x}{\varepsilon}\right)}{\psi_\infty\left(\frac{x}{\varepsilon}\right) u_N^\delta\left(\frac{1}{\varepsilon}\right)} = \frac{\psi_\infty\left(\frac{1}{\varepsilon}\right) e^{\frac{\theta_N^\delta x}{\varepsilon}} w_N^\delta\left(\frac{x}{\varepsilon}\right)}{\psi_\infty\left(\frac{x}{\varepsilon}\right) e^{\frac{\theta_N^\delta}{\varepsilon}} w_N^\delta(\delta)}, \\ \theta_\delta = \theta_N^\delta - \theta_\infty, \quad u^\varepsilon(1) = \psi_\infty\left(\frac{1}{\varepsilon}\right)\phi^\varepsilon(1), \quad \mu_\delta = \lambda_N^\delta - \lambda_\infty, \end{gathered} \tag{12.10}$$

as well as

$$\frac{\phi_\delta\left(\frac{x}{\varepsilon}\right)}{\phi_0(0)} = \frac{e^{\frac{\theta_\delta x}{\varepsilon}}\tilde{\varphi}^\delta_{\theta_\delta}\left(\frac{x}{\varepsilon}\right)}{\tilde{\varphi}^0_{\theta_0}(0)} = \frac{\psi_\infty(\delta-1)u^\delta_N\left(\frac{x}{\varepsilon}\right)}{\psi_\infty\left(\frac{x}{\varepsilon}\right)\psi_\infty(-1)u^0_N(0)} = \frac{\psi_\infty(\delta-1)e^{\frac{\theta^\delta_N x}{\varepsilon}}w^\delta_N\left(\frac{x}{\varepsilon}\right)}{\psi_\infty(-1)\psi_\infty\left(\frac{x}{\varepsilon}\right)w^0_N(0)}, \tag{12.11}$$

the statements in Theorem 12.2.1 are equivalent to those in Corollary 12.6.1. A more precise corrector result is stated in Proposition 12.6.2. □

The last case, $\theta^0_N \geq \theta_\infty$ and $\theta^\delta_N \leq \theta_\infty$, not covered by Theorem 12.2.1, corresponds to a homogenization regime. In such a case, the first eigensolution does not localize at the endpoints. Its precise asymptotic form is given by the following result.

**Theorem 12.2.2.** *For $\theta^0_N \geq \theta_\infty$ and $\theta^\delta_N \leq \theta_\infty$, the first eigenpair of (12.1) is of the form*

$$u^\varepsilon(x) \approx \psi_\infty\left(\frac{x}{\varepsilon}\right)u(x) \text{ and } \lambda_\varepsilon = \lambda_\infty + \varepsilon^2(\lambda^*_0 + o(1)),$$

*where $\psi_\infty$ is a periodic function and $(u, \lambda^*_0)$ is the first eigenpair of an homogenized problem*

$$\begin{cases} -d^*\Delta u & = & \lambda^*_0 s^* u \text{ in } \Omega, \\ u \in H^1(\Omega) & \text{and either} & u(0) = 0 \text{ or } u(1) = 0, \text{ or both.} \end{cases}$$

*where $d^*$ and $s^*$ are positive constants. (See Theorem 12.4.1 for a more precise statement and for the proof).*

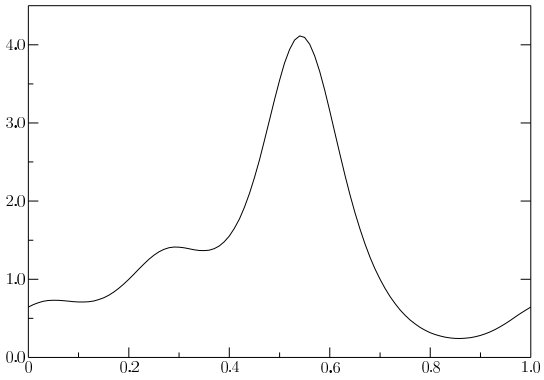


FIGURE 12.1 – The diffusion coefficient $a$ over $Y = (0, 1)$.

It is interesting to notice that, in the case of Dirichlet boundary conditions, Theorem 12.2.2 gives the only possible asymptotic behavior, for any $\varepsilon$, i.e., for any $\delta$, and in any space dimension (see [7]). Therefore, the case of Neumann boundary conditions is very sensitive to the precise geometry as is already well known in homogenization theory.

To illustrate our main results, we provide numerical examples of each possible asymptotic behavior described in Theorem 12.2.1 and 12.2.2. We will show in the next section that non-selfadjoint problems can be reduced to selfadjoint ones, thus we chose $b(y) = 0$ for our numerical tests. For simplicity we also take $\rho(y) = 1$. Not all possible behavior can be observed with only one pair of coefficient. We use two pairs $(a(y), c_1(y))$ and $(a(y), c_2(y))$, represented in Figure 12.1 and 12.2.

The coefficients (chosen very arbitrarily) are given by

$$\begin{aligned} \log\left(a\left(y - \frac{3}{10}\right)\right) &= -\sin(2\pi y) - \frac{1}{2}\sin(4\pi y) - \frac{1}{6}\sin(6\pi y) + \frac{1}{4}\sin(8\pi y), \\ \sqrt{c_1(y)} &= \exp\left(-\frac{c_2(y)^2}{4}\right) + \frac{1}{2}, \\ c_2(y) &= \sin(2\pi y) + \cos(4\pi y) + 3. \end{aligned}$$

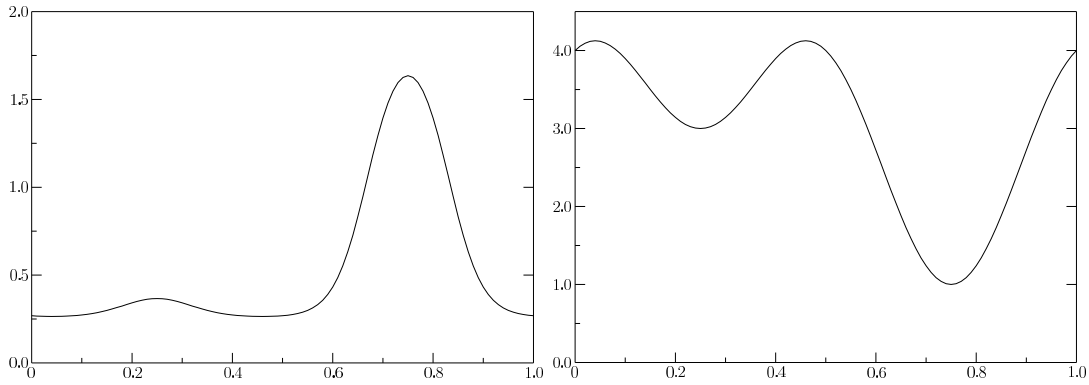


FIGURE 12.2 – The zero-order $c_1$ (left) and $c_2$ (right) over $Y = (0, 1)$.

In all three Figures 12.3, 12.4 and 12.5 we plot the first eigenfunction of (12.1) for $n = 30$ (dashed line) and $n = 70$ (solid line) to show the trend of convergence as $\varepsilon$ goes to zero. Figures 12.3 and 12.4 are obtained using the first pair $(a(y), c_1(y))$ and three different values of $\delta$, corresponding to the three configurations identified in Theorem 12.2.1. In particular the first eigenfunction converges pointwise to zero in the interior of the domain.

Figure 12.5 was obtained using the second pair $(a(y), c_2(y))$ and $\delta = 0.2$ : it illustrates the homogenization effect characterized in Theorem 12.2.2. In particular the values of the first eigenfunction at the two boundary points converge to zero.

Note that the influence of the $\delta$ parameter on the first-order corrector to the eigenvalue of a non singularly perturbed homogenization problem was already observed in [15], [12].

The purely periodic character of the coefficients in (12.1) is crucial for our results to hold true. Actually, a completely different behavior can arise if the coefficients depend on the macroscopic variable $x$ too, namely localization inside $\Omega$ can appear [4], [5].

The content of our paper is the following. In the next section, by using a factorization principle (in the spirit of [16], [1, 2]) we reduce the original problem (12.1) to a selfadjoint one. It thus allows us to write a variational characterization of the first eigenvalue. Of course, this "miracle" is possible only in one space dimension. Then, Section 12.4 adresses the homogenization regime of Theorem 12.2.2. Section 12.5 is concerned with the exponential convergence of the eigenvalues in Theorem 12.2.1. Eventually Section 12.6 deals with the convergence and localization of the eigenfunctions.

## 12.3 Transformation into a self-adjoint problem

A remarkable feature of this eigenvalue problem is that it can be reformulated, after a suitable change of unknowns, as a self-adjoint problem with compact resolvent. Among the many advantages of working with self-adjoint problems, we shall use in the sequel the fact that the first eigenvalue is characterized as the minimizer of a Rayleigh quotient, and that the normalized eigenvectors span the space $L^2(\Omega)$. This change of unknowns will be made thanks to the exponential-periodic functions introduced in (12.6), as in [6, 7, 14].

### 12.3.1 Factorization

To transform the problem into a self-adjoint one, we perform a change of unknown and consider instead of $u^\varepsilon$ the function $\phi^\varepsilon$ defined by

$$\phi^\varepsilon(x) = \frac{u^\varepsilon(x)}{\psi_\infty\left(\frac{x}{\varepsilon}\right)} \tag{12.12}$$

where $\psi_\infty$ is the first cell eigenfunction defined in Proposition 12.2.1. Because $x \to \psi_\infty\left(\frac{x}{\varepsilon}\right)$ is a solution of the equation (with different boundary conditions) it was proved in [1, 2] that (12.12) is indeed a change of variable from $H^1(\Omega)$ to $H^1(\Omega)$.

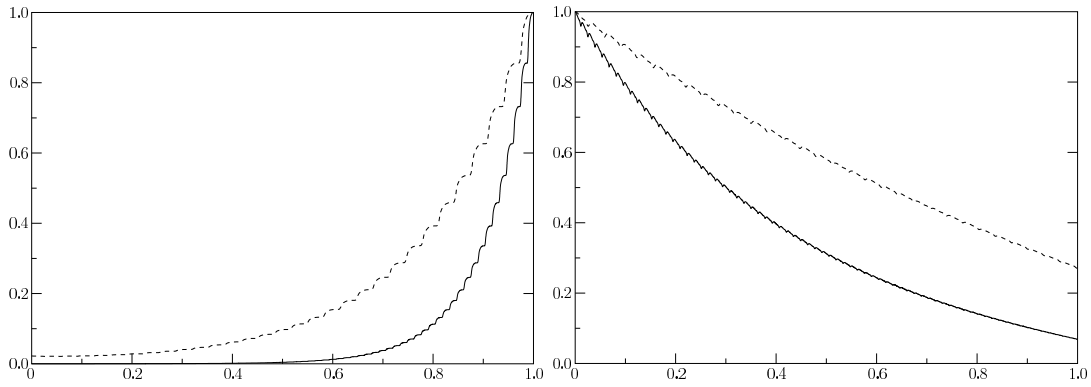


FIGURE 12.3 – Left : concentration at $x = 1$, for $\delta = 0.6$. Right : concentration at $x = 0$, for $\delta = 0.9$ ($n = 30$ dashed line, $n = 70$ solid line).

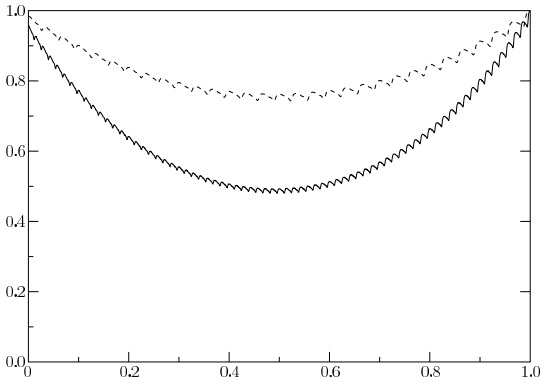


FIGURE 12.4 – Concentration at both end points, for $\delta = 0.2$ ($n = 30$ dashed line, $n = 70$ solid line).

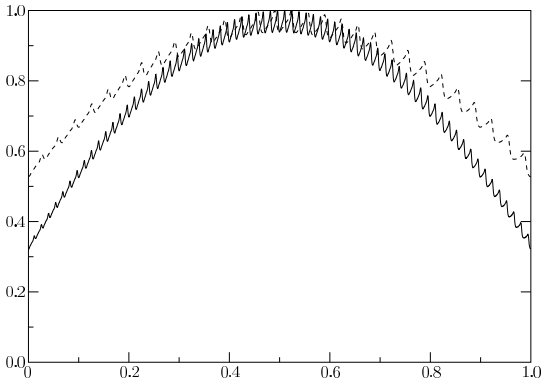


FIGURE 12.5 – The homogenization regime ($n = 30$ dashed line, $n = 70$ solid line).

*Proposition* 12.3.1. If $u^\varepsilon$ is a solution of the original problem (12.1), then the function $\phi^\varepsilon$, defined by (12.12), is an eigensolution for the following self-adjoint problem

$$\begin{cases} -\mathrm{div}(d\left(\frac{x}{\varepsilon}\right)\nabla\phi^\varepsilon) = \mu^\varepsilon s\left(\frac{x}{\varepsilon}\right)\phi^\varepsilon \text{ in } \Omega, \\ \\ d\left(\frac{x}{\varepsilon}\right)\nabla\phi^\varepsilon + \frac{1}{\varepsilon}\phi^\varepsilon m\left(\frac{x}{\varepsilon}\right) = 0 \quad \text{ on } \partial\Omega. \end{cases} \tag{12.13}$$

The new periodic coefficients are given by

$$d(y) = a(y)\psi_\infty(y)\psi^*_\infty(y), \quad s(y) = \rho(y)\psi_\infty(y)\psi^*_\infty(y), \quad m(y) = d(y)\frac{\nabla\psi_\infty(y)}{\psi_\infty(y)}$$

and the eigenvalues $\mu_\varepsilon$ are related to the ones of (12.1) by

$$\mu^\varepsilon = \frac{\lambda^\varepsilon - \lambda_\infty}{\varepsilon^2}.$$

*Remarque* 12.3.1. There are other transformations which map a non self-adjoint problem into a self-adjoint one in the theory of Hill's equation (see chapter III in [11]).

*Démonstration.* As in [1, 7, 10, 16], replacing $u^\varepsilon(x)$ by $\phi^\varepsilon(x)\psi_\infty\left(\frac{x}{\varepsilon}\right)$ in (12.1) gives

$$-\varepsilon^2\mathrm{div}\left(a\psi_\infty\nabla\phi^\varepsilon\right) - \varepsilon\mathrm{div}\left(a\phi^\varepsilon\nabla_y\psi_\infty\right) + \varepsilon b\psi_\infty\nabla\phi^\varepsilon + b\phi^\varepsilon\nabla_y\psi_\infty + c\psi_\infty\phi^\varepsilon = \lambda^\varepsilon\psi_\infty\phi^\varepsilon. \tag{12.14}$$

Using the fact that $\psi_\infty$ is solution of a cell problem, we note that

$$b\phi^\varepsilon\nabla_y\psi_\infty + c\psi_\infty\phi^\varepsilon = \lambda_\infty\psi_\infty\phi^\varepsilon - \mathrm{div}_y\left(a\nabla_y\psi_\infty\right)\phi^\varepsilon.$$

Therefore (12.14) becomes

$$-\varepsilon^2\mathrm{div}\left(a\psi_\infty\nabla\phi^\varepsilon\right) - \varepsilon a\nabla\phi^\varepsilon\nabla_y\psi_\infty + \varepsilon b\psi_\infty\nabla\phi^\varepsilon = (\lambda^\varepsilon - \lambda_\infty)\psi_\infty\phi^\varepsilon.$$

Multiplying this last identity by $\psi^*_\infty$, we obtain

$$-\varepsilon^2\psi^*_\infty\mathrm{div}\left(a\psi_\infty\nabla\phi^\varepsilon\right) - \varepsilon a\psi^*_\infty\nabla\phi^\varepsilon\nabla_y\psi_\infty + \varepsilon b\psi^*_\infty\psi_\infty\nabla\phi^\varepsilon = (\lambda^\varepsilon - \lambda_\infty)\psi^*_\infty\psi_\infty\phi^\varepsilon$$

which becomes

$$-\varepsilon^2\mathrm{div}(a\psi^*_\infty\psi_\infty\nabla\phi^\varepsilon) + \varepsilon\left(-a\psi^*_\infty\nabla_y\psi_\infty + a\psi_\infty\nabla_y\psi^*_\infty + b\psi^*_\infty\psi_\infty\right)\nabla\phi^\varepsilon = (\lambda^\varepsilon - \lambda_\infty)\psi^*_\infty\psi_\infty\phi^\varepsilon.$$

Thanks to (12.7), the first order term cancels, and we obtain (12.12). □

*Remarque* 12.3.2. Note that because of the regularity and positivity of $\psi_\infty$ and $\psi^*_\infty$ the coefficients $d$, $s$ and $m$ are continuous and satisfy, for some constant $C > 0$,

$$C < d(y) < C^{-1},\ C < s(y) < C^{-1} \text{ and } -C < m(y) < C \text{ for all } y \in Y.$$

The coefficients $d(y), s(y), m(y)$ are indeed $Y$-periodic functions. As $\psi_\infty(y) = \exp(\theta_\infty y)g_\infty(y)$, with $g_\infty$ $Y$-periodic, and $\psi^*_\infty(y) = \exp(-\theta_\infty y)g^*_\infty(y)$, with $g^*_\infty$ $Y$-periodic, we have $\psi_\infty\psi^*_\infty = g_\infty g^*_\infty$, and also

$$\frac{\nabla\psi_\infty(y)}{\psi_\infty(y)} = \theta_\infty + \frac{\nabla g_\infty(y)}{g_\infty(y)} = \frac{\nabla\psi_\infty(y+1)}{\psi_\infty(y+1)}.$$

*Remarque* 12.3.3. The above factorization principle can actually be applied in any space dimension. However it yields an additional convective term in equation (12.13) with a periodic velocity which is divergence free and has zero average. It is only in the one-dimensional case that it implies that the velocity is zero. This is the main reason why we restrict ourselves to a one-dimensional setting.

We have transformed a non-selfadjoint problem into a selfadjoint one, at the cost of changing the Neumann boundary condition into a Fourier or Robin boundary condition. Since we work in one space dimension, we did not write the unit external normal vector in the Fourier boundary condition which thus changes the usual sign convention for the boundary condition at the left end of the interval $\Omega$. Remark that (12.13) is still singularly perturbed because of the factor $\varepsilon^{-1}$ in the boundary condition. Nevertheless, this transformation enables us to characterize the first eigenpair as minimizers of a Rayleigh quotient.

*Proposition* 12.3.2. The first eigenvalue of problem (12.13) $\mu^\varepsilon$ is given by

$$\mu^\varepsilon = \min_{\phi \in H^1(\Omega)} \frac{\displaystyle\int_\Omega d\left(\frac{x}{\varepsilon}\right) |\nabla \phi|^2 \, dx + \frac{1}{\varepsilon}\left(m(\delta)\phi^2(1) - m(0)\phi^2(0)\right)}{\displaystyle\int_\Omega s\left(\frac{x}{\varepsilon}\right) \phi^2(x) \, dx}. \tag{12.15}$$

Furthermore, the minimum in (12.15) is achieved by any multiple of the first eigenfunction of (12.13).

The proof of Proposition 12.3.2 is obvious : simply note that, whatever the signs of $m(0)$ and $m(\delta)$, the boundary terms cause no problems in the coercivity, for fixed $\varepsilon$, of the Rayleigh quotient since, for any small $\kappa > 0$, there exists a constant $C_\kappa > 0$ such that

$$\phi^2(0) \leq \kappa \int_\Omega |\nabla \phi|^2 \, dx + C_\kappa \int_\Omega \phi^2(x) \, dx \quad \forall\, \phi \in H^1(\Omega).$$

### 12.3.2 Cell Problems

After the factorization (12.12) we can again introduce exponential-periodic cell problems, adapted to the new spectral problem (12.13). For each $\theta \in \mathbb{R}$, define $\varphi_\theta^t$ as the first eigenfunction of

$$\begin{cases} -\mathrm{div}(d(y)\nabla \varphi_\theta^t) = \nu_\theta s(y) \varphi_\theta^t \quad \text{in } Y, \\ \\ y \to e^{-\theta y} \varphi_\theta^t(y) \quad Y - \text{periodic}, \end{cases} \tag{12.16}$$

normalized by $\varphi_\theta^t(t-1) = 1$. Since (12.16) is self-adjoint, there is no need to introduce an adjoint problem. In the periodic case, i.e., $\theta = 0$, the explicit solution of (12.16) is $\nu_0 = 0$ and $\varphi_0 \equiv 1$.

In the same spirit, we can perform a factorization, similar to (12.12), for the solution $u_N^t$ of (12.4) and define

$$\phi_t(y) = \psi_\infty(t-1)\frac{u_N^t(y)}{\psi_\infty(y)} \quad \text{and } \mu_t = \lambda_N^t - \lambda_\infty. \tag{12.17}$$

Thus $\phi_t$ is the first eigenfunction of

$$\begin{cases} -\mathrm{div}_y(d(y)\nabla_y \phi_t) = \mu_t s(y) \phi_t \quad \text{in } (t-1, t), \\ \\ d(t-1)\nabla_y \phi_t(t-1) + m(t-1)\phi_t(t-1) = 0, \\ \\ d(t)\nabla_y \phi_t(t) + m(t)\phi_t(t) = 0, \end{cases} \tag{12.18}$$

normalized by $\phi_t(t-1) = 1$. Alternatively, (12.18) can be motivated by a formal study of the influence of the boundary condition in (12.13). As usual, the simplicity of the first eigenvalue as well as the uniqueness and positivity of the first normalized eigenfunctions of (12.16) and (12.18) follows from the Krein-Rutman theorem. The problems (12.16) and (12.18) play a role in the final result.

We now show that the eigenvalue problem (12.18) can be interpreted as an exponential-periodic problem.

*Proposition* 12.3.3. For each $t \in [0,1]$ there exists a unique $\theta_t \in \mathbf{R}$ such that $\varphi_{\theta_t}^t = \phi_t$ and $\nu_{\theta_t} = \mu_t$. The sign of $\theta_t$ is the opposite of that of $m(t)$. Furthermore, $\mu_t < 0$ if $m(t) \neq 0$.

As a consequence, if $m(0) > 0$ then there exists $\theta_0 < 0$ and $C > 0$ such that for all $x$,

$$0 < C < e^{-\theta_0 x}\phi_0(x) < \frac{1}{C} \text{ and } 0 > -C > \frac{\nabla \phi_0(x)}{\phi_0(x)} > -\frac{1}{C}.$$

If $m(\delta) < 0$ then there exists $\theta_\delta > 0$ and $C > 0$ such that for all $x$,

$$0 < C < e^{-\theta_\delta x}\phi_\delta(x) < \frac{1}{C} \text{ and } 0 < C < \frac{\nabla \phi_\delta(x)}{\phi_\delta(x)} < \frac{1}{C}.$$

*Démonstration.* Recall from Remark 12.3.2 that $d$ and $m$ are periodic continuous functions. On the same token, $y \to \frac{\nabla\varphi_\theta^t(y)}{\varphi_\theta^t(y)}$ is also $Y$-periodic. Thanks to Proposition 12.2.1 (which can also be applied to the spectral problem (12.16)) we know that there exists a unique $\theta_t$ such that $\frac{\nabla\varphi_{\theta_t}^t(t-1)}{\varphi_{\theta_t}^t(t-1)} = -\frac{m(t-1)}{d(t-1)} = -\frac{m(t)}{d(t)}$. Thus, $\varphi_{\theta_t}^t$ satisfies the boundary conditions of (12.18). Since $\varphi_{\theta_t}^t(t-1) = \phi_t(t-1) = 1$, the uniqueness of the positive normalized first eigenfunction of (12.18) implies that $\varphi_{\theta_t}^t \equiv \phi_t$.

Finally, note that the maximum of the map $\theta \to \nu_\theta$ is attained at $\theta = 0$, since the maximizer is characterized by (12.7), which is clearly satisfied for $\varphi_0 = \varphi_0^* = 1$. Therefore, for all $\theta_t \neq 0$, $\mu_t = \nu_{\theta_t} < \nu_0 = 0$.

We have proved that $\phi_0 = \varphi_{\theta_0}^0$ for some $\theta_0$. Note that, thanks to Proposition 12.2.1, for every $x \in [0,1]$, the map $L(x,\cdot) : \theta_t \to \nabla\varphi_{\theta_t}^t(x)/\varphi_{\theta_t}^t(x)$ is increasing. Since $L(0,0) = 0$ and $L(0,\theta_0) = -m(0)/d(0) < 0$, we conclude that $\theta_0 < 0$. Since $x \to \exp(-\theta_0 x)\phi_0(x)$ is a positive continuous periodic function, it is bounded above and below by positive constants.

Next, notice that, $L(x,0) = 0$ for all $x \in [0,1]$, therefore $L(x,\theta_0) < 0$ since $\theta_0 < 0$. Finally, since $L(\cdot,\theta_0)$ is a negative continuous $Y$-periodic function, it is therefore bounded above and below by negative constants. The second statement involving $\theta_\delta$ is proved in a similar way. □

## 12.4 The homogenization regime

In this section we show that the assumption $m(0) \leq 0 \leq m(\delta)$ implies that the spectral problem (12.13) admits a homogenized limit.

*Remarque* 12.4.1. The equality $m(0) = m(\delta) = 0$ is a very special case which is easy to analyze. In this case, the minimum of the Rayleigh quotient (12.15) is zero, attained by $\phi^\varepsilon = \varphi_0 \equiv 1$, and we deduce that

$$\lambda_\varepsilon = \lambda_\infty \text{ and } u^\varepsilon(x) = \psi_\infty\left(\frac{x}{\varepsilon}\right).$$

From now on we shall further assume that $m(\delta) \neq m(0)$ since $m(0) = m(\delta)$ together with the assumption $m(0) \leq 0 \leq m(\delta)$ implies that both term vanish.

*Proposition* 12.4.1. Assume $m(0) \leq 0 \leq m(\delta)$. The eigenvalue $\mu_\varepsilon$ satisfies

$$0 \leq \mu_\varepsilon \leq \frac{\max(d)}{\min(s)}\pi^2.$$

*Démonstration.* Since $H_0^1(\Omega) \subset H^1(\Omega)$,

$$\mu_\varepsilon \leq \min_{\phi\in H_0^1(\Omega)} \frac{\displaystyle\int_\Omega d\left(\frac{x}{\varepsilon}\right)|\nabla\phi|^2(x)}{\displaystyle\int_\Omega s\left(\frac{x}{\varepsilon}\right)\phi^2(x)} \leq \frac{\max(d)}{\min(s)}\pi^2.$$

When $m(\delta) \geq 0 \geq m(0)$ all terms in the numerator of the Rayleigh quotient (12.15) are non-negative, and therefore $\mu_\varepsilon \geq 0$. □

This shows that the sequence $\mu_\varepsilon$ is bounded independently of $\varepsilon$. In this case, following a well-established strategy (see e.g. [1, 2, 3, 13]) we consider the operator $S^\varepsilon$ defined as follows

*Proposition* 12.4.2. Assume $m(0) \leq 0 \leq m(\delta)$, and $m(\delta) \neq m(0)$. Let $S^\varepsilon : L^2(\Omega) \to L^2(\Omega)$ be the self-adjoint operator defined, for $f \in L^2(\Omega)$, by $S^\varepsilon f = w^\varepsilon$ which is the unique solution in $H^1(\Omega)$ of

$$\int_\Omega d\left(\frac{x}{\varepsilon}\right)\nabla w^\varepsilon\nabla\zeta\,dx + \frac{1}{\varepsilon}\left(m(\delta)w^\varepsilon(1)\zeta(1) - m(0)w^\varepsilon(0)\zeta(0)\right) = \int_\Omega f\zeta dx \tag{12.19}$$

for all $\zeta \in H^1(\Omega)$. Then, for each $\varepsilon > 0$, $S^\varepsilon$ is a compact operator in $L^2(\Omega)$. Furthermore, as $\varepsilon$ tends to zero, $S^\varepsilon$ converges uniformly to the operator $S$ which to $f$ associates $w \in \mathcal{H}$ given by

$$-d^*\Delta w = f \text{ in } \Omega,$$

where $d^* = \left(\int_Y d^{-1}(y)dy\right)^{-1}$ and

$$\mathcal{H} = \left\{u \in H^1(\Omega) \text{ s.t. } u(0)m(0) = u(1)m(\delta) = 0\right\}.$$

*Démonstration.* This is a classical homogenization result [1, 2, 3, 13], which stems from the following a priori estimate

$$\|\nabla w^\varepsilon\|^2_{L^2(\Omega)} + \varepsilon^{-1}|m(\delta)|\left(w^\varepsilon(1)\right)^2 + \varepsilon^{-1}|m(0)|\left(w^\varepsilon(0)\right)^2 \leq C\|f\|^2_{L^2(\Omega)}.$$

We will therefore only establish this estimate. Choosing $w^\varepsilon$ as a test function in (12.19) we obtain

$$\int_\Omega d\left(\frac{x}{\varepsilon}\right)|\nabla w^\varepsilon|^2 dx + \frac{1}{\varepsilon}\left(m(\delta)\left(w^\varepsilon(1)\right)^2 - m(0)\left(w^\varepsilon(0)\right)^2\right) = \int_\Omega f w^\varepsilon dx.$$

Since each term on the left hand side is non-negative, $d(y) > C > 0$, $m(\delta)$ and $m(0)$ are not both zero, the estimate follows from the Poincaré inequality, for any $\zeta \in H^1(\Omega)$

$$\|\zeta\|^2_{L^2(\Omega)} \leq C\left(\alpha|\zeta(0)|^2 + (1-\alpha)|\zeta(1)|^2 + \|\nabla\zeta\|^2_{L^2(\Omega)}\right)$$

where $\alpha = 0$ or $1$. □

**Theorem 12.4.1.** *Assume $m(0) \leq 0 \leq m(\delta)$, and $m(\delta) \neq m(0)$. Then*

$$u^\varepsilon(x) = \psi_\infty\left(\frac{x}{\varepsilon}\right)(u(x) + r^\varepsilon(x)) \text{ and } \lambda_\varepsilon = \lambda_\infty + \varepsilon^2\lambda_0^* + o(\varepsilon^2),$$

*where $r^\varepsilon$ tends to zero weakly in $H^1(\Omega)$ and $(u, \lambda_0^*)$ is the first eigenpair of the problem*

$$\begin{cases} -d^*\Delta u & = & \lambda_0^* s^* u \text{ in } \Omega, \\ u \in H^1(\Omega) & \text{and} & m(0)u(0) = m(\delta)u(1) = 0, \end{cases}$$

*with $s^* = \int_Y s(y)dy$.*

*Démonstration.* We write (12.13) as

$$S^\varepsilon\left(\mu^\varepsilon s\left(\frac{x}{\varepsilon}\right)\phi^\varepsilon\right) = \phi^\varepsilon$$

Since $\mu^\varepsilon s\left(\frac{x}{\varepsilon}\right)$ is bounded in $L^\infty(\Omega)$, and $\phi^\varepsilon$ is normalized in $L^2(\Omega)$, we can extract a weakly converging subsequence. Since $S^\varepsilon$ is compact, $\phi^\varepsilon$ converges strongly in $L^2(\Omega)$ to a limit $u$. Thus $\mu^\varepsilon s\left(\frac{x}{\varepsilon}\right)\phi^\varepsilon$ converges weakly to $\mu s^* u$ in $L^2(\Omega)$. The conclusion follows from Proposition 12.4.2. □

## 12.5 The localization regime : convergence of the eigenvalues

We now turn to the other cases, that is, either $m(0) > 0$ or $m(\delta) < 0$, or both. We shall use two auxiliary cell problems. We introduce $p_\delta$ and $q_\delta$ as the first normalized eigenfunctions (and $l_p, l_q$ their corresponding first eigenvalues) of the following problems, posed on partial cells,

$$\begin{cases} -\text{div}(d(y)\nabla p_\delta) = l_p s(y) p_\delta \text{ in } (0,\delta) \\ \\ d(0)\nabla p_\delta(0) + m(0)p_\delta(0) = 0, \\ \\ d(\delta)\nabla p_\delta(\delta) + m(\delta)p_\delta(\delta) = 0, \text{ and } p_\delta(0) = 1, \end{cases} \tag{12.20}$$

and

$$\begin{cases} -\text{div}(d(y)\nabla q_\delta) = l_q s(y) q_\delta \text{ in } (\delta,1) \\ \\ d(\delta)\nabla q_\delta(\delta) + m(\delta)q_\delta(\delta) = 0, \\ \\ d(1)\nabla q_\delta(1) + m(1)q_\delta(1) = 0, \text{ and } q_\delta(\delta) = 1. \end{cases} \tag{12.21}$$

Note that both $p_\delta$ and $q_\delta$ are $C^1$ functions, and satisfy the uniform bounds

$$0 < C < p_\delta < C^{-1} \text{ and } 0 < C < q_\delta < C^{-1}.$$

*Proposition* 12.5.1. The first eigenvalues $\mu_0, \mu_\delta$ of (12.18) for $t = 0, \delta$ satisfy

$$\min(l_p, l_q) \le \mu_0 \le \max(l_p, l_q)\,, \quad \min(l_p, l_q) \le \mu_\delta \le \max(l_p, l_q),$$

and the inequalities are strict except when $l_p = l_q$.

*Démonstration.* Define a test function $w(y) = p_\delta(y)$ for $0 \le y \le \delta$ and $w(y) = p_\delta(\delta) q_\delta(y)$ for $\delta \le y \le 1$. It is easy to see that this function is $C^1$. We have

$$\begin{aligned}
\mu_0 &\le \frac{1}{\int_Y s(y) w^2(y)} \left( \int_0^1 d(y) (\nabla w)^2(y) + w^2(1) m(0) - w^2(0) m(0) \right) \\
&= \frac{1}{\int_Y s(y) w^2(y)} \left( \int_0^\delta d(y) (\nabla w)^2(y) + w^2(\delta) m(\delta) - w^2(0) m(0) \right) \\
&+ \frac{1}{\int_Y s(y) w^2(y)} \left( \int_\delta^1 d(y) (\nabla w)^2(y) + w^2(1) m(0) - w^2(\delta) m(\delta) \right) \\
&= \frac{1}{\int_Y s(y) w^2(y)} \left( l_p \int_0^\delta s(y) w^2(y) + l_q \int_\delta^1 s(y) w^2(y) \right) \\
&\le \max(l_p, l_q).
\end{aligned}$$

Alternatively

$$\begin{aligned}
\mu_0 &= \frac{1}{\int_Y s(y) \phi_0^2(y)} \left( \int_0^1 d(y) (\nabla \phi_0)^2(y) + \phi_0^2(1) m(0) - \phi_0^2(0) m(0) \right) \\
&= \frac{1}{\int_Y s(y) \phi_0^2(y)} \left( \int_0^\delta d(y) (\nabla \phi_0)^2(y) + \phi_0^2(\delta) m(\delta) - \phi_0^2(0) m(0) \right) \\
&+ \frac{1}{\int_Y s(y) \phi_0^2(y)} \left( \int_\delta^1 d(y) (\nabla \phi_0)^2(y) + \phi_0^2(1) m(0) - \phi_0^2(\delta) m(\delta) \right) \\
&\ge \frac{1}{\int_Y s(y) \phi_0^2(y)} \left( l_p \int_0^\delta s(y) \phi_0^2(y) + l_q \int_\delta^1 s(y) \phi_0^2(y) \right) \\
&\ge \min(l_p, l_q).
\end{aligned}$$

Furthermore, the inequalities above show that $\mu_0$ is bounded from above and below by two strictly convex combinations of $l_p$ and $l_q$. It implies that any inequality becomes an equality if and only if $l_p = l_q$. Indeed, if, for example, $l_p = \mu_0$, the previous inequalities imply $\mu_0 = l_q$, then if an inequality is not strict, we get immediately $l_p = l_q$.

The proof for $\mu_\delta$ is similar. □

The goal of this section is to prove that $\varepsilon^2 \mu_\varepsilon$ converges to a limit which is either $\min(\mu_0, \mu_\delta)$ or $\max(\mu_0, \mu_\delta)$ depending on the sign of $l_p - l_q$.

*Proposition* 12.5.2. Assume either $m(0) > 0$ or $m(\delta) < 0$, or both. Then, if $l_p \ge l_q$, $\varepsilon^2 \mu_\varepsilon$ is a decreasing sequence converging to a limit $L$ given by

$$L = \inf_{\varepsilon > 0} \varepsilon^2 \mu_\varepsilon = \max(\mu_0, \mu_\delta),$$

whereas, if $l_q \ge l_p$, then $\varepsilon^2 \mu_\varepsilon$ is an increasing sequence converging to

$$L = \sup_{\varepsilon > 0} \varepsilon^2 \mu_\varepsilon = \min(\mu_0, \mu_\delta).$$

Furthermore,

$$\left| \varepsilon^2 \mu_\varepsilon - L \right| \le C \exp\left( -\frac{C}{\varepsilon} \right).$$

Proposition 12.5.2 involves four parameters, namely the sign of $m(0)$, the sign of $m(\delta)$, the sign of $l_p - l_q$, and the sign of $\mu_0 - \mu_\delta$. Not all combinations of signs are possible, and in fact the sign of one of the parameters can be determined by the others. We now give a variant of Proposition 12.5.2, which gives the convergence of the eigenvalues without referring to $l_p$ or $l_q$.

*Proposition* 12.5.3. If $m(0) > 0$, or $m(\delta) < 0$, or both, then $\varepsilon^2 \mu_\varepsilon$ converges monotonically to a limit $L$, and

$$|\varepsilon^2 \mu_\varepsilon - L| \leq C \exp\left(-\frac{C}{\varepsilon}\right).$$

If $m(0) > 0$ and $m(\delta) \geq 0$, then $L = \mu_0$.
If $m(0) \leq 0$ and $m(\delta) < 0$, then $L = \mu_\delta$.
If both $m(0) > 0$ and $m(\delta) < 0$, then $\varepsilon^2 \mu_\varepsilon$ increases monotonically to $\min(\mu_0, \mu_\delta)$.

To prove Proposition 12.5.2, we rely on several lemmas, that will be proved at the end of this section.

First, we derive an upper bound when $l_q \geq l_p$, and a lower bound when $l_p \geq l_q$.

*Lemme* 12.5.1. Suppose $m(0) > 0$, or $m(\delta) < 0$, or both.
Then for $\varepsilon$ small enough, $\varepsilon^2 \mu_\varepsilon < -C < 0$.
If $l_q \geq l_p$, then $\varepsilon^2 \mu_\varepsilon \leq \min(\mu_0, \mu_\delta)$.
If $l_p \geq l_q$, then $\varepsilon^2 \mu_\varepsilon \geq \max(\mu_0, \mu_\delta)$.

Second, we make use of the dependence on $n$ of the sequence $\varepsilon$. Specifically, in the following lemma we denote $\varepsilon_n = (n+\delta)^{-1}$, and $\mu_n = \mu_{\varepsilon_n}$, for all $n$. We derive lower and upper bounds for differences between two consecutive terms of the sequence $(\varepsilon_n^2 \mu_n)$.

*Lemme* 12.5.2. The following two lower bounds hold :

$$\varepsilon_{n+1}^2 \mu_{n+1} \geq \varepsilon_n^2 \mu_n \left(1 - \kappa^1_{\varepsilon_{n+1}}\right) + \kappa^1_{\varepsilon_{n+1}} \mu_\delta, \tag{12.22}$$

and

$$\varepsilon_{n+1}^2 \mu_{n+1} \geq \varepsilon_n^2 \mu_n \left(1 - \kappa^0_{\varepsilon_{n+1}}\right) + \kappa^0_{\varepsilon_{n+1}} \mu_0, \tag{12.23}$$

where

$$0 < \kappa^1_\varepsilon = \frac{\int_{1-\varepsilon}^1 s\left(\frac{x}{\varepsilon}\right) \phi^\varepsilon(x)^2 dx}{\int_0^1 s\left(\frac{x}{\varepsilon}\right) \phi^\varepsilon(x)^2 dx} < 1, \text{ and } 0 < \kappa^0_\varepsilon = \frac{\int_0^\varepsilon s\left(\frac{x}{\varepsilon}\right) \phi^\varepsilon(x)^2 dy}{\int_0^1 s\left(\frac{x}{\varepsilon}\right) \phi^\varepsilon(x)^2 dx} < 1. \tag{12.24}$$

The following two upper bounds hold :

$$\varepsilon_{n+1}^2 \mu_{n+1} \leq \varepsilon_n^2 \mu_n \left(1 - \chi^1_{\varepsilon_n}\right) + \mu_\delta \chi^1_{\varepsilon_n}, \tag{12.25}$$

and

$$\varepsilon_{n+1}^2 \mu_{n+1} \leq \varepsilon_n^2 \mu_n \left(1 - \chi^0_{\varepsilon_n}\right) + \mu_0 \chi^0_{\varepsilon_n}, \tag{12.26}$$

where

$$0 < \chi^0_\varepsilon = \frac{\left(\frac{\phi^\varepsilon(0)}{\phi_0(1)}\right)^2 \int_0^1 s(y) \phi_0(y)^2 dy}{\frac{1}{\varepsilon}\int_0^1 s\left(\frac{x}{\varepsilon}\right) \phi^\varepsilon(x)^2 dx + \left(\frac{\phi^\varepsilon(0)}{\phi_0(1)}\right)^2 \int_0^1 s(y) \phi_0(y)^2 dy} < 1, \tag{12.27}$$

and

$$0 < \chi^1_\varepsilon = \frac{\left(\frac{\phi^\varepsilon(1)}{\phi_\delta(\delta-1)}\right)^2 \int_{\delta-1}^\delta s(y) \phi_\delta(y)^2 dy}{\frac{1}{\varepsilon}\int_0^1 s\left(\frac{x}{\varepsilon}\right) \phi^\varepsilon(x)^2 dx + \left(\frac{\phi^\varepsilon(1)}{\phi_\delta(\delta-1)}\right)^2 \int_{\delta-1}^\delta s(y) \phi_\delta(y)^2 dy} < 1. \tag{12.28}$$

Finally, we show that lower bounds on the weights $\kappa^{0,1}_\varepsilon$ and $\chi^{0,1}_\varepsilon$ can be obtained depending on the boundary conditions $m(0)$ and $m(\delta)$.

*Lemme* 12.5.3. The following relations hold

$$\kappa_\varepsilon^0 \int_0^1 s\left(\frac{x}{\varepsilon}\right)\phi^\varepsilon(x)^2dx > \varepsilon C\phi^\varepsilon(0)^2 \text{ and } \kappa_\varepsilon^1 \int_0^1 s\left(\frac{x}{\varepsilon}\right)\phi^\varepsilon(x)^2dx > \varepsilon C\phi^\varepsilon(1)^2.$$

If $m(0) > 0$ and $m(\delta) \geq 0$,

$$\phi^\varepsilon(0)^2 > \frac{C}{\varepsilon}\int_0^1 s\left(\frac{x}{\varepsilon}\right)\phi^\varepsilon(x)^2dx. \tag{12.29}$$

As a consequence, $\kappa_\varepsilon^0 > C > 0$ and $\chi_\varepsilon^0 > C > 0$.

If $m(0) \leq 0$ and $m(\delta) < 0$,

$$\phi^\varepsilon(1)^2 > \frac{C}{\varepsilon}\int_0^1 s\left(\frac{x}{\varepsilon}\right)\phi^\varepsilon(x)^2dx. \tag{12.30}$$

As a consequence, $\kappa_\varepsilon^1 > C > 0$ and $\chi_\varepsilon^1 > C > 0$.

If $m(0) > 0$ and $m(\delta) < 0$,

$$\phi^\varepsilon(0)^2 + \phi^\varepsilon(1)^2 > \frac{C}{\varepsilon}\int_0^1 s\left(\frac{x}{\varepsilon}\right)\phi^\varepsilon(x)^2dx. \tag{12.31}$$

as a consequence, $\min\left(\kappa_\varepsilon^0, \kappa_\varepsilon^1\right) > C > 0$, $\min\left(\chi_\varepsilon^0, \chi_\varepsilon^1\right) > C > 0$.

We are now in a position to prove Proposition 12.5.2.

*Proof of Proposition 12.5.2.* Suppose $l_p \geq l_q$. Then, Lemma 12.5.1 shows that $\varepsilon^2\mu_\varepsilon \geq \max(\mu_0, \mu_\delta)$. Using the upper bound $\mu_0 \leq \max(\mu_0, \mu_\delta) \leq \varepsilon_n^2\mu_n$ in (12.26) yields

$$\varepsilon_{n+1}^2\mu_{n+1} \leq \varepsilon_n^2\mu_n,$$

therefore the sequence $\varepsilon^2\mu_\varepsilon$ is decreasing. Now rewrite (12.26) under the form

$$0 \leq \varepsilon_{n+1}^2\mu_{n+1} - \mu_0 \leq \left(1 - \chi_{\varepsilon_n}^0\right)\left(\varepsilon_n^2\mu_n - \mu_0\right).$$

This geometric relation implies, for $n \geq 1$, noting that $\mu_{n=0} = l_p$,

$$0 \leq \varepsilon_n^2\mu_n - \mu_0 \leq \left(1 - \min_{m\leq n}\chi_{\varepsilon_m}^0\right)^n (l_p - \mu_0),$$

or in other words,

$$0 \leq \varepsilon^2\mu_\varepsilon - \mu_0 \leq Ce^{-\varepsilon^{-1}\min_{\zeta\geq\varepsilon}\chi_\zeta^0}. \tag{12.32}$$

Similarly, using (12.25) instead, we obtain

$$0 \leq \varepsilon^2\mu_\varepsilon - \mu_\delta \leq Ce^{-\varepsilon^{-1}\min_{\zeta\geq\varepsilon}\chi_\zeta^1}. \tag{12.33}$$

Now, Lemma 12.5.3 says that when $m(0) > 0$, or $m(\delta) < 0$, or both, then

$$\max(\min_{\varepsilon>0}\chi_\varepsilon^0, \min_{\varepsilon>0}\chi_\varepsilon^1) > C > 0.$$

So at least one of inequalities (12.32) and (12.33) implies convergence of $\varepsilon^2\mu_\varepsilon$ to either $\mu_0$ or $\mu_\delta$, and since $\varepsilon^2\mu_\varepsilon > \max(\mu_0, \mu_\delta)$, this in fact shows

$$0 \leq \varepsilon^2\mu_\varepsilon - \max(\mu_0, \mu_\delta) < C\exp(-C/\varepsilon),$$

as announced.

Suppose now $l_q > l_p$. Then, Lemma 12.5.1 shows that $\varepsilon^2\mu_\varepsilon < \min(\mu_0, \mu_\delta)$. Using the upper bound $\mu_0 \geq \min(\mu_0, \mu_\delta) > \varepsilon_n^2\mu_n$ in (12.23) yields

$$\varepsilon_{n+1}^2\mu_{n+1} > \varepsilon_n^2\mu_n,$$

therefore the sequence $\varepsilon^2\mu_\varepsilon$ is increasing. Now rewrite (12.23) under the form

$$0 > \varepsilon_{n+1}^2\mu_{n+1} - \mu_0 \geq \left(1 - \kappa^0_{\varepsilon_n}\right)\left(\varepsilon_n^2\mu_n - \mu_0\right).$$

As above this geometric relation implies

$$0 < \mu_0 - \varepsilon^2\mu_\varepsilon \leq Ce^{-\varepsilon^{-1}\min_{\zeta\geq\varepsilon}\kappa^0_\zeta}. \tag{12.34}$$

Similarly, using (12.22) instead of (12.23), we obtain

$$0 < \mu_\delta - \varepsilon^2\mu_\varepsilon \leq Ce^{-\varepsilon^{-1}\min_{\zeta\geq\varepsilon}\kappa^1_\zeta}. \tag{12.35}$$

And, again, Lemma 12.5.3 says that when $m(0) > 0$, or $m(\delta) < 0$, or both, at least one of the two terms $\min_{\varepsilon>0}\kappa^0_\varepsilon$ and $\min_{\varepsilon>0}\kappa^1_\varepsilon$ is positive. So at least one of inequalities (12.34) and (12.35) implies

$$0 < \min(\mu_0, \mu_\delta) - \varepsilon^2\mu_\varepsilon < C\exp(-C/\varepsilon),$$

as announced. □

We now turn to the proof of the different Lemmas.

*Proof of Lemma 12.5.2.* Let us prove the two lower bounds (12.22) and (12.23). Take two successive small positive parameters $\varepsilon_{n+1} < \varepsilon_n$. Let us denote by $\phi^{n+1} = \phi^{\varepsilon_{n+1}}$ the first eigenfunction of (12.13) or the minimizer of (12.15). We make the change of variables $y = x/\varepsilon_{n+1}$ and we define $\tilde\phi^{n+1}(y) = \phi^{n+1}(\varepsilon_{n+1}y)$. Recalling that $\varepsilon_{n+1} = (n+1+\delta)^{-1}$, we get

$$\begin{aligned}
\varepsilon_{n+1}^2\mu_{n+1} &= \varepsilon_{n+1}^2 \frac{\int_\Omega d(\frac{x}{\varepsilon_{n+1}})\left(\nabla\phi^{n+1}\right)^2(x)dx + \frac{1}{\varepsilon_{n+1}}\left(m(\delta)\phi^{n+1}(1)^2 - m(0)\phi^{n+1}(0)^2\right)}{\int_\Omega s(\frac{x}{\varepsilon_{n+1}})\phi^{n+1}(x)^2dx} \\
&= \frac{\int_0^{\varepsilon_{n+1}^{-1}} d(y)\left(\nabla\tilde\phi^{n+1}\right)^2(y)dy + m(\delta)\tilde\phi^{n+1}(\varepsilon_{n+1}^{-1})^2 - m(0)\tilde\phi^{n+1}(0)^2}{\int_0^{\varepsilon_{n+1}^{-1}} s(y)\tilde\phi^{n+1}(y)^2dy} \\
&= \frac{\int_0^{n+\delta} d(y)\left(\nabla\tilde\phi^{n+1}\right)^2(y)dy + m(\delta)\tilde\phi^{n+1}(n+\delta)^2 - m(0)\tilde\phi^{n+1}(0)^2}{\int_0^{n+1+\delta} s(y)\tilde\phi^{n+1}(y)^2dy} \\
&+ \frac{\int_{n+\delta}^{n+1+\delta} d(y)\left(\nabla\tilde\phi^{n+1}\right)^2(y)dy + m(\delta)\left(\tilde\phi^{n+1}(n+1+\delta)^2 - \tilde\phi^{n+1}(n+\delta)^2\right)}{\int_0^{n+1+\delta} s(y)\tilde\phi^{n+1}(y)^2dy}
\end{aligned}$$

From the minimizing properties of $\mu_n$, we get

$$\begin{aligned}
&\frac{\int_0^{n+\delta} d(y)\left(\nabla\tilde\phi^{n+1}\right)^2(y)dy + m(\delta)\tilde\phi^{n+1}(n+\delta)^2 - m(0)\tilde\phi^{n+1}(0)^2}{\int_0^{n+1+\delta} s(y)\tilde\phi^{n+1}(y)^2dy} \\
&\geq \varepsilon_n^2\mu_n \frac{\int_0^{n+\delta} s(y)\tilde\phi^{n+1}(y)^2dy}{\int_0^{n+1+\delta} s(y)\tilde\phi^{n+1}(y)^2dy}.
\end{aligned}$$

On the other hand, the segment $[n+\delta, n+1+\delta]$ is a translation of $[\delta-1, \delta]$ and from the minimizing property of $\mu_\delta$ we deduce

$$\frac{\int_{n+\delta}^{n+1+\delta} d\left(\nabla\tilde\phi^{n+1}\right)^2dy + m(\delta)\left(\tilde\phi^{n+1}(n+1+\delta)^2 - \tilde\phi^{n+1}(n+\delta)^2\right)}{\int_0^{n+1+\delta} s\tilde\phi^{n+1}(y)^2dy} \geq \mu_\delta \frac{\int_{n+\delta}^{n+1+\delta} s\tilde\phi^{n+1}(y)^2dy}{\int_0^{n+1+\delta} s\tilde\phi^{n+1}(y)^2dy}.$$

Thus we obtain the lower bound (12.22),

$$\varepsilon_{n+1}^2\mu_{n+1} \geq \varepsilon_n^2\mu_n(1 - \kappa^1_{\varepsilon_{n+1}}) + \kappa^1_{\varepsilon_{n+1}}\mu_\delta,$$

where $\kappa^1_{\varepsilon_{n+1}}$ is defined by (12.24). By a symmetric argument, exchanging the two endpoints, we obtain in a similar way (12.23).

Let us now turn to the upper bounds. Since $\varepsilon_{n+1} < \varepsilon_n$, we define a test function

$$w^{n+1} = \begin{cases} \phi^n\left(\frac{\varepsilon_n}{\varepsilon_{n+1}}x\right) \text{ on } [0, \varepsilon_{n+1}/\varepsilon_n], \\ \frac{\phi^n(1)}{\phi_\delta(\delta-1)}\phi_\delta\left(\frac{x}{\varepsilon_{n+1}} + \delta - 1 - \frac{1}{\varepsilon_n}\right) \text{ on } [\varepsilon_{n+1}/\varepsilon_n, 1], \end{cases}$$

which is clearly continuous on $\Omega$ (it is even $C^1(\Omega)$ by further inspection). Taking $w^{n+1}$ as a test function in the Rayleigh quotient for $\mu_{n+1}$, and arguing as above, we deduce (12.25), namely,

$$\varepsilon^2_{n+1}\mu_{n+1} \le \varepsilon^2_n\mu_n(1-\chi^1_{\varepsilon_n}) + \mu_\delta\chi^1_{\varepsilon_n},$$

where

$$\chi^1_{\varepsilon_n} = \frac{\dfrac{|\phi^n(1)|^2}{|\phi_\delta(\delta-1)|^2}\displaystyle\int_{\frac{\varepsilon_{n+1}}{\varepsilon_n}}^1 s\left(\frac{x}{\varepsilon_{n+1}}\right)\phi_\delta\left(\frac{x}{\varepsilon_{n+1}} + \delta - 1 - \frac{1}{\varepsilon_n}\right)^2 dx}{\displaystyle\int_0^{\frac{\varepsilon_{n+1}}{\varepsilon_n}} s\left(\frac{x}{\varepsilon_{n+1}}\right)\phi^n\left(\frac{\varepsilon_n x}{\varepsilon_{n+1}}\right)^2 dx + \frac{|\phi^n(1)|^2}{|\phi_\delta(\delta-1)|^2}\displaystyle\int_{\frac{\varepsilon_{n+1}}{\varepsilon_n}}^1 s\left(\frac{x}{\varepsilon_{n+1}}\right)\phi_\delta\left(\frac{x}{\varepsilon_{n+1}} + \delta - 1 - \frac{1}{\varepsilon_n}\right)^2 dx}.$$

By the change of variables $y = x/\varepsilon_{n+1}$, we obtain that $\chi^1_{\varepsilon_n}$ is indeed given by (12.28).

To prove the other upper bound (12.26), the argument is similar, using in this case the test function

$$w^{n+1} = \begin{cases} \frac{\phi^n(0)}{\phi_0\left(\frac{1}{\varepsilon_{n+1}} - \frac{1}{\varepsilon_n}\right)}\phi_0\left(\frac{x}{\varepsilon_{n+1}}\right) \text{ on } [0, 1-\varepsilon_{n+1}/\varepsilon_n], \\ \phi^n\left(\frac{\varepsilon_n}{\varepsilon_{n+1}}x + 1 - \frac{\varepsilon_n}{\varepsilon_{n+1}}\right) \text{ on } [1-\varepsilon_{n+1}/\varepsilon_n, 1]. \end{cases}$$

□

*Proof of Lemma 12.5.3.* If either $m(\delta) < 0$ or $m(0) > 0$, or both, Lemma 12.5.1 shows that $\mu_\varepsilon < -\varepsilon^{-2}C < 0$. Integrating directly (12.13) we obtain, for $t \in (0,1)$,

$$d\left(\frac{t}{\varepsilon}\right)\nabla\phi^\varepsilon(t) + \frac{1}{\varepsilon}m(0)\phi^\varepsilon(0) = -\mu_\varepsilon\int_0^t s\left(\frac{x}{\varepsilon}\right)\phi^\varepsilon dx$$

Dividing by $d\left(\frac{t}{\varepsilon}\right)$ and integrating again

$$\phi^\varepsilon(t) - \phi^\varepsilon(0) + \frac{1}{\varepsilon}\left(\int_0^t d^{-1}\left(\frac{\tau}{\varepsilon}\right)d\tau\right)m(0)\phi^\varepsilon(0) = -\mu_\varepsilon\int_0^t d^{-1}\left(\frac{u}{\varepsilon}\right)\int_0^u s\left(\frac{x}{\varepsilon}\right)\phi^\varepsilon dx du. \tag{12.36}$$

The right-hand-side of (12.36) is positive because $\mu_\varepsilon < -\varepsilon^{-2}C < 0$ and $\phi^\varepsilon > 0$. If $m(0) \le 0$, this implies that, for $0 \le t \le 1$,

$$\phi^\varepsilon(t) \ge \phi^\varepsilon(0).$$

On the other hand, if $m(0) > 0$, we write

$$\phi^\varepsilon(t) \ge \phi^\varepsilon(0)\left(1 - \frac{1}{\varepsilon}\left(\int_0^t d^{-1}\left(\frac{u}{\varepsilon}\right)du\right)m(0)\right) \ge \phi^\varepsilon(0)\left(1 - \frac{t}{\varepsilon}\frac{m(0)}{\min d}\right),$$

which implies, for $0 \le t \le \frac{\varepsilon}{2}\min(\frac{m(0)}{\min d}, 1)$, that

$$\phi^\varepsilon(t) \ge \frac{1}{2}\phi^\varepsilon(0).$$

Consequently, in either case

$$\kappa_\varepsilon^0 \int_0^1 s\left(\frac{x}{\varepsilon}\right)\phi^\varepsilon(x)^2dx = \int_0^\varepsilon s\left(\frac{x}{\varepsilon}\right)\phi^\varepsilon(x)^2dx \geq \varepsilon\frac{\min(s)}{4}\phi^\varepsilon(0)^2.$$

The proof of $\int_{1-\varepsilon}^1 s\left(\frac{x}{\varepsilon}\right)\phi^\varepsilon(x)^2dx \geq C\varepsilon\phi^\varepsilon(1)^2$ is similar.

Let us now prove the lower bounds (12.29-12.31). The variational formulation of (12.13) with $\phi^\varepsilon$ as a test function yields

$$-\int_0^1 d\left(\frac{x}{\varepsilon}\right)(\nabla\phi^\varepsilon)^2dx - \frac{1}{\varepsilon}m(\delta)\phi^\varepsilon(1)^2 + \frac{1}{\varepsilon}m(0)\phi^\varepsilon(0)^2 = -\mu_\varepsilon\int_0^1 s\left(\frac{x}{\varepsilon}\right)(\phi^\varepsilon)^2dx.$$

Since the first term is negative and $\mu_\varepsilon < -\varepsilon^{-2}C < 0$, we deduce

$$\max\left(-m(\delta)\phi^\varepsilon(1)^2, m(0)\phi^\varepsilon(0)^2\right) \geq -\frac{\varepsilon\mu_\varepsilon}{2}\int_0^1 s\left(\frac{x}{\varepsilon}\right)(\phi^\varepsilon)^2dx \geq \frac{C}{\varepsilon}\int_0^1 s\left(\frac{x}{\varepsilon}\right)(\phi^\varepsilon)^2dx. \tag{12.37}$$

If $m(\delta) \geq 0$, and $m(0) > 0$ the maximum is $m(0)\phi^\varepsilon(0)^2$, which proves (12.29). Conversely, if $m(\delta) < 0$, and $m(0) \leq 0$, the maximum is $-m(\delta)\phi^\varepsilon(1)^2$, which proves (12.30). If $m(\delta) < 0$, and $m(0) > 0$, the maximum is attained by at least one of the points, or both, which proves (12.31). Finally, notice that for $i = 0, 1$,

$$\chi_\varepsilon^i = \left(\frac{\int_0^1 s\left(\frac{x}{\varepsilon}\right)\phi^\varepsilon(x)^2dx}{\varepsilon\phi^\varepsilon(i)^2}c_i + 1\right)^{-1},$$

where $c_i$ is a positive constant, therefore the bound (12.37) implies the desired lower bound on $\min(\chi_\varepsilon^0, \chi_\varepsilon^1) > C > 0$.

Finally, note that $\chi_\varepsilon^i < C\frac{\varepsilon\phi^\varepsilon(i)^2}{\int_0^1 s\left(\frac{x}{\varepsilon}\right)\phi^\varepsilon(x)^2dx} \leq C\kappa_\varepsilon^i$, therefore $\min(\chi_\varepsilon^0, \chi_\varepsilon^1) > C > 0$ implies $\min(\kappa_\varepsilon^0, \kappa_\varepsilon^1) > C > 0$. □

Lemma 12.5.1 will be a consequence of the following Lemma.

*Lemme* 12.5.4. There exist two parameters $0 < \tau_\varepsilon^0 < 1$ and $0 < \tilde{\kappa}_\varepsilon^1 < \kappa_\varepsilon^1 < 1$ such that

$$\mu_0\left(1 - \tilde{\kappa}_\varepsilon^1\right) + l_p\tilde{\kappa}_\varepsilon^1 \leq \varepsilon^2\mu_\varepsilon \leq \mu_0\left(1 - \tau_\varepsilon^0\right) + l_p\tau_\varepsilon^0. \tag{12.38}$$

Similarly, there exist two parameters $0 < \tau_\varepsilon^\delta < 1$ and $0 < \tilde{\kappa}_\varepsilon^0 < \kappa_\varepsilon^0 < 1$ such that

$$\mu_\delta\left(1 - \kappa_\varepsilon^0\right) + l_p\kappa_\varepsilon^0 \leq \varepsilon^2\mu_\varepsilon \leq \mu_\delta\left(1 - \tau_\varepsilon^\delta\right) + l_p\tau_\varepsilon^\delta. \tag{12.39}$$

This allows to prove Lemma 12.5.1.

*Proof of Lemma 12.5.1.* Proposition 12.5.1 implies that $\min(l_p, l_q) \leq \mu_0, \mu_\delta \leq \max(l_p, l_q)$.

If $l_p \leq l_q$, then the upper bound in (12.38) shows that $\varepsilon^2\mu_\varepsilon \leq \mu_0$, whereas the upper bound in (12.39) shows that $\varepsilon^2\mu_\varepsilon \leq \mu_\delta$. Thus, $\varepsilon^2\mu_\varepsilon \leq \min(\mu_0, \mu_\delta) < 0$ by virtue of Proposition 12.3.3.

Symmetrically if $l_p \geq l_q$ using the lower bounds in (12.38) and (12.39) we obtain $\varepsilon^2\mu_\varepsilon \geq \max(\mu_0, \mu_\delta)$.

Finally, let us show that $\varepsilon^2\mu_\varepsilon < -C < 0$ for $\varepsilon$ small enough. Suppose $m(0) > 0$. Choosing as a test function $\exp(-\alpha x/\varepsilon)$ with $\alpha > 0$, in the Rayleigh quotient (12.15) defining $\mu_\varepsilon$, we obtain

$$\begin{aligned}\mu_\varepsilon &\leq \frac{\dfrac{\alpha^2}{\varepsilon^2}\displaystyle\int_0^1 d\left(\frac{x}{\varepsilon}\right)\exp(-2\alpha x/\varepsilon)dx - \frac{1}{\varepsilon}m(0) + \frac{1}{\varepsilon}m(\delta)\exp(-2\alpha/\varepsilon)}{\displaystyle\int_0^1 s\left(\tfrac{x}{\varepsilon}\right)\exp(-2\alpha x/\varepsilon)dx},\\ &\leq \frac{1}{\varepsilon^2}\frac{\alpha\max(d)/2 - m(0) + m(\delta)\exp(-2\alpha/\varepsilon)}{\min(s)(1-\exp(-2\alpha/\varepsilon))},\end{aligned}$$

Pick for example $\alpha = m(0)/\max(d)$, to obtain $\mu_\varepsilon \leq \frac{-m(0)}{2\varepsilon^2\min(s)}(1 + C\exp(-C/\varepsilon))$, which shows that $\varepsilon^2\mu_\varepsilon < -C < 0$ for $\varepsilon$ small enough. The argument is similar for $m(\delta) < 0$, choosing instead a test function $\exp(-\alpha(1-x)/\varepsilon)$ with $\alpha > 0$. □

*Proof of Lemma 12.5.4.* Let us focus on the proof of the first bound (12.38). To obtain an upper bound, we construct a continuous (actually $C^1$) test function for the Rayleigh quotient (12.15) as follows. Recall that $\varepsilon^{-1} = n + \delta$, so that $\varepsilon^{-1} - 1 < n < \varepsilon^{-1}$ and $n\varepsilon \leq x \leq 1 \Leftrightarrow 0 \leq (x - n\varepsilon)\varepsilon^{-1} \leq \delta$. We define $w^\varepsilon$ as

$$w^\varepsilon(x) = \begin{cases} \phi_0\left(\frac{x}{\varepsilon}\right) & \text{for } 0 \leq x \leq n\varepsilon,\\ \\ \phi_0(n)p_\delta\left(\frac{x-n\varepsilon}{\varepsilon}\right) & \text{for } n\varepsilon \leq x \leq 1.\end{cases}$$

Recall that, by virtue of Proposition 12.3.3, $\phi_0$ is equal to an exponential-periodic function $\varphi_{\theta_0}$ and thus is defined everywhere in $\mathbf{R}$ and not only on the interval $(0,1)$. By construction, $w^\varepsilon$ is continuous and we can use it as a test function in (12.15) to obtain

$$\begin{aligned}\mu^\varepsilon &\leq \frac{\displaystyle\int_0^{n\varepsilon} d\left(\frac{x}{\varepsilon}\right)|\nabla w^\varepsilon|^2 dx + \frac{1}{\varepsilon}\left(m(0)\,|w^\varepsilon(n\varepsilon)|^2 - m(0)\,|w^\varepsilon(0|^2\right)}{\displaystyle\int_0^1 s\left(\frac{x}{\varepsilon}\right)w^\varepsilon(x)^2 dx}\\ &+ \frac{\displaystyle\int_{n\varepsilon}^1 d\left(\frac{x}{\varepsilon}\right)|\nabla w^\varepsilon|^2 dx + \frac{1}{\varepsilon}\left(m(\delta)\,|w^\varepsilon(1)|^2 - m(0)\,|w^\varepsilon(n\varepsilon)|^2\right)}{\displaystyle\int_0^1 s\left(\frac{x}{\varepsilon}\right)w^\varepsilon(x)^2 dx}\\ &\leq \frac{\varepsilon^{-2}\mu_0\displaystyle\int_0^{n\varepsilon} s\left(\frac{x}{\varepsilon}\right)\phi_0\left(\frac{x}{\varepsilon}\right)^2 dx + \varepsilon^{-2}l_p\phi_0(n)^2\displaystyle\int_{n\varepsilon}^1 s\left(\frac{x}{\varepsilon}\right)p_\delta\left(\frac{x-n\varepsilon}{\varepsilon}\right)^2 dx}{\displaystyle\int_0^1 s\left(\frac{x}{\varepsilon}\right)w^\varepsilon(x)^2 dx}\\ &\leq \varepsilon^{-2}\mu_0\left(1-\tau_\varepsilon^0\right) + \varepsilon^{-2}l_p\tau_\varepsilon^0,\end{aligned}$$

where, using the change of variables $y = (x - n\varepsilon)/\varepsilon$, we defined

$$\tau_\varepsilon^0 = \phi_0(n)^2\frac{\varepsilon\displaystyle\int_0^\delta s(y)p_\delta(y)^2 dy}{\displaystyle\int_0^1 s\left(\tfrac{x}{\varepsilon}\right)w^\varepsilon(x)^2 dx}.$$

Let us now turn to the lower bound in (12.38). The idea is to get a lower bound in the Rayleigh quotient (12.15), using the fact that $\mu_0$ and $l_p$ are themselves given as minima of Rayleigh quotients. In (12.38) the coefficient $\tilde{\kappa}^1_\varepsilon$ is going to be defined by

$$\tilde{\kappa}^1_\varepsilon = \frac{\int_{1-\delta\varepsilon}^{1} s\left(\frac{x}{\varepsilon}\right)\phi^\varepsilon(x)^2 dx}{\int_0^1 s\left(\frac{x}{\varepsilon}\right)\phi^\varepsilon(x)^2 dx}. \tag{12.40}$$

Indeed,

$$\begin{aligned}
&\int_0^1 d\left(\frac{x}{\varepsilon}\right)|\nabla\phi^\varepsilon|^2 dx + \frac{1}{\varepsilon}\left(m(\delta)\left(\phi^\varepsilon(1)\right)^2 - m(0)\left(\phi^\varepsilon(0)\right)^2\right)\\
= &\int_0^{n\varepsilon} d\left(\frac{x}{\varepsilon}\right)|\nabla\phi^\varepsilon|^2 dx + \frac{1}{\varepsilon}\left(m(0)\left(\phi^\varepsilon(n\varepsilon)\right)^2 - m(0)\left(\phi^\varepsilon(0)\right)^2\right)\\
+ &\int_{n\varepsilon}^{n\varepsilon+\delta\varepsilon} d\left(\frac{x}{\varepsilon}\right)|\nabla\phi^\varepsilon|^2 dx + \frac{1}{\varepsilon}\left(m(\delta)\left(\phi^\varepsilon(n\varepsilon+\delta\varepsilon)\right)^2 - m(0)\left(\phi^\varepsilon(n\varepsilon)\right)^2\right)\\
\geq\ &\varepsilon^{-2}\mu_0\int_0^{n\varepsilon} s\left(\frac{x}{\varepsilon}\right)\phi^\varepsilon(x)^2 dx + \varepsilon^{-2} l_p \int_{n\varepsilon}^{n\varepsilon+\delta\varepsilon} s\left(\frac{x}{\varepsilon}\right)\phi^\varepsilon(x)^2 dx,
\end{aligned}$$

thanks to the minimizing properties of $\mu_0$ and $l_p$. So, altogether,

$$\mu_\varepsilon = \frac{\int_0^1 d\left(\frac{x}{\varepsilon}\right)|\nabla\phi^\varepsilon|^2 dx + \frac{1}{\varepsilon}\left(m(\delta)\left(\phi^\varepsilon\right)^2(1) - m(0)\left(\phi^\varepsilon\right)^2(0)\right)}{\int_0^1 s\left(\frac{x}{\varepsilon}\right)\left(\phi^\varepsilon\right)^2(x)dx} \geq \varepsilon^{-2}((1-\tilde{\kappa}^1_\varepsilon)\mu_0 + \tilde{\kappa}^1_\varepsilon l_p).$$

The proof of the inequalities (12.39), involving $\mu_\delta$, is similar. We use instead

$$w^\varepsilon(x) = \begin{cases} p_\delta\left(\frac{x}{\varepsilon}\right) & \text{for } 0 \leq x \leq \delta\varepsilon,\\ \\ p_\delta(\delta)\phi_\delta\left(\frac{x-\delta\varepsilon}{\varepsilon}\right) & \text{for } \delta\varepsilon \leq x \leq 1. \end{cases}$$

□

*Proof of Proposition 12.5.3.* The fact that the convergence is exponential in all cases is already established in Proposition 12.5.2. When $m(0) > 0$ and $m(\delta) \geq 0$, let us check that the limit of $\mu_\varepsilon$ is always $\mu_0$. In the course of the proof of Proposition 12.5.2, we have established (12.32) and (12.34) which prove that the limit is $\mu_0$ if either $\min_{\varepsilon>0}\kappa^0_\varepsilon$ or $\min_{\varepsilon>0}\chi^0_\varepsilon$ is positive. Lemma 12.5.3 provides such a result when $m(0) > 0$ and $m(\delta) \geq 0$.

The case $m(0) \leq 0$ and $m(\delta) < 0$ is handled by similar arguments using (12.33) and (12.35).

If $m(0) > 0$ and $m(\delta) < 0$, we have

$$l_q = \min_{\phi\in H^1(\delta,1)} \frac{\int_\delta^1 d(y)|\nabla\phi|^2 dy + \left(m(0)\phi(1)^2 - m(\delta)\phi(\delta)^2\right)}{\int_\delta^1 s(y)\phi^2 dy} \geq 0.$$

From Proposition 12.5.1, $\min(l_p, l_q) \leq \mu_0 < 0$, therefore $l_p < 0 < l_q$. Then Proposition 12.5.2 shows that $\varepsilon^2\mu_\varepsilon$ is an increasing sequence converging to $\min(\mu_0, \mu_\delta)$. □

## 12.6 The localization regime : a corrector result

In this section, we show that, in the self adjoint case, the first eigenfunction must localize at one of the end-points when, either $m(0) > 0$ or $m(\delta) < 0$, or both. More precisely, if $\mu_0 \neq \mu_\delta$, then localization occurs at only one end point. On the other hand, if $\mu_0 = \mu_\delta$, then two cases can happen : when $m(0)m(\delta) < 0$ localization takes place at both endpoints, while, when $m(0)m(\delta) > 0$ the first eigenfunction can be computed exactly and localization occurs at only one end point.

We start with this last case which is peculiar because it is equivalent to $\phi_0 = \phi_\delta$ – up to a renormalization.

*Proposition* 12.6.1. If $\mu_0 = \mu_\delta$ and $m(\delta)m(0) > 0$, then $\phi_0 = \frac{\phi_\delta}{\phi_\delta(-1)}$, and we have the exact relation

$$\mu_\varepsilon = \mu_0, \text{ and } u^\varepsilon(x) = \psi_\infty\left(\frac{x}{\varepsilon}\right)\frac{\phi_0\left(\frac{x}{\varepsilon}\right)}{\phi_0(0)}.$$

Conversely, if $\phi_0 = \frac{\phi_\delta}{\phi_\delta(-1)}$ then $\mu_0 = \mu_\delta$ and $m(\delta)m(0) > 0$.

*Remarque* 12.6.1. Proposition 12.6.1 is very similar to Proposition 12.2.2 when the two Neumann eigenfunctions coincide $u_N^0 = \frac{u_N^\delta}{u_N^\delta(-1)}$.

*Démonstration.* Recall that, in view of Proposition 12.3.3, $\phi_0$ and $\phi_\delta$ are exponential-periodic functions, namely $\phi_0 = \varphi_{\theta_0}^0$ and $\phi_\delta = \varphi_{\theta_\delta}^\delta$. Since $\mu_0 = \mu_\delta$ they are also solutions of the same equation,

$$-\mathrm{div}_y\left(d(y)\nabla_y\phi\right) = \mu_0 s(y)\phi \text{ in } \mathbf{R}.$$

If $m(\delta)$ and $m(0)$ have the same sign, then the exponent $\theta_0$ and $\theta_\delta$ have the same sign too. But the maps $\theta \to \varphi_\theta^t$ and $\theta \to \nu_\theta$, where $(\nu_\theta, \varphi_\theta)$ is the solution of the spectral problem (12.16) are one-to-one when restricted to $\theta \in \mathbf{R}^+$ or $\theta \in \mathbf{R}^-$. Thus, it implies that $\phi_0 = \frac{\phi_\delta}{\phi_\delta(-1)}$. This implies in turn that $x \to \phi_0\left(\frac{x}{\varepsilon}\right)$ is positive, and satisfies both $\nabla\phi_0(0) = -\frac{1}{\varepsilon}m(0)\phi_0(0)$ and $\nabla\phi_0(1/\varepsilon) = -\frac{1}{\varepsilon}m(\delta)\phi_0(1/\varepsilon)$, i.e., it is the first eigensolution of problem (12.13) and then is equal to $\phi^\varepsilon$ after a renormalization. □

To handle the other cases, we shall now make full use of the one-dimensional nature of the problem. Notice that problem (12.13) can be viewed as a linear second order ordinary differential equation, thus $\phi^\varepsilon$ is a combination of any two other linear independent solutions of (12.13) with different boundary conditions.

We first need the following lemmas.

*Lemme* 12.6.1. Assume $m(0) > 0$ or $m(\delta) < 0$, or both. Then, there exists $\theta_\varepsilon \neq 0$ such that $\mu_\varepsilon = \nu_{\theta_\varepsilon} = \nu_{-\theta_\varepsilon}$ where $\nu_\theta$ is the first eigenvalue of (12.16).

*Démonstration.* According to Lemma 12.5.1 we have $\mu_\varepsilon < 0$ since either $m(0) > 0$ or $m(\delta) < 0$. Proposition 12.2.1, applied to the selfadjoint case (12.16), tells us that $(\max_\theta \nu_\theta) = \nu_0 = 0$ and thus the range of $\nu_\theta$ is $\mathbf{R}^-$. Therefore, there exists $\theta_\varepsilon \neq 0$ such that $\mu_\varepsilon = \nu_{\theta_\varepsilon} = \nu_{-\theta_\varepsilon}$. □

*Lemme* 12.6.2. Suppose $m(0) > 0$ or $m(\delta) < 0$, or both. Then,

$$\phi^\varepsilon(0) + \phi^\varepsilon(1) \leq \frac{C}{\varepsilon}\|\phi_\varepsilon\|_{L^1(\Omega)}.$$

*Remarque* 12.6.2. Note that in the case of constant coefficients, $\phi_0(\cdot/\varepsilon)$ would be the form $\exp(-B\cdot/\varepsilon)$, and $\|\exp(-B\cdot/\varepsilon)\|_{L^1(\Omega)} \leq \varepsilon/B$, so in this sense this estimate is sharp.

*Démonstration.* Integrating by part (12.13) against $D_\varepsilon(x) = \int_0^x d\left(\frac{z}{\varepsilon}\right)^{-1} dz$ shows that

$$\varepsilon\mu_\varepsilon \int_0^1 s\left(\frac{x}{\varepsilon}\right) D_\varepsilon(x)\phi^\varepsilon dx = m(\delta)\phi^\varepsilon(1)D_\varepsilon(1) + \varepsilon\int_0^1 \nabla\phi^\varepsilon dx = m(\delta)\phi^\varepsilon(1)D_\varepsilon(1) + \varepsilon(\phi^\varepsilon(1) - \phi^\varepsilon(0)),$$

since the left hand side is negative and since $-C' < \varepsilon^2\mu_\varepsilon < -C$ by Proposition 12.5.2, we obtain

$$0 \le (-m(\delta)\int_0^1 d\left(\frac{x}{\varepsilon}\right)^{-1} dx - \varepsilon)\phi^\varepsilon(1) + \varepsilon\phi^\varepsilon(0)) \le \frac{C}{\varepsilon}\|\phi_\varepsilon\|_{L^1(\Omega)},$$

thus, if $m(\delta) < 0$, $\phi^\varepsilon(1) \le \frac{C}{\varepsilon}\|\phi_\varepsilon\|_{L^1(\Omega)}$. Symmetrically, integrating by part (12.13) against $D_\varepsilon(x) = \int_x^1 d\left(\frac{z}{\varepsilon}\right)^{-1} dz$ we obtain

$$\varepsilon\mu_\varepsilon \int_0^1 s\left(\frac{x}{\varepsilon}\right) D_\varepsilon(x)\phi^\varepsilon dx = -m(0)\phi^\varepsilon(0)D_\varepsilon(0) - \varepsilon\int_0^1 \nabla\phi^\varepsilon dx,$$

so, if $m(0) > 0$, we deduce $\phi^\varepsilon(0) \le \frac{C}{\varepsilon}\|\phi_\varepsilon\|_{L^1(\Omega)}$. Therefore, when either $m(0) > 0$ or $m(\delta) < 0$, or both, we obtain

$$\phi^\varepsilon(0) + \phi^\varepsilon(1) \le \frac{C}{\varepsilon}\|\phi_\varepsilon\|_{L^1(\Omega)},$$

□

*Lemme* 12.6.3. The first eigencouple $(\nu_\theta, \varphi^t_\theta)$ of (12.16) is real analytic as function of $\theta \in \mathbf{R}$ with values in $\mathbf{R} \times L^2(Y)$. If the sequence $\theta_\varepsilon$, defined in Lemma 12.6.1, converges to a limit $\theta_t$, then the eigenfunction $\varphi^t_{\theta_\varepsilon}$ can be expanded as follows

$$||\varphi^t_{\theta_\varepsilon}(y) - \varphi^t_{\theta_t}(y) - (\theta_\varepsilon - \theta_t)v_{\theta_t}(y)||_{L^\infty(Y)} = O((\theta_\varepsilon - \theta_t)^2) \tag{12.41}$$

where the function $v_{\theta_t} \in L^2(Y)$ is defined by (12.44), and

$$d(t-1)\nabla v_{\theta_t}(t-1) + m(t-1)v_{\theta_t}(t-1) \neq 0. \tag{12.42}$$

*Remarque* 12.6.3. Recall that, according to Proposition 12.3.3, $\phi_t = \varphi^t_{\theta_t}$.

*Démonstration.* The analyticity property is well-known by changing the unknown $\varphi^t_\theta$ into $\tilde\varphi^t_\theta = e^{-\theta y}\varphi^t_\theta$ which is a 1-periodic function, defined in a space independent of $\theta$, satisfying an elliptic equation with coefficients that depend quadratically on $\theta$. The variational formulation for $\tilde\varphi^t_\theta$ is

$$\int_Y d(y)(\nabla\tilde\varphi^t_\theta + \theta\tilde\varphi^t_\theta)(\nabla\tilde\phi - \theta\tilde\phi) = \nu_\theta\int_Y s(y)\tilde\varphi^t_\theta\tilde\phi, \tag{12.43}$$

for any 1-periodic test function $\tilde\phi \in H^1(Y)$. We conclude using Kato's Theorem [9] to prove the analyticity of the eigenvector $\tilde\varphi^t_\theta$. Since $\varphi^t_\theta = e^{\theta y}\tilde\varphi^t_\theta$, (12.41) holds in the $L^\infty$ norm by Sobolev embedding. To characterize the function $v_{\theta_t}$ we differentiate (12.43) with respect to $\theta$ and obtain for the value $\theta_t$

$$\int_Y d(y)(\nabla\tilde v_{\theta_t} + \theta_t\tilde v_{\theta_t})(\nabla\tilde\phi - \theta_t\tilde\phi) + \int_Y [d(y)\tilde\varphi^t_{\theta_t}(\nabla\tilde\phi - \theta_t\tilde\phi) - d(y)(\nabla\tilde\varphi^t_{\theta_t} + \theta_t\tilde\varphi^t_{\theta_t})\tilde\phi]$$
$$= \frac{d\nu}{d\theta}(\theta_t)\int_Y s(y)\tilde\varphi^t_{\theta_t}\tilde\phi + \nu_{\theta_t}\int_Y s(y)\tilde v_{\theta_t}\tilde\phi.$$

Introducing the test function $\phi = e^{-\theta_t y}\tilde\phi$ and defining $v_{\theta_t} = e^{-\theta_t y}\tilde v_{\theta_t}$ we deduce

$$\int_Y d(y)\nabla v_{\theta_t}\nabla\phi + \int_Y [d(y)\varphi^t_{\theta_t}\nabla\phi - d(y)\nabla\varphi^t_{\theta_t}\phi] = \frac{d\nu}{d\theta}(\theta_t)\int_Y s(y)\varphi^t_{\theta_t}\phi + \nu_{\theta_t}\int_Y s(y)v_{\theta_t}\phi. \tag{12.44}$$

To prove (12.42), we argue by contradiction. Assume $d(t-1)\nabla v_{\theta_t}(t-1) + m(t-1)v_{\theta_t}(t-1) = 0$. Since $v_{\theta_t}(t-1) = 0$, it implies that $\nabla v_{\theta_t}(t-1) = 0$. As a consequence, the 1-periodic function $\tilde v_{\theta_t} = e^{-\theta_t y}v_{\theta_t}$ satisfies the following boundary conditions

$$\tilde v_{\theta_t}(t-1) = \tilde v_{\theta_t}(t) = 0 \quad \text{and} \quad \nabla\tilde v_{\theta_t}(t-1) = \nabla\tilde v_{\theta_t}(t) = 0.$$

Returning back to the function $v_{\theta_t}$ we deduce

$$v_{\theta_t}(t-1) = v_{\theta_t}(t) = 0 \quad \text{and} \quad \nabla v_{\theta_t}(t-1) = \nabla v_{\theta_t}(t) = 0.$$

In other words, $v_{\theta_t}$ is solution of the over-determined boundary value problem

$$\begin{cases} -\mathrm{div}_y\left(d(y)\nabla_y v_{\theta_t}\right) - \nu_{\theta_t} s(y) v_{\theta_t} = \mathrm{div}_y(d(y)\varphi^t_{\theta_t}) + d(y)\nabla_y\varphi^t_{\theta_t} + \frac{d\nu}{d\theta}(\theta_t)s(y)\varphi^t_{\theta_t} \\ v_{\theta_t}(t-1) = v_{\theta_t}(t) = 0 \\ \nabla_y v_{\theta_t}(t-1) = \nabla_y v_{\theta_t}(t) = 0 \end{cases}$$

Multiplying the above equation by $\varphi^t_{\theta_t}$, integrating two times by parts (without any boundary contribution) and using the spectral equation satisfied by $\varphi^t_{\theta_t}$, we deduce

$$\frac{d\nu}{d\theta}(\theta_t)\int\limits_Y s(y)|\varphi^t_{\theta_t}|^2 = 0, \quad \text{that is,} \quad \frac{d\nu}{d\theta}(\theta_t) = 0,$$

which leads to a contradiction since $\theta \to \nu(\theta)$ is strictly concave and the only root of $\frac{d\nu}{d\theta}(\theta) = 0$ is $\theta = 0$. □

We are now in a position to evaluate how close the solution $\phi^\varepsilon$ is to a linear combination of $\varphi_{\pm\theta_\varepsilon}$. Recall that Proposition 12.5.3 implies that the only possible limits of the sequence $\theta_\varepsilon$ is $\theta_0$ or $\theta_\delta$.

*Proposition* 12.6.2. Suppose $m(0) > 0$, or $m(\delta) < 0$, or both.

(*i*) If $\mu_0 \neq \mu_\delta$ and $\theta_\varepsilon \to \theta_0$, we have

$$\left|\phi^\varepsilon(x) - \left(\frac{\phi^\varepsilon(0)}{\varphi^0_{\theta_\varepsilon}(0)}\varphi^0_{\theta_\varepsilon}\left(\frac{x}{\varepsilon}\right) - \frac{\phi^\varepsilon(0)}{\varphi^0_{\theta_\varepsilon}(0)}e^{2\theta_\varepsilon n}K(\delta)\varphi^0_{-\theta_\varepsilon}\left(\frac{x}{\varepsilon}\right)\right)\right| \leq \frac{C}{\varepsilon}\|\phi_\varepsilon\|_{L^1(\Omega)}e^{2\theta_0/\varepsilon}$$

and

$$|\mu_\varepsilon - \mu_0| = \gamma_0 e^{2\theta_0/\varepsilon}(1 + o(1)), \tag{12.45}$$

with

$$\gamma_0 = \left|\left(\frac{K(\delta)}{k(0)}\right)\frac{d(0)\nabla\varphi^0_{-\theta_0}(0)\phi_0(0) + m(0)\varphi^0_{-\theta_0}(0)\phi_0(0)}{\int\limits_Y s(y)\phi_0\varphi^0_{-\theta_0}}\right|. \tag{12.46}$$

(*ii*) If $\mu_0 \neq \mu_\delta$ and $\theta_\varepsilon \to \theta_\delta$, we have

$$\left|\phi^\varepsilon(x) - \left(\frac{\phi^\varepsilon(1)}{\varphi^\delta_{\theta_\varepsilon}\left(\frac{1}{\varepsilon}\right)}\varphi^\delta_{\theta_\varepsilon}\left(\frac{x}{\varepsilon}\right) - \frac{\phi^\varepsilon(1)}{\varphi^\delta_{\theta_\varepsilon}\left(\frac{1}{\varepsilon}\right)}K(0)\varphi^\delta_{-\theta_\varepsilon}\left(\frac{x}{\varepsilon}\right)\right)\right| \leq \frac{C}{\varepsilon}\|\phi_\varepsilon\|_{L^1(\Omega)}e^{-2\theta_\delta/\varepsilon},$$

and

$$|\mu_\varepsilon - \mu_\delta| = \gamma_1 e^{-2\theta_\delta/\varepsilon}(1 + o(1)). \tag{12.47}$$

with

$$\gamma_1 = \left|\left(\frac{K(0)}{k(\delta)}\right)\frac{d(\delta)\nabla\varphi^\delta_{-\theta_\delta}(\delta)\phi_\delta(\delta) + m(\delta)\varphi^\delta_{-\theta_\delta}(\delta)\phi_\delta(\delta)}{\int\limits_Y s(y)\phi_\delta\varphi^\delta_{-\theta_\delta}}\right|. \tag{12.48}$$

(*iii*) If $\mu_0 = \mu_\delta$, we have

$$\left|\phi^\varepsilon(x) - \left(\frac{\phi^\varepsilon(0)}{\varphi^0_{\theta_\varepsilon}(0)}\varphi^0_{\theta_\varepsilon}\left(\frac{x}{\varepsilon}\right) + \frac{\phi^\varepsilon(0)}{\varphi^0_{\theta_\varepsilon}(0)}c_\delta e^{\theta_\varepsilon n}\varphi^\delta_{-\theta_\varepsilon}\left(\frac{x}{\varepsilon}\right)\right)\right| \leq \frac{C}{\varepsilon}\|\phi_\varepsilon\|_{L^1(\Omega)}e^{\theta_0/\varepsilon},$$

with

$$c_\delta = -\sqrt{-\left(\frac{d(0)\nabla v_{\theta_0}(0) + m(0)v_{\theta_0}(0)}{d(\delta)\nabla v_{\theta_\delta}(\delta) + m(\delta)v_{\theta_\delta}(\delta)}\right)\left(\frac{d(\delta)\nabla\phi_0(\delta) + m(\delta)\phi_0(\delta)}{d(0)\nabla\phi_\delta(0) + m(0)\phi_\delta(0)}\right)} \tag{12.49}$$

and

$$|\mu_\varepsilon - \mu_0| = \gamma_\delta e^{\theta_0/\varepsilon}(1+o(1)), \tag{12.50}$$

with

$$\gamma_\delta = \left| \sqrt{-\frac{k(\delta)}{k(0)}}\, \frac{d(0)\nabla\phi_\delta(0)\phi_0(0) + m(0)\phi_\delta(0)\phi_0(0)}{\int\limits_Y s(y)\phi_0\phi_\delta} \right|. \tag{12.51}$$

We used the following notations

$$K(\delta) = \frac{d(\delta)\nabla\phi_0(\delta) + m(\delta)\phi_0(\delta)}{d(\delta)\nabla\varphi^0_{-\theta_0}(\delta) + m(\delta)\varphi^0_{-\theta_0}(\delta)},\ K(0) = \frac{d(0)\nabla\phi_\delta(0) + m(0)\phi_\delta(0)}{d(0)\nabla\varphi^\delta_{-\theta_\delta}(0) + m(0)\varphi^\delta_{-\theta_\delta}(0)},$$

and

$$k(0) = \frac{d(0)\nabla v_{\theta_0}(0) + m(0)v_{\theta_0}(0)}{d(0)\nabla\varphi_{-\theta_0}(0) + m(0)\varphi_{-\theta_0}(0)} \quad k(\delta) = \frac{d(\delta)\nabla\varphi^\delta_{-\theta_\delta}(\delta) + m(\delta)\varphi^\delta_{-\theta_\delta}(\delta)}{d(\delta)\nabla v_{-\theta_\delta}(\delta) + m(\delta)v_{-\theta_\delta}(\delta)}.$$

Proposition 12.6.2 provides a detailed description of the first order correctors for the first eigenpair. The following corollary limit the results of Proposition 12.6.2 to the leading order term. This highlights the main trend of the first eigenvectors, at the cost of an exponentially small loss of accuracy. The case when a double localization occurs is a limit case when zero and first order terms are of the same strength. In that case, characterizing the main trend means calculating first order correctors.

*Corollaire* 12.6.1. Suppose $\theta_0 > 0$, or $\theta_\delta < 0$, or both. Let $\tilde\varphi^t_\theta$ be the positive, bounded and $Y$-periodic function given by $\tilde\varphi^t_\theta = e^{-\theta y}\varphi^t_\theta$ where $\varphi^t_\theta$ is the first eigenfunction of (12.16).

(*i*) If $\mu_0 \neq \mu_\delta$, the first eigenvector localize in one of the endpoints. Indeed when $\theta_0 < 0$ and either $\theta_\delta \leq 0$, or $\theta_\delta > 0$ and $\mu_0 < \mu_\delta$, we have $L = \mu_0$ and

$$\left\| \phi^\varepsilon(x) - \phi^\varepsilon(0) e^{\frac{\theta_0 x}{\varepsilon}} \frac{\tilde\varphi^0_{\theta_0}\left(\frac{x}{\varepsilon}\right)}{\tilde\varphi^0_{\theta_0}(0)} \right\|_{L^\infty(\Omega)} \leq \frac{C}{\varepsilon} e^{\theta_0/\varepsilon} \|\phi^\varepsilon\|_{L^1(\Omega)} \text{ and } |\mu_\varepsilon - \mu_0| = \gamma_0 e^{2\theta_0/\varepsilon}(1+o(1))$$

where $\gamma_0$ is defined by (12.46).

Alternatively when $\theta_\delta > 0$ and either $\theta_0 \geq 0$, or $\theta_0 < 0$ and $\mu_\delta < \mu_0$, we have $L = \mu_\delta$ and

$$\left\| \phi^\varepsilon(x) - \phi^\varepsilon(1) e^{\frac{\theta_\delta(x-1)}{\varepsilon}} \frac{\tilde\varphi^\delta_{\theta_\delta}\left(\frac{x}{\varepsilon}\right)}{\tilde\varphi^\delta_{\theta_\delta}(\delta)} \right\|_{L^\infty(\Omega)} \leq \frac{C}{\varepsilon} e^{-\theta_\delta/\varepsilon} \|\phi^\varepsilon\|_{L^1(\Omega)} \text{ and } |\mu_\varepsilon - \mu_\delta| = \gamma_1 e^{-2\theta_\delta/\varepsilon}(1+o(1))$$

where $\gamma_1$ is defined by (12.48).

(*ii*) If $\mu_0 = \mu_\delta$, then the eigenvector could mix both boundary layers. We obtain

$$\begin{aligned} & \left\| \phi^\varepsilon(x) - \frac{\phi^\varepsilon(0)}{\tilde\varphi^0_{\theta_0}(0)} e^{\frac{\theta_0 x}{\varepsilon}} \tilde\varphi^0_{\theta_0}\left(\frac{x}{\varepsilon}\right) - \frac{\phi^\varepsilon(0)}{\tilde\varphi^0_{\theta_0}(0)} c_\delta e^{\frac{\theta_0}{\varepsilon}} e^{\frac{\theta_\delta x}{\varepsilon}} \tilde\varphi^\delta_{\theta_\delta}\left(\frac{x}{\varepsilon}\right) \right\|_{L^\infty(\Omega)} \\ = & \left\| \phi^\varepsilon(x) - \frac{\phi^\varepsilon(0)}{\tilde\varphi^0_{\theta_0}(0)} e^{\frac{\theta_0 x}{\varepsilon}} \tilde\varphi^0_{\theta_0}\left(\frac{x}{\varepsilon}\right) - \frac{\phi^\varepsilon(0)}{\tilde\varphi^0_{\theta_0}(0)} c_\delta e^{\frac{\theta_\delta(x-1)}{\varepsilon}} \tilde\varphi^\delta_{\theta_\delta}\left(\frac{x}{\varepsilon}\right) \right\|_{L^\infty(\Omega)} \leq C \frac{e^{\theta_0/\varepsilon}}{\varepsilon} \|\phi^\varepsilon\|_{L^1(\Omega)} \end{aligned}$$

and

$$|\mu_\varepsilon - \mu_0| = \gamma_\delta e^{\theta_0/\varepsilon}(1+o(1))$$

where $\gamma_\delta$ is defined by (12.51) and $c_\delta$ is defined by (12.49). Note that in this last case $\theta_0 = -\theta_\delta < 0$.

*Proof of Corollary 12.6.1.* To prove this corollary starting from Proposition 12.6.2, we notice that, when by Proposition 12.5.3, if $\theta_0 < 0$ and $\theta_\delta \leq 0$, or if $\theta_0 < 0$, $\theta_\delta > 0$ and $\mu_0 < \mu_\delta$, we have $L = \mu_0$, $\theta_\varepsilon$ tends to $\theta_0 < 0$ and that $\theta_\varepsilon - \theta_0 = O(e^{2\theta_0/\varepsilon})$. This implies that

$$\varphi^0_{\theta_\varepsilon}\left(\frac{x}{\varepsilon}\right) = \phi_0\left(\frac{x}{\varepsilon}\right) + O(e^{2\theta_0/\varepsilon}), \quad {\varphi^0}^{-1}_{\theta_\varepsilon} = \phi_0^{-1} + O(e^{\theta_0/\varepsilon}), \text{ and } e^{2\theta_\varepsilon n} K(\delta) \varphi^0_{-\theta_\varepsilon}\left(\frac{x}{\varepsilon}\right) = O(e^{\theta_0/\varepsilon}).$$

Since

$$\frac{\phi_0\left(\frac{x}{\varepsilon}\right)}{\phi_0(0)} = e^{\frac{\theta_0 x}{\varepsilon}} \frac{\tilde\varphi^0_{\theta_0}\left(\frac{x}{\varepsilon}\right)}{\tilde\varphi^0_{\theta_0}(0)}, \tag{12.52}$$

we have proved the first estimate.

In a same way, when either $\theta_0 \geq 0$ and $\theta_\delta > 0$, or $\theta_0 < 0$, $\theta_\delta > 0$ and $\mu_\delta < \mu_0$, by Proposition 12.5.3, $L = \mu_\delta$, $\theta_\varepsilon$ tends to $\theta_\delta > 0$, and $\theta_\varepsilon - \theta_\delta = O(e^{-2\theta_\delta/\varepsilon})$. We then obtain

$$\varphi^\delta_{\theta_\varepsilon} = \phi_\delta + O(e^{-2\theta_\delta/\varepsilon}), \quad {\varphi^\delta}^{-1}_{\theta_\varepsilon} = \phi_\delta^{-1} + O(e^{-\theta_\delta/\varepsilon}), \text{ and } \frac{K(0)}{\varphi^\delta_{\theta_\varepsilon}\left(\frac{1}{\varepsilon}\right)} \varphi^\delta_{-\theta_\varepsilon}\left(\frac{x}{\varepsilon}\right) = O(e^{-\theta_\delta/\varepsilon}).$$

As before, we write

$$\frac{\phi_\delta\left(\frac{x}{\varepsilon}\right)}{\phi_\delta\left(\frac{1}{\varepsilon}\right)} = e^{\frac{\theta_\delta(x-1)}{\varepsilon}} \frac{\tilde\varphi^\delta_{\theta_\delta}\left(\frac{x}{\varepsilon}\right)}{\tilde\varphi^\delta_{\theta_\delta}\left(\frac{1}{\varepsilon}\right)} = e^{\frac{\theta_\delta(x-1)}{\varepsilon}} \frac{\tilde\varphi^\delta_{\theta_\delta}\left(\frac{x}{\varepsilon}\right)}{\tilde\varphi^\delta_{\theta_\delta}(\delta)}. \tag{12.53}$$

Finally, when $\mu_0 = \mu_\delta$, $m(0) > 0$ and $m(\delta) < 0$, $\theta_\varepsilon$ tends to $\theta_0 < 0$ and $\theta_\varepsilon - \theta_0 = O(e^{\theta_0/\varepsilon})$. This implies that $\exp(\theta_\varepsilon n) = \exp(\theta_0 n)(1 + o(1))$, and therefore that

$$\varphi^0_{\theta_\varepsilon}\left(\frac{x}{\varepsilon}\right) = \phi_0\left(\frac{x}{\varepsilon}\right) + O(e^{\theta_0/\varepsilon}) \quad \text{and} \quad \varphi^\delta_{-\theta_\varepsilon}\left(\frac{x}{\varepsilon}\right) = \phi_\delta\left(\frac{x}{\varepsilon}\right) + O(e^{\theta_0/\varepsilon}).$$

Together with the observation that

$$\varphi^0_{\theta_\varepsilon}(0) = \phi_0(0)(1 + O(e^{2\theta_0/\varepsilon})),$$

this shows that

$$\frac{1}{\varphi^0_{\theta_\varepsilon}(0)} = \frac{1}{\phi_0(0)} + O(e^{2\theta_0/\varepsilon}).$$

This allows us to conclude.
□

*Proof of Proposition 12.6.2.* Since $\varphi^{t_1}_{+\theta_\varepsilon}(y)$ and $\varphi^{t_2}_{-\theta_\varepsilon}(y)$ are linearly independent solutions of (12.13), we have

$$\phi^\varepsilon = \alpha^\varepsilon \varphi^{t_1}_{\theta_\varepsilon}\left(\frac{x}{\varepsilon}\right) + \beta^\varepsilon \varphi^{t_2}_{-\theta_\varepsilon}\left(\frac{x}{\varepsilon}\right).$$

Inserting the boundary conditions of problem (12.13), the existence of of a non trivial pair $(\alpha^\varepsilon, \beta^\varepsilon)$ implies

$$\left| \begin{array}{ll} d(0)\nabla\varphi^{t_1}_{\theta_\varepsilon}(0) + m(0)\varphi^{t_1}_{\theta_\varepsilon}(0) & d(0)\nabla\varphi^{t_2}_{-\theta_\varepsilon}(0) + m(0)\varphi^{t_2}_{-\theta_\varepsilon}(0) \\ \left(d(\delta)\nabla\varphi^{t_1}_{\theta_\varepsilon}(\delta) + m(\delta)\varphi^{t_1}_{\theta_\varepsilon}(\delta)\right)\frac{\varphi^{t_1}_{\theta_\varepsilon}(\varepsilon^{-1})}{\varphi^{t_1}_{\theta_\varepsilon}(\delta)} & \left(d(\delta)\nabla\varphi^{t_2}_{-\theta_\varepsilon}(\delta) + m(\delta)\varphi^{t_2}_{-\theta_\varepsilon}(\delta)\right)\frac{\varphi^{t_2}_{-\theta_\varepsilon}(\varepsilon^{-1})}{\varphi^{t_2}_{-\theta_\varepsilon}(\delta)} \end{array} \right| = 0.$$

This identity can also be written as

$$\begin{aligned} \frac{d(0)\nabla\varphi^{t_1}_{\theta_\varepsilon}(0) + m(0)\varphi^{t_1}_{\theta_\varepsilon}(0)}{d(0)\nabla\varphi^{t_2}_{-\theta_\varepsilon}(0) + m(0)\varphi^{t_2}_{-\theta_\varepsilon}(0)} &= \frac{d(\delta)\nabla\varphi^{t_1}_{\theta_\varepsilon}(\delta) + m(\delta)\varphi^{t_1}_{\theta_\varepsilon}(\delta)}{d(\delta)\nabla\varphi^{t_2}_{-\theta_\varepsilon}(\delta) + m(\delta)\varphi^{t_2}_{-\theta_\varepsilon}(\delta)} \frac{\varphi^{t_1}_{\theta_\varepsilon}(\varepsilon^{-1})}{\varphi^{t_1}_{\theta_\varepsilon}(\delta)} \frac{\varphi^{t_2}_{-\theta_\varepsilon}(\delta)}{\varphi^{t_2}_{-\theta_\varepsilon}(\varepsilon^{-1})} \\ &= \frac{d(\delta)\nabla\varphi^{t_1}_{\theta_\varepsilon}(\delta) + m(\delta)\varphi^{t_1}_{\theta_\varepsilon}(\delta)}{d(\delta)\nabla\varphi^{t_2}_{-\theta_\varepsilon}(\delta) + m(\delta)\varphi^{t_2}_{-\theta_\varepsilon}(\delta)} e^{2\theta_\varepsilon n}, \end{aligned} \tag{12.54}$$

the second relation being a consequence of the relation

$$\frac{\varphi^{t_1}_{\theta_\varepsilon}(\varepsilon^{-1})}{\varphi^{t_1}_{\theta_\varepsilon}(\delta)} \frac{\varphi^{t_2}_{-\theta_\varepsilon}(\delta)}{\varphi^{t_2}_{-\theta_\varepsilon}(\varepsilon^{-1})} = \frac{\varphi^{t_1}_{\theta_\varepsilon}(n+\delta)}{\varphi^{t_1}_{\theta_\varepsilon}(\delta)} \frac{\varphi^{t_2}_{-\theta_\varepsilon}(\delta)}{\varphi^{t_2}_{-\theta_\varepsilon}(n+\delta)} = e^{2\theta_\varepsilon n}.$$

At $x = 0$, we obtain the following additional relation

$$\alpha^\varepsilon \varphi^{t_1}_{\theta_\varepsilon}(0) + \beta^\varepsilon \varphi^{t_2}_{-\theta_\varepsilon}(0) = \phi^\varepsilon(0).$$

The key point of the proof will be the computation of $\alpha^\varepsilon$ and $\beta^\varepsilon$.

We will now consider three cases. In the first one, $\theta_\varepsilon$ tends to $\theta_0$, with $\theta_0 < 0$, and $\mu_0 \neq \mu_\delta$. In the second one, $\theta_\varepsilon$ tends to $\theta_\delta$, with $\theta_\delta > 0$, and $\mu_0 \neq \mu_\delta$. Finally, we will consider the limit case when $\theta_\varepsilon$ tends to $\theta_0$, with $\theta_0 < 0$, and $\mu_0 = \mu_\delta$. Proposition 12.5.3 shows that these are the only possible cases when concentration occurs.

**Case 1.** Assume that $\mu_0 \neq \mu_\delta$ and $\theta_\varepsilon$ tends to $\theta_0$, with $\theta_0 < 0$. This implies that $\varphi^0_{\theta_\varepsilon}$ tends to $\phi_0$. Define $\eta := \theta_\varepsilon - \theta_0$. Thanks to Lemma 12.6.3, the following first order expansions in $\eta$ hold

$$\begin{aligned} \varphi^0_{\theta_\varepsilon}(y) &= \phi_0(y) + \eta v_{\theta_0}(y) + O(\eta^2) \\ \varphi^0_{-\theta_\varepsilon}(y) &= \varphi_{-\theta_0}(y) + O(\eta). \end{aligned}$$

Inserting this ansatz in (12.54), we obtain

$$\frac{\eta\left(d(0)\nabla v_{\theta_0}(0) + m(0)v_{\theta_0}(0)\right) + O(\eta^2)}{d(0)\nabla\varphi^0_{-\theta_0}(0) + m(0)\varphi^0_{-\theta_0}(0) + O(\eta)} = \frac{d(\delta)\nabla\phi_0(\delta) + m(\delta)\phi_0(\delta) + O(\eta)}{d(\delta)\nabla\varphi^0_{-\theta_0}(\delta) + m(\delta)\varphi^0_{-\theta_0}(\delta) + O(\eta)} e^{2\theta_\varepsilon n}.$$

Note that $d(\delta)\nabla\varphi^0_{-\theta_0}(\delta) + m(\delta)\varphi^0_{-\theta_0}(\delta) \neq 0$, as this would imply $\mu_0 = \mu_\delta$, which we assume does not hold. Thanks to Lemma 12.6.3 we know that, $d(0)\nabla v_{\theta_0}(0) + m(0)v_{\theta_0}(0) \neq 0$, therefore we can write

$$\eta = \frac{d(\delta)\nabla\phi_0(\delta) + m(\delta)\phi_0(\delta)}{d(\delta)\nabla\varphi^0_{-\theta_0}(\delta) + m(\delta)\varphi^0_{-\theta_0}(\delta)} \frac{d(0)\nabla\varphi^0_{-\theta_0}(0) + m(0)\varphi^0_{-\theta_0}(0)}{d(0)\nabla v_{\theta_0}(0) + m(0)v_{\theta_0}(0)} e^{2\theta_\varepsilon n} + o(e^{2\theta_\varepsilon n}). \tag{12.55}$$

This provides a first order correction (in exponential terms) for $\theta_\varepsilon$. This value of $\eta$ allows us to compute $\alpha^\varepsilon$ and $\beta^\varepsilon$, namely

$$\begin{aligned} \beta^\varepsilon &= -\eta\alpha^\varepsilon \frac{d(0)\nabla v_{\theta_0}(0) + m(0)v_{\theta_0}(0)}{d(0)\nabla\varphi^0_{-\theta_0}(0) + m(0)\varphi^0_{-\theta_0}(0)} + O(\alpha^\varepsilon\eta^2) \\ \alpha^\varepsilon &= \frac{\phi^\varepsilon(0)}{\varphi^0_{\theta_\varepsilon}(0)} + \phi^\varepsilon(0)O(\eta). \end{aligned}$$

Turning now to the solution $\phi^\varepsilon$, we have obtained

$$\begin{aligned} \phi^\varepsilon &= \alpha^\varepsilon\varphi^0_{\theta_\varepsilon}\left(\frac{x}{\varepsilon}\right) + \beta^\varepsilon\varphi^0_{-\theta_\varepsilon}\left(\frac{x}{\varepsilon}\right), \\ &= \frac{\phi^\varepsilon(0)}{\varphi^0_{\theta_\varepsilon}(0)}\varphi^0_{\theta_\varepsilon}\left(\frac{x}{\varepsilon}\right) - \frac{\phi^\varepsilon(0)}{\varphi^0_{\theta_\varepsilon}(0)} \frac{d(\delta)\nabla\phi_0(\delta) + m(\delta)\phi_0(\delta)}{d(\delta)\nabla\varphi^0_{-\theta_0}(\delta) + m(\delta)\varphi^0_{-\theta_0}(\delta)} e^{2\theta_\varepsilon n}\varphi^0_{-\theta_\varepsilon}\left(\frac{x}{\varepsilon}\right) + \phi^\varepsilon(0)O\left(e^{2\theta_\varepsilon n}\right). \end{aligned}$$

Using Lemma 12.6.2, the proof of the asymptotic formula for the eigenvector is complete. Let us now turn to the eigenvalue. Testing (12.44) against $\phi_\theta = \varphi^0_{-\theta_0}$, we obtain

$$\int_Y \left(d(y)\phi_0\nabla\varphi^0_{-\theta_0} - d(y)\nabla\phi_0\varphi^0_{-\theta_0}\right) dy = \frac{d\mu}{d\theta}(\theta_0) \int_Y s(y)\phi_0\varphi^0_{-\theta_0} dy. \tag{12.56}$$

Note that the wronskian $d\phi_0\nabla\varphi^0_{-\theta_0} - d\nabla\phi_0\varphi^0_{-\theta_0}$ is a constant, therefore

$$\int_Y d(y)\left(\phi_0\nabla\varphi^0_{-\theta_0} - \nabla\phi_0\varphi^0_{-\theta_0}\right) dy = d(0)\nabla\varphi^0_{-\theta_0}(0) + m(0)\varphi^0_{-\theta_0}(0).$$

Thanks to Lemma 12.6.3, $\nu_\theta$ is analytic, with $\nu_{\theta_0} = \mu_0$ and $\nu_{\theta_\varepsilon} = \mu_\varepsilon$. We write

$$\mu_\varepsilon = \mu_0 + \eta\frac{d\mu}{d\theta}(\theta_0) + O(\eta^2).$$

and inserting (12.55) and (12.56) we obtain

$$\mu_\varepsilon = \mu_0 + \frac{K(\delta)}{k(0)} \frac{d(0)\nabla\varphi^0_{-\theta_0}(0) + m(0)\varphi^0_{-\theta_0}(0)}{\int\limits_Y s(y)\phi_0\varphi^0_{-\theta_0}} e^{2\theta_0/\varepsilon}(1+o(1))$$

which is (12.45).

**Case 2.** If $\mu_0 \neq \mu_\delta$ and $\theta_\varepsilon$ tends to $\theta_\delta$, then $\varphi^\delta_{\theta_\varepsilon}$ tends to $\phi_\delta$. The same strategy and similar arguments shows that

$$\eta = \frac{d(0)\nabla\phi_\delta(0) + m(0)\phi_\delta(0)}{d(0)\nabla\varphi^\delta_{-\theta_\delta}(0) + m(0)\varphi^\delta_{-\theta_\delta}(0)} \frac{d(\delta)\nabla\varphi^\delta_{-\theta_\delta}(\delta) + m(\delta)\varphi^\delta_{-\theta_\delta}(\delta)}{d(\delta)\nabla v_{\theta_\delta}(\delta) + m(\delta)v_{\theta_\delta}(\delta)} e^{-2\theta_\varepsilon n} + o(e^{-2\theta_\varepsilon n}),$$

and, in turn, using $\phi^\varepsilon(1) = \alpha^\varepsilon\varphi^\delta_{\theta_\varepsilon}\left(\frac{1}{\varepsilon}\right) + \beta^\varepsilon\varphi^\delta_{-\theta_\varepsilon}\left(\frac{1}{\varepsilon}\right)$ we obtain,

$$\beta^\varepsilon = -\eta\alpha^\varepsilon e^{2\theta_\varepsilon n} \frac{d(\delta)\nabla v_{\theta_\delta}(\delta) + m(\delta)v_{\theta_\delta}(\delta)}{d(\delta)\nabla\varphi^\delta_{-\theta_\delta}(\delta) + m(\delta)\varphi^\delta_{-\theta_\delta}(\delta)} + O(e^{2\theta_\varepsilon n}\alpha^\varepsilon\eta^2), \quad \text{and } \alpha^\varepsilon = \frac{\phi^\varepsilon(1)}{\varphi^\delta_{\theta_\varepsilon}\left(\frac{1}{\varepsilon}\right)} + \phi^\varepsilon(1)O(\eta).$$

This implies

$$\phi^\varepsilon = \frac{\phi^\varepsilon(1)}{\varphi^\delta_{\theta_\varepsilon}\left(\frac{1}{\varepsilon}\right)}\varphi^\delta_{\theta_\varepsilon}\left(\frac{x}{\varepsilon}\right) - \frac{\phi^\varepsilon(1)}{\varphi^\delta_{\theta_\varepsilon}\left(\frac{1}{\varepsilon}\right)} \frac{d(0)\nabla\phi_\delta(0) + m(0)\phi_\delta(0)}{d(0)\nabla\varphi^\delta_{-\theta_\delta}(0) + m(0)\varphi^\delta_{-\theta_\delta}(0)}\varphi^\delta_{-\theta_\varepsilon}\left(\frac{x}{\varepsilon}\right) + \phi^\varepsilon(0)O\left(e^{-2\theta_\varepsilon n}\right),$$

which is the announced result. The proof of (12.47) follows that of the first case.

**Case 3.** If $\mu_0 = \mu_\delta$, then $\phi_\delta = \varphi^\delta_{-\theta_0}$ and we can rewrite the expansion as follows

$$\begin{aligned} \varphi^0_{\theta_\varepsilon} &= \phi_0 + \eta v_{\theta_0} + O(\eta^2) \\ \varphi^\delta_{-\theta_\varepsilon} &= \phi_\delta - \eta v_{\theta_\delta} + O(\eta^2). \end{aligned}$$

In this case $\varphi^0_{\theta_0} = \phi_0$ satisfies the boundary condition at 0 whereas $\varphi^\delta_{-\theta_0} = \phi_\delta$ satisfies the boundary conditions at $\delta$, and equation (12.54) shows that

$$\frac{\eta^2\left(d(0)\nabla v_{\theta_0}(0) + m(0)v_{\theta_0}(0) + O(\eta)\right)}{d(0)\nabla\phi_\delta(0) + m(0)\phi_\delta(0) + O(\eta)} = -\frac{d(\delta)\nabla\phi_0(\delta) + m(\delta)\phi_0(\delta) + O(\eta)}{d(\delta)\nabla v_{\theta_\delta}(\delta) + m(\delta)v_{\theta_\delta}(\delta) + O(\eta)} e^{2\theta_\varepsilon n}.$$

Thus

$$\eta = \sqrt{-\frac{d(\delta)\nabla\phi_0(\delta) + m(\delta)\phi_0(\delta)}{d(\delta)\nabla v_{\theta_\delta}(\delta) + m(\delta)v_{\theta_\delta}(\delta)} \frac{d(0)\nabla\phi_\delta(0) + m(0)\phi_\delta(0)}{d(0)\nabla v_{\theta_0}(0) + m(0)v_{\theta_0}(0)}} e^{\theta_\varepsilon n} + o(e^{\theta_\varepsilon n}).$$

Following the same steps as in the first case, we obtain

$$\phi^\varepsilon(x) = \frac{\phi^\varepsilon(0)}{\varphi^0_{\theta_\varepsilon}(0)}\varphi^0_{\theta_\varepsilon}\left(\frac{x}{\varepsilon}\right) - \frac{\phi^\varepsilon(0)}{\varphi^0_{\theta_\varepsilon}(0)}\eta \frac{d(0)\nabla v_{\theta_0}(0) + m(0)v_{\theta_0}(0)}{d(0)\nabla\phi_\delta(0) + m(0)\phi_\delta(0)}\varphi^\delta_{-\theta_\varepsilon}\left(\frac{x}{\varepsilon}\right) + \phi^\varepsilon(0)O(e^{\theta_\varepsilon n}),$$

and finally

$$\phi^\varepsilon(x) = \frac{\phi^\varepsilon(0)}{\varphi^0_{\theta_\varepsilon}(0)}\varphi^0_{\theta_\varepsilon}\left(\frac{x}{\varepsilon}\right) + \frac{\phi^\varepsilon(0)}{\varphi^0_{\theta_\varepsilon}(0)}c_\delta e^{\theta_\varepsilon n}\varphi^\delta_{-\theta_\varepsilon}\left(\frac{x}{\varepsilon}\right) + \phi^\varepsilon(0)O(e^{\theta_\varepsilon n}),$$

with

$$\begin{aligned} c_\delta &= -\sqrt{-\frac{d(\delta)\nabla\phi_0(\delta) + m(\delta)\phi_0(\delta)}{d(\delta)\nabla v_{\theta_\delta}(\delta) + m(\delta)v_{\theta_\delta}(\delta)} \frac{d(0)\nabla\phi_\delta(0) + m(0)\phi_\delta(0)}{d(0)\nabla v_{\theta_0}(0) + m(0)v_{\theta_0}(0)}} \frac{d(0)\nabla v_{\theta_0}(0) + m(0)v_{\theta_0}(0)}{d(0)\nabla\phi_\delta(0) + m(0)\phi_\delta(0)} \\ &= \sqrt{-\frac{d(0)\nabla v_{\theta_0}(0) + m(0)v_{\theta_0}(0)}{d(\delta)\nabla v_{\theta_\delta}(\delta) + m(\delta)v_{\theta_\delta}(\delta)} \frac{d(\delta)\nabla\phi_0(\delta) + m(\delta)\phi_0(\delta)}{d(0)\nabla\phi_\delta(0) + m(0)\phi_\delta(0)}} \end{aligned}$$

as claimed. The proof of (12.50) follows that of the first case. □

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# Chapitre 13

# The optimal mass transport problem for relativistic costs

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**abstract**

In this paper, we study the optimal mass transportation problem in $\mathbf{R}^d$ for a class of cost functions that we call relativistic cost functions. Consider as a typical example, the cost function $c(x,y) = h(x-y)$ being the restriction of a strictly convex and differentiable function to a ball and infinite outside this ball. We show the existence and uniqueness of the optimal map given a relativisc cost function and two measures with compact support, one of the two being absolutely continuous with respect to the Lebesgue measure. With an additional assumption on the support of the initial measure and for supercritical speed of propagation, we also prove the existence of a Kantorovich potential and study the regularity of this map. Besides these general results, a particular attention is given to a specific cost because of its connections with a relativistic heat equation as pointed out by Brenier in [14].

## 13.1 Setting of the problem

In the last ten years, the Monge-Kantorovich problem has been widely investigated. So far, there was no general result that could be compared to the Gangbo-McCann theorem for strictly convex and real-valued cost functions on $\mathbf{R}^d$ [29]. In this paper, we provide the first result of this kind for a large class of strictly convex functions that are infinite outside a bounded convex set. The cost functions we consider are not even continuous maps from $\mathbf{R}^d$ to $[0,+\infty]$. We show that under quite standard assumptions including the compactness of the supports, there exists a unique optimal transport map relative to such a cost function. We refer to Theorem 13.2.6 for a precise statement. Among all these cost functions, a particular attention is given to a cost function that we call relativistic heat cost function. This relativistic heat cost, introduced by Brenier in [14], is related to a relativistic heat equation as explained below. In [14], Brenier mentions the question of the existence of an optimal map for this cost function, putting the emphasis on the case where only a part of the mass can be transported. This cost function was our initial motivation to study this kind of problems. Let us now describe more precisely the problem we are interested in.

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### 13.1.1 The general optimal transportation problem

Let us first recall the Monge problem.

Given two probability measures $\mu_0 \in \mathcal{P}(\mathbf{R}^d)$ and $\mu_1 \in \mathcal{P}(\mathbf{R}^d)$, we consider the maps $F$, if any, that pushes $\mu_0$ forward to $\mu_1$, meaning the maps that satisfy $F_{\#}\mu_0 = \mu_1$. In terms of test functions, the equation $F_{\#}\mu_0 = \mu_1$ amounts to

$$\int_{\mathbf{R}^d} \varphi(F(x))d\mu_0(x) = \int_{\mathbf{R}^d} \varphi(y)d\mu_1(y)$$

for any test function $\varphi \in C^0(\mathbf{R}^d)$.

Monge asked whether the infimum

$$\inf_{F_{\#}\mu_0=\mu_1} \int_{\mathbf{R}^d} c(x, F(x))d\mu_0(x)$$

is attained among the maps $F$ such that $F_{\#}\mu_0 = \mu_1$. The function $c$ above is called the cost function. Note that there might be no map that pushes $\mu_0$ forward to $\mu_1$ and that the Monge problem is nonlinear in general. This leads Kantorovich to consider a relaxation of the Monge problem, now called the Kantorovich problem [29]. In this problem, we consider transport plans instead of transport maps. A transport plan $\pi$ is a probability measure on $\mathbf{R}^d \times \mathbf{R}^d$ with first marginal $\mu_0$ and second marginal $\mu_1$. This means that $\mu_0(U) = \pi(U \times \mathbf{R}^d)$ and $\mu_1(U) = \pi(\mathbf{R}^d \times U)$ for any Borel set $U \subset \mathbf{R}^d$. Note that a transport map $F$ induces a transport plan $(Id, F)_{\#}\mu_0$. The set of transport plans between $\mu_0$ and $\mu_1$ is denoted by $\Pi(\mu_0, \mu_1)$.

The Kantorovich problem then consists in studying, given two probability measures $\mu_0$ and $\mu_1$ in $\mathcal{P}(\mathbf{R}^d)$, the optimal transport plans, namely the plans that achieve the minimum

$$\min_{\pi\in\Pi(\mu_0,\mu_1)} \int_{\mathbf{R}^d\times\mathbf{R}^d} c(x, y)d\pi(x, y).$$

Indeed, contrary to the Monge problem, the Kantorovich problem always admits solutions as soon as the cost function is a lower semi continuous and non-negative map, see [43, 40] for a proof. We refer to [15, 16, 28, 29, 43, 40] for existence and uniqueness of solution to the Monge-Kantorovich problem in the quadratic (or $L^2$) case and for strictly convex cost functions and to [26, 19, 4, 5, 22] for the study of the $L^1$ case.

### 13.1.2 The relativistic heat cost

The original motivation of this paper comes from what we call the relativistic heat cost given by $c(x, y) = h(x - y)$ where

$$h(z) = \begin{cases} 1 - \sqrt{1 - |z|^2} & |z| \leq 1 \\ \infty & |z| > 1. \end{cases} \tag{13.1}$$

The peculiarity of this cost function is that even if it is strictly convex, it is neither real-valued nor continuous and its gradient goes to infinity when $|z| \to 1$.

The name of this cost function comes from the fact that it prevents the displacement – or the velocity if we take the Lagrangian point of view – to be larger than one which corresponds to a relativistic behavior. This cost has been proposed by Brenier in [14] to obtain a relativistic heat equation as a gradient flow of the Boltzmann entropy (see [2, 3, 43] for references on the notion of gradient flows). Several Cauchy problems for PDE's has been studied via the JKO time discrete scheme [1, 30, 38]... For example, the heat equation can be obtained by computing the time discrete solution at the time step $i$ in the following way

$$\rho^i = \inf_{\rho\in P(\mathbf{R}^d)} \int \rho \log \rho + \varepsilon\, W^2_{2,\varepsilon}(\rho^{i-1}, \rho) \quad \text{with} \quad W^2_{2,\varepsilon}(\rho^{i-1}, \rho) = \min_{\pi\in\Pi(\rho^{i-1},\rho)} \int \left(\frac{|x - y|}{\varepsilon}\right)^2 d\pi(x, y) \tag{13.2}$$

and by passing to the limit when the time step $\varepsilon$ goes to zero. Analogously, a Cauchy result has been proved by R. McCann and the second author in [31] for the equation

$$\partial_t \rho = \operatorname{div}(\rho \frac{\frac{\nabla \rho}{\rho}}{\sqrt{1+|\frac{\nabla \rho}{\rho}|^2}}) = \operatorname{div}(\rho \nabla h^*(\nabla(\log \rho))) \tag{13.3}$$

replacing $W_{2,\varepsilon}^2$ in (13.2) by

$$W_c = \min_{\pi \in \Pi(\rho^{i-1},\rho)} \int h(\frac{x-y}{\varepsilon}) d\pi(x,y) \quad \text{where } c(x,y) = h(x-y) \text{ is the relativistic heat cost.}$$

Equation (13.3) is called relativistic heat equation since it can be seen as a relativistic version of the heat equation interpreted [37, 40] as a transport of mass equation with velocity $\dfrac{\nabla \rho}{\rho}$. It has previously been investigated via more PDE's oriented method in the series of papers [7, 9, 8, 10, 11, 12, 13, 19]. The proof of the main result of [31] already requires the existence of an optimal map relative to the problem $W_c$ with $\rho^{i-1}$ and $\rho^i$. This map is obtained by a regularization argument applied to the cost function and based on the Yosida transform (we refer to [18] for the definition). However, the limiting process is strongly based on the double minimization involved in the scheme, in particular the compactness of the sequence of maps comes from the fact that the target measure $\rho^i$ also minimizes the entropy. In this paper, we propose another method to prove the existence and uniqueness of the optimal map that goes beyond the particular setting described above.

The method we found applies to more general relativistic cost functions including the cost given by formula (13.1) and another important class of cost functions given by

$$c(x,y) = f(x-y) + \delta(x-y|\mathbf{C}) \tag{13.4}$$

where $\delta(\cdot|\mathbf{C})$ is the indicator function of a strictly convex body $\mathbf{C}$ containing $0$, namely $\delta(x|\mathbf{C}) = 0$ if $x \in \mathbf{C}$ and $+\infty$ otherwise and $f : \mathbf{R}^d \to \mathbf{R}^+$ is a strictly convex and differentiable function such that $f(0) = 0$.

The question of the existence of an optimal map for cost functions as in (13.4) has been mentioned in [20]. In that paper, Carlier, De Pascale, and Santambrogio propose a different strategy based on the existence of a maximizer in the Kantorovich dual problem. As they point out, the existence of such a maximizer is a delicate question. We are nonetheless able to prove its existence under additional assumptions on the initial measure. Note that the case where $f = |\cdot|^2$ is also considered in [27] with the weaker assumption that $\mathbf{C}$ is a convex body.

### 13.1.3 The parametrized optimal transportation problem for relativistic costs

In this work, we introduce an additional parameter called the speed (of light) which takes into account the relativistic behaviour of the cost functions we consider. To explain this, it is convenient to use a Lagrangian point of view, namely given $c(x,y) = h(x-y)$ a relativistic cost function, observe that

$$c(x,y) = \inf_{C_u} \int_0^1 h(\gamma'(s)) ds$$

where $C_u = \{\gamma : [0,1] \to \mathbf{R}^d; \gamma(0) = x, \gamma(1) = y \text{ and } \gamma \text{ is } C^1 \text{ on } (0,1)\}$.

When $h(x-y)$ is finite, we can only consider curves whose energy $\int_0^1 h(\gamma'(s)) ds$ is finite, in other terms whose speed $\gamma'$ is confined to $\mathbf{C}$. Thus, let us introduce the one-parameter family of cost functions

$$c_t(x,y) = h(\frac{x-y}{t})$$

for any positive number $t$.

Now, given two probability measures $\mu_0$ and $\mu_1$ with compact support, and a relativistic cost function, we can always change the speed of light $t$ so that the minimum in the Kantorovich problem

relative to $c_t$ is finite. The study of the variation of this minimum in terms of the speed also gives us some useful informations. Thus, let us define the total cost function

$$C(t) = \min_{\pi \in \Pi(\mu_0,\mu_1)} \int_{\mathbf{R}^{2d}} c(\frac{x-y}{t})\, d\pi(x,y).$$

The overall idea is that the total cost function should be infinite when $t$ is small (at least if $\mu_0 \neq \mu_1$), whereas when $t$ is large, the transportation plans should not be affected by the discontinuity of the cost function and we should get as in [29], existence and uniqueness of the optimal transport map. These properties are proved as part of a more general result in the next sections together with the existence of a critical speed of light $T$ defined below.

**Definition 13.1.1** (Critical speed)**.**

There exists a speed $T$ such that for any $t \geq T$, $C(t) < +\infty$ whereas $C(t) = +\infty$ otherwise. (13.5)

## 13.2 setting of the result

In order to state the main result of that paper, we need to introduce a few definitions and notations.

**Definition 13.2.1.** A probability measure $\mu = \rho dx$ on $\mathbf{R}^d$ is said to be regular if there exists a domain $\Omega$ (*i.e.* an open bounded connected set) on which $\mu$ is concentrated and such that

$$\exists c > 0; \quad \rho(x) \geq c$$

for almost every $x \in \Omega$.

**Definition 13.2.2** (Directional derivatives)**.** Let $h : \mathbf{R}^d \to [0, +\infty]$ be a convex function such that $h^{-1}([0,+\infty)) = \mathbf{C}$ is the closure of an open bounded convex set. We set $h'$ the directional derivative map

$$\begin{array}{rcl} h' : IDom & \longrightarrow & \mathbf{R} \cup \{-\infty\} \\ (x,w) & \longmapsto & h'(x;w) \end{array}$$

where $IDom = \{(x,w) \in \mathbf{C} \times \mathbf{R}^d; x + w \in \overset{\circ}{\mathbf{C}}\}$ and

$$h'(x;w) = \lim_{\lambda \downarrow 0} \frac{h(x+\lambda w) - h(x)}{\lambda}.$$

*Remark* 13.2.3. Observe that $IDom$ is an open bounded subset of $\mathbf{C} \times \mathbf{R}^d$ for the induced topology.

**Definition 13.2.4** (Relativistic cost functions)**.** We say that $c(x,y) = h(x-y)$ is a relativistic cost function if $h : \mathbf{R}^d \to [0,+\infty]$ is a strictly convex function such that $h^{-1}([0,+\infty)) = \mathbf{C}$ is the closure of an open bounded and strictly convex set. We further assume that $h(0) = 0$ and $0 \in \overset{\circ}{\mathbf{C}}$. Last, we assume that $h$ (respectively $h'$) is continuous on $\mathbf{C}$ (respectively on the subset $(h')^{-1}(\mathbf{R}) \subset IDom$).

*Remark* 13.2.5. Notice that the condition on the directional derivatives amounts to $h$ differentiable on $\overset{\circ}{\mathbf{C}}$ when the differential blows up on all the boundary of $\mathbf{C}$ as for the relativistic heat cost for instance.

**Notations :** we set

$$||c||_\infty = \sup_{x \in \mathbf{C}} c(x).$$

Moreover, using that $0 \in \overset{\circ}{\mathbf{C}}$, we set $r_m > 0$ a positive number such that the open ball of radius $r_m$ centered at the origin satisfies

$$B(0, r_m) \subset \mathbf{C}. \tag{13.6}$$

Last, given two probability measures $\mu_0$ and $\mu_1$ on $\mathbf{R}^d$, we denote by

$$\Gamma_0^t(\mu_0, \mu_1)$$

the set of optimal transport plans relative to the cost function $c_t$ where $t \geq T$ ($T$ being defined in (13.5)).

The aim of this paper is the proof of the following theorem.

**Theorem 13.2.6.** *Let $\mu_0$ and $\mu_1$ be two probability measures with compact support on $\mathbf{R}^d$ and $c(x,y) = h(x-y)$ be a relativistic cost function. We assume that $\mu_0 = \rho_0 dx$ is absolutely continuous with respect to the Lebesgue measure. Then, for any critical or supercritical speed $t \geq T$, there exists a unique optimal transport plan $\pi_0^t$ for the cost $c_t = h(\frac{\cdot}{t})$ and this plan is induced by a map*

$$\pi_0^t = (Id, F)_\sharp \mu_0.$$

*Moreover, if we further assume that $\mu_0$ is regular and that $t > T$, there exists a $c_t$-concave map $\phi$ such that $F$ is defined $\pi_0^t$ almost everywhere on $p_x(\{x - y \in t\overset{\circ}{\mathbf{C}}\} \cap \operatorname{supp} \pi_0^t)$ by the formula*

$$F(x) = x - t\nabla h^*(|\tilde{\nabla}\phi(x)|) \tag{13.7}$$

*where $h^*$ denotes the Legendre transform of $h$ and $\tilde{\nabla}\phi$ is the approximate gradient of $\phi$ (see Definition 13.5.14 for a definition) .*

*Remark* 13.2.7. More can be said in the case of the relativistic heat cost by using that the differential of the cost blows up on the unit sphere. Indeed, Corollary 13.3.6 implies that for almost every supercritical speed, the map is almost everywhere given by (13.7) since the subset made of pairs of points at maximal distance is a negligible set.

In the next section, we study the parametrized function $C(t)$ and we prove the existence of a critical speed of light $T$. Then, in section 13.4, for $t \geq T$ fixed, we use the notion of $c$-cyclical monotonicity to prove existence and uniqueness of the optimal transport map. In the last section, we give for supercritical speed, the expression of the optimal map in terms of the Kantorovich potential.

## 13.3 Properties of the parametrized total cost function

In this part, we are given two probability measures $\mu_0 \neq \mu_1 \in \mathcal{P}(\mathbf{R}^d)$ with compact support. Let us recall the parametrized cost function defined above

$$C(t) = \min_{\pi \in \Pi(\mu_0, \mu_1)} \int_{\mathbf{R}^{2d}} h(\frac{x-y}{t})\, d\pi(x,y). \tag{13.8}$$

**Proposition 13.3.1** (Properties of $C(t)$)**.** *The total cost function satisfies :*

*a) The speed $T = \inf\{s\ ;\ C(s) < \infty\}$ is positive and finite. $T$ is called the critical speed.*

*b) The function $C(t)$ is a decreasing function on $(T, +\infty)$.*

*c) The total cost $C(T)$ is finite.*

*d) The fonction $C(t)$ is right continuous on $[T, +\infty)$.*

*Démonstration.* a) Let us first prove that $T > 0$. Since $\mu_0$ and $\mu_1$ are not equal, there exists a Borelian set $B$ such that $\mu_0(B) \neq \mu_1(B)$. Without loss of generality we can assume that $B$ is a compact set and

$$\mu_0(B) < \mu_1(B). \tag{13.9}$$

The proof of $T > 0$ is by contradiction. First of all, note that the assumption $C(t) < +\infty$ implies the two following inequalities

$$\mu_0(B) \leq \mu_1(B + t\mathbf{C}) \text{ and } \mu_0(B + t\mathbf{C}) \geq \mu_1(B). \tag{13.10}$$

Indeed when $C(t) < +\infty$, there exists a transport plan whose support is in $\{(x,y)\ ;\ x - y \in t\mathbf{C}\}$. Therefore, if $C(t) < +\infty$ for any $t > 0$, we get by passing to the limit in $\mu_0(B + t\mathbf{C}) \geq \mu_1(B)$ that $\mu_0(B) \geq \mu_1(B)$. Here, we have used the fact that $B$ is a closed subset and $\mathbf{C}$ is a bounded set. This contradicts (13.9).

Now, let us prove that $T < +\infty$. Since the supports of $\mu_0$ and $\mu_1$ are compact sets, the quantity $\sup\{|x-y|; (x,y) \in \operatorname{supp}\mu_0 \times \operatorname{supp}\mu_1\}$ is finite.

Thanks to (13.6), there exists $t_0$ such that $\{x-y; (x,y) \in \operatorname{supp}\mu_0 \times \operatorname{supp}\mu_1\} \subset t_0\mathbf{C}$. Consequently,

$$\int c_{t_0}\, d\mu_0 \otimes d\mu_1 < +\infty$$

and $C(t_0) < +\infty$.

b) Let $t_1 > t_2 > T$. By definition of $T$, $C(t_2) < +\infty$. Let $\pi_2 \in \Pi(\mu_0, \mu_1)$ be an optimal plan for the cost $c_{t_2}$. By definition of the cost function, we have $\operatorname{supp} \pi_2 \subset \{(x,y); x - y \in t_2 \mathbf{C}\}$. Now, since $h(0) = 0$ and $h$ is strictly convex, for a fixed vector $u \neq 0 \in \mathbf{C}$ and $0 < a < b \leq 1$, we have $\frac{h(au)}{a} < \frac{h(bu)}{b}$. This yields for any $(x,y) \in \operatorname{supp} \pi_2$ such that $x \neq y$, $h(\frac{x-y}{t_1}) < h(\frac{x-y}{t_2})$. Therefore, since $\mu_0 \neq \mu_1$, we get

$$\int c_{t_1} \, d\pi_2 < \int c_{t_2} \, d\pi_2$$

hence the result, by definition of $\pi_2$.

c) Let us consider a sequence of speed $t_n \searrow T$ and $\pi_n$ a sequence of optimal plans for $c_{t_n}$. By definition of $T$, the total cost for any $t_n$ is finite and then for all $n \in \mathbf{N}$, $\pi_{t_n}(\{(x,y); x - y \in t_n \mathbf{C}\}) = 1$. Since $\pi_{t_n}$ is a sequence of probability measures in $\Gamma(\mu_0, \mu_1)$, up to extracting a subsequence, we can assume that $\pi_{t_n}$ converges weakly to $\pi_\infty \in \Pi(\mu_0, \mu_1)$ thanks to Prokhorov's theorem. Given $\varepsilon > 0$, by noticing that $\{(x,y); x - y \notin (T + \varepsilon)\mathbf{C}\}$ is an open set, we get

$$\pi_\infty(\{(x,y); x - y \notin (T+\varepsilon)\mathbf{C}\}) \leq \liminf_{n \to 0} \pi_{t_n}(\{(x,y); x - y \notin (T+\varepsilon)\mathbf{C}\}) = 0.$$

Thus, $\pi_\infty(\{(x,y); x - y \in (T+\varepsilon)\mathbf{C}\}) = 1$ for any $\varepsilon > 0$. Since $\mathbf{C}$ is a closed bounded set, this yields $\pi_\infty(\{(x,y); x - y \in T\mathbf{C}\}) = 1$ and, therefore, $\int c_T \, d\pi_\infty < +\infty$.

d) Fix $s \geq T$. Let us consider a sequence $t_n \searrow s$ and for each $n$, let $\pi_{t_n} \in \Pi(\mu_0, \mu_1)$ be an optimal transport plan for $c_{t_n}$. Thanks to Prokhorov's theorem, we can assume that $\pi_{t_n}$ converges to $\pi_\infty \in \Pi(\mu_0, \mu_1)$. Since $t_n > s$,

$$\int_{\mathbf{R}^{2d}} c(\frac{x-y}{t_n}) d\pi_{t_n} \geq \int_{\{x-y \in s\mathbf{C}\}} c(\frac{x-y}{t_n}) d\pi_{t_n}.$$

Moreover, since $s\mathbf{C}$ is a compact set and $h$ is assumed to be continuous on $\mathbf{C}$, for any $\varepsilon > 0$, there exists $n_0$ such that for $n \geq n_0$ and $x - y \in s\mathbf{C}$,

$$c_{t_n}(x,y) = h(\frac{x-y}{t_n}) \geq h(\frac{x-y}{s}) - \varepsilon.$$

By combining this together with the continuity of $c_s$ when restricted to $\{(x,y); x - y \in s\mathbf{C}\}$, we get

$$\liminf_{t_n \to s} C(t_n) = \liminf_{t_n \to s} \int c_{t_n} \, d\pi_{t_n} \geq \int c_s \, d\pi_\infty.$$

On the other hand,

$$C(t_n) \leq \int_{\mathbf{R}^{2d}} h(\frac{x-y}{t_n}) \, d\pi_\infty(x,y) \leq \int_{\mathbf{R}^{2d}} h(\frac{x-y}{s}) \, d\pi_\infty(x,y).$$

Thus

$$C(s) \geq \lim_{n \to +\infty} C(t_n) = \int c_s \, d\pi_\infty \geq C(s)$$

and the statement is proved. □

Now, we give some properties relative to the critical speed $T$.

**Lemma 13.3.2.** *Let $\pi \in \Gamma(\mu_0, \mu_1)$ such that $\int c_T \, d\pi < +\infty$. Then, this plan satisfies*

$$\operatorname{supp} \pi \cap \{(x,y); x - y \in T\partial\mathbf{C}\} \neq \emptyset.$$

*Moreover, the critical speed is characterized by the equality*

$$T = \min_{\pi \in \Gamma(\mu_0, \mu_1)} \pi - ess \sup |x - y|_{\mathbf{C}}$$

*where $|x|_{\mathbf{C}} = \inf\{\lambda; x \in \lambda \mathbf{C}\}$ and $\pi - ess \sup |x-y|_{\mathbf{C}} = \inf\{s; \pi(\{(x,y); |x-y|_{\mathbf{C}} > s\}) = 0\}$.*

*Remark* 13.3.3. In the case of the relativistic heat cost (or any other radial relativistic cost function), $|\cdot|_{\mathbf{C}}$ is nothing but the Euclidean norm (up to scaling). Thus in this case, $\min_{\pi\in\Gamma(\mu_0,\mu_1)} \pi - ess\sup |x-y|_{\mathbf{C}}$ coincides with the $\infty-$Wasserstein distance studied in [23].

*Remark* 13.3.4. This lemma means that for the critical speed $T$, the support of any optimal transport plan must intersect the critical set $\{(x,y)\ ;\ x-y\in T\mathbf{C}\}$ but we do not know whether or not all the optimal transport plans give some mass to this critical set. See below for more on this set in the case of the relativistic heat cost.

*Démonstration.* We prove the first statement by contradiction. Suppose on the contrary that $\operatorname{supp}\pi\cap\{(x,y); x-y\in T\partial\mathbf{C}\}=\emptyset$. By assumption on $\pi$, we have $\operatorname{supp}\pi\subset\{(x,y); x-y\in T\mathbf{C}\}$. Thus, by compactness of the support of $\pi$, there exists $\delta>0$ such that

$$\operatorname{supp}\pi\subset\{(x,y); x-y\in (T-\delta)\mathbf{C}\}.$$

This implies $C(T-\delta)<+\infty$, a contradiction.

Now, we prove the second statement. Let us start with the following equalities.

$$\begin{aligned}
T &= \inf\{s; C(s)<+\infty\}\\
&= \inf\{s;\ \exists\pi\in\Gamma(\mu_0,\mu_1) \text{ s.t. } \pi(\{(x,y); x-y\in s\mathbf{C}\})=1\}\\
&= \inf\{s;\ \exists\pi\in\Gamma(\mu_0,\mu_1) \text{ s.t. } \pi(\{(x,y); |x-y|_{\mathbf{C}}\le s\})=1\}\\
&= \inf\{s;\ \exists\pi\in\Gamma(\mu_0,\mu_1) \text{ s.t. } \pi(\{(x,y); |x-y|_{\mathbf{C}}> s\})=0\}\\
&= \inf_{\pi\in\Gamma(\mu_0,\mu_1)} \inf\{s;\ \pi(\{(x,y); |x-y|_{\mathbf{C}}> s\})=0\}\\
&= \inf_{\pi\in\Gamma(\mu_0,\mu_1)} \pi - ess\sup |x-y|_{\mathbf{C}}.
\end{aligned}$$

To conclude, let us prove that the infimum above is actually a minimum. To this aim, consider a minimizing sequence $(\pi_i)_{i\in\mathbb{N}}$ and assume thanks to Prokhorov's theorem, that $\pi_i\rightharpoonup\pi_\infty\in\Gamma(\mu_0,\mu_1)$. We need the following fact.

Fact : $\forall(x,y)\in\operatorname{supp}\pi_\infty,\ \exists(x_i,y_i)_{i\in\mathbb{N}}\in\operatorname{supp}\pi_i; (x_i,y_i)\to(x,y)$.

A proof is given in [23, Lemma 2.3]. Thus, since $\mathbf{C}$ satisfies (13.6), this gives for any $\varepsilon>0$,

$$\operatorname{supp}\pi_\infty\subset\operatorname{supp}\pi_i+\varepsilon B(0,1)\subset\operatorname{supp}\pi_i+\frac{\varepsilon}{r_m}\mathbf{C}$$

for large $i$. This yields

$$\sup_{\operatorname{supp}\pi_\infty}|x-y|_{\mathbf{C}}\le\liminf_{i\to+\infty}\sup_{\operatorname{supp}\pi_i}|x-y|_{\mathbf{C}}$$

thus $\pi_\infty$ is actually a minimizer. □

In the case of the relativistic heat cost, we obtain the following additional proposition and its straightforward corollary (similar statements are probably within reach for other relativistic cost functions).

**Proposition 13.3.5.** *In the case of the relativistic heat cost denoted by $c_t(x,y)=h(\frac{|x-y|}{t})$, the following property holds true for any $t\ge T$*

$$C'_+(t)>-\infty\Longrightarrow\forall\pi\in\Gamma_0^t(\mu_0,\mu_1),\ \ \pi(\{(x,y); |x-y|=t\})=0$$

*where $C'_+(t)$ denotes the right derivative of $C$ at $t$.*

*Démonstration.* Recall that $T = \inf\{t\,;\ c_t < \infty\}$. Let $\pi_t \in \Gamma_0^t(\mu_0, \mu_1)$. For any $s > t$, let us compute

$$\begin{aligned} C(t) - C(s) &\geq \int h(\frac{|x-y|}{t})\, d\pi_t - \int h(\frac{|x-y|}{s})\, d\pi_t \\ &\geq \int_{\{(x,y);|x-y|=t\}} h(\frac{|x-y|}{t}) - h(\frac{|x-y|}{s})\, d\pi_t \\ &\geq \int_{\{(x,y);|x-y|=t\}} h(1) - h(\frac{t}{s})\, d\pi_t \\ &\geq \frac{s-t}{s} h'(\frac{t}{s}) \pi_t(\{|x-y| = t\}). \end{aligned}$$

We get the result by letting $s$ decrease to $t$. □

Since $C$ is a monotone function, the proposition implies readily the following property on the critical set $\{(x,y); |x-y| = t\}$.

**Corollary 13.3.6.** *In the case of the relativistic heat cost, any optimal transport plan $\pi_0^t \in \Gamma_0^t(\mu_0, \mu_1)$ satisfies*

$$\pi_0^t(\{(x,y); |x-y| = t\}) = 0$$

*for almost every $t \geq T$.*

## 13.4 Existence and uniqueness of optimal map

In this section, we prove the uniqueness of the optimal transport plan and that this plan is induced by a map. Let us recall the definition of a relativistic cost function.

**Definition 13.4.1** (Relativistic cost functions)**.** We say that $c(x,y) = h(x-y)$ is a relativistic cost function if $h : \mathbf{R}^d \to [0, +\infty]$ is a strictly convex function such that $h^{-1}([0,+\infty)) = \mathbf{C}$ is the closure of an open bounded and strictly convex set. We further assume that $h(0) = 0$ and $0 \in \overset{\circ}{\mathbf{C}}$. Last, we assume that $h$ (respectively $h'$) is continuous on $\mathbf{C}$ (respectively on the subset $(h')^{-1}(\mathbf{R}) \subset IDom$ where $IDom = \{(x,w) \in \mathbf{C} \times \mathbf{R}^d; x + w \in \overset{\circ}{\mathbf{C}}\}$).

*Remark* 13.4.2. In general, the function $h'$ is an upper semi-continuous function on $IDom$ (this follows from a straightforward adaption of the proof of [34, Theorem 24.5]). Thus, the assumption above actually implies that $h'$ is continuous on $IDom$. Note also that if $h'(x;w) = -\infty$ then $h'(x;w') = -\infty$ for all $w'$ such that $(x,w') \in IDom$. Under our assumptions these points $x$ are exactly those in $\mathbf{C}$ such that $\partial h(x) = \emptyset$ (we refer to [34, Theorem 23.3] for a proof of the last two statements).

The main result of this part is the following theorem.

**Theorem 13.4.3.** *Let $\mu_0$ and $\mu_1$ be two probability measures with compact support on $\mathbf{R}^d$. We assume that $\mu_0 = \rho_0 dx$ is absolutely continuous with respect to the Lebesgue measure. Then, for any critical or supercritical speed $t \geq T$, there exists a unique optimal transport plan $\pi_0^t$ for the cost $c_t$ and this plan is induced by a map*

$$\pi_0^t = (Id, F^t)_\sharp \mu_0.$$

The proof of this result is based on a method introduced by Champion, de Pascale and Juutinen [23]. A cornerstone of their proof is the following lemma.

**Lemma 13.4.4.** *Assume that $\mu << \mathcal{L}^d$. Let $\pi \in \Pi_0^t(\mu, \nu)$ and $\Gamma$ a set on which $\pi$ is concentrated. Then $\pi$ is concentrated on a $\sigma-$compact set $R(\pi) \subset \Gamma$ such that for all $(x,y) \in R(\pi)$ the point $x$ is a Lebesgue point of $p_x(\Omega_0 \times B(y,r) \cap \Gamma)$ for all $r > 0$.*

This precise version is proved in [22, Lemma 4.3].

**Definition 13.4.5.** A couple $(x,y) \in \Gamma$ is said to be $\pi$-regular if $x$ is a Lebesgue point of $p_x(\Omega_0 \times B(y_0, r)) \cap \Gamma)$ for any positive $r$.

Consequently, we get the corollary below.

**Corollary 13.4.6.** *Let $(x_0, y_0) \in R(\pi)$, $r > 0$, $\alpha \in (0,1)$, $\xi \in \mathbb{S}^{d-1}$ and $\delta > 0$. Then for $\varepsilon > 0$ sufficiently small, the set of points $x \in p_x(\Omega_0 \times B(y,r)) \cap \Gamma)$ such that $x \in \hat{C}(x_0, \xi, \delta) \cap (B(x_0, \varepsilon) \setminus B(x_0, \alpha\varepsilon))$ has positive measure where*

$$\hat{C}(x_0, \xi, \delta) = \{x \in \mathbf{R}^d \setminus \{x_0\};\ \frac{x - x_0}{|x - x_0|} \cdot \xi \geq 1 - \delta\}.$$

Before we start the proof of Theorem 13.4.3, let us recall the definition of $c-$cyclically monotone subset. This notion was introduced by Knott and Smith [38] in order to detect the optimality of a given transport plan.

**Definition 13.4.7** ($c$-monotone set)**.** A subset $S \subset \mathbf{R}^d \times \mathbf{R}^d$ is called a $c$-cyclically monotone set if and only if for any positive integer $m$ and any pairs $(x_1, y_1), \cdots, (x_m, y_m) \in S$, the following inequality is satisfied :

$$c(x_1, y_1) + \cdots + c(x_m, y_m) \leq c(x_1, y_2) + c(x_2, y_3) + \cdots + c(x_m, y_1).$$

It is well-known that an optimal plan is concentrated on a measurable $c$-cyclically monotone set when (the transport problem is well-posed and) the cost function $c : \mathbf{R}^d \times \mathbf{R}^d \to \mathbf{R}^+ \cup \{+\infty\}$ is a lower semi-continuous map [4, Theorem 3.2].

### 13.4.1 Proof of Theorem 13.4.3

We are given $\pi_0^t \in \Pi_0^t(\mu_0, \mu_1)$. Thanks to Lemma 13.4.4 and the result above, there exists a measurable $c$-cyclically monotone set $\Gamma_0^t \subset \operatorname{supp} \pi_0^t$ made of regular points such that $\pi_0^t(\Gamma_0^t) = 1$. Following the approach in [23] and the subsequent papers, we are going to prove by contradiction that $\Gamma_0^t$ is a graph.

Let us assume that $(x_0, y_0)$ and $(x_0, y_1)$ belong to $\Gamma_0^t$ with $y_0 \neq y_1$ and set $u_0 = (x_0 - y_0)/t$ and $u_1 = (x_0 - y_1)/t$. Note that $u_0 \neq u_1$ and $u_0$ and $u_1$ belong to $\mathbf{C}$ since $t \geq T$. We also claim that

$$h'(u_0; \overrightarrow{u_0u_1}) + h'(u_1; \overrightarrow{u_1u_0}) < 0. \tag{13.11}$$

Indeed, $h'(u_0; \overrightarrow{u_0u_1}) + h'(u_1; \overrightarrow{u_1u_0}) = \theta'_+(0) - \theta'_-(1)$ where $\theta'_+$ (respectively $\theta'_-$) denotes the right derivative (respectively left derivative) of the convex real-valued function of a real variable $\theta(s) = h(u_0 + s\overrightarrow{u_0u_1})$. Now, since $\mathbf{C}$ is a strictly convex set, the open segment $(u_0, u_1) = \{tu_0 + (1-t)u_1; t \in (0,1)\}$ is actually within $\overset{\circ}{\mathbf{C}}$. Therefore, since $IDom$ is an open subset of $\mathbf{C} \times \mathbf{R}^d$ with respect to the induced topology, we get the existence of numbers $r > 0$ and $r' > 0$ such that for all $v \in B(u_0, r) \cap \mathbf{C}, w \in B(u_1, r) \cap \mathbf{C}$ and $z \in B(0, r')$

$$v + \overrightarrow{u_0u_1}/2 + z \in \overset{\circ}{\mathbf{C}} \text{ and } w - \overrightarrow{u_0u_1}/2 - z \in \overset{\circ}{\mathbf{C}}.$$

Let us recall that $y \to h'(x; y)$ is a positively homogeneous function. Consequently, we get that

$$v + \lambda(\overrightarrow{u_0u_1}/2 + z) \in \overset{\circ}{\mathbf{C}} \text{ and } w - \lambda(\overrightarrow{u_0u_1}/2 - z) \in \overset{\circ}{\mathbf{C}}$$

whenever $v \in B(u_0, r) \cap \mathbf{C}$, $0 < \lambda \leq 1$ and $z \in B(0, r')$. For the sake of clarity, let us introduce a notation for some truncated cones :

$$C(0, \overrightarrow{u_0u_1}/2, r'; \eta) = \{\lambda x \in \mathbf{R}^d;\ \lambda > 0 \text{ and } x \in \overrightarrow{u_0u_1}/2 + B(0, r')\} \cap \overline{B}(0, \eta) \setminus B(0, \eta/2).$$

In particular, when $\eta$ is small enough (and $r, r' << |\overrightarrow{u_0u_1}/2|$), we have

$$B(u_0, r) \cap \mathbf{C} + C(0, \overrightarrow{u_0u_1}/2, r'; \eta) \subset \overset{\circ}{\mathbf{C}} \text{ and } B(u_1, r) \cap \mathbf{C} - C(0, \overrightarrow{u_0u_1}/2, r'; \eta) \subset \overset{\circ}{\mathbf{C}}.$$

Now, since $h$ is continuous on $\mathbf{C}$ and $h'$ is continuous on $IDom$ (see Remark 13.4.2), for any $\varepsilon > 0$ there exist $0 < \overline{r} < r, 0 < \overline{r'} < r'$ and $\eta > 0$ small such that

$$\sup \left\{ \left| \frac{h(v+z) - h(v)}{|z|} - h'(u_0, \overrightarrow{u_0u_1}/|\overrightarrow{u_0u_1}|) \right|; v \in B(u_0, \overline{r}) \cap \mathbf{C}, z \in C(0, \overrightarrow{u_0u_1}/2, \overline{r'}; \eta) \right\} < \epsilon \tag{13.12}$$

and

$$\sup\left\{\Big|\frac{h(w-z)-h(w)}{|z|}-h'(u_1,-\overrightarrow{u_0u_1}|\overrightarrow{u_0u_1}|)\Big|;w\in B(u_1,\overline{r})\cap\mathbf{C},z\in C(0,\overrightarrow{u_0u_1}/2,\overline{r'};\eta)\right\}<\varepsilon. \tag{13.13}$$

In the second part of the proof, we use the fact that $(x_0,y_1)$ is a regular pair in the sense of Definition 13.4.5 to get a contradiction. Given $\varepsilon>0$, let $\overline{r},\overline{r'},\eta>0$ such that (13.12) and (13.13) hold. We also assume that $\eta<t\overline{r}/2$. Now, applying Corollary 13.4.6 to $(x_0,y_1)$ with $r=t\overline{r}/2$, we get the existence of $(\tilde{x},\tilde{y})\in\Gamma_0^t$ such that $\tilde{x}\in\{x_0\}+C(0,\overrightarrow{u_0u_1}/2,\overline{r'};\eta)$ and $\tilde{y}\in B(y_1,t\overline{r}/2)$ (up to shrinking $\eta$). By construction we have $(\tilde{x}-\tilde{y})/t\in B(u_1,\overline{r})$ and $(\tilde{x}-\tilde{y})\in t\mathbf{C}$ since $t\geq T$. Now, by applying the estimates (13.12) and (13.13) with $v=u_0$, $w=(\tilde{x}-\tilde{y})/t$ and $z=(\tilde{x}-x_0)/t$ , we get

$$\begin{aligned}h((\tilde{x}-y_0)/t)+h((x_0-\tilde{y})/t) &\leq h((x_0-y_0)/t)+h((\tilde{x}-\tilde{y})/t)\\ &\quad+|(\tilde{x}-x_0)/t|\big(2\varepsilon+h'(u_0;\overrightarrow{u_0u_1}/|\overrightarrow{u_0u_1}|)+h'(u_1;\overrightarrow{u_1u_0}/|\overrightarrow{u_0u_1}|)\big).\end{aligned}$$

Combining this together with (13.11) contradicts the $c$-cyclical monotonicity of $\Gamma_0^t$ for small $\varepsilon$. Thus, we have proved that any optimal plan is concentrated on a measurable graph.

The uniqueness proof is more classical. Consider two opimal plans, by what precedes these plans are induced by maps. The arithmetic mean of these plans is still an optimal plan but it cannot be induced by a map unless the two plans coincide.

## 13.5 Supercritical speed

This section is valid only for supercritical speed. The goal of this part is to obtain (13.7) linking the optimal map and the Kantorovich potential. We recall that a domain is an open bounded connected set.

**Theorem 13.5.1.** *Let $\mu_0$ and $\mu_1$ be two probability measures with compact support on $\mathbf{R}^d$. We assume that $\mu_0=\rho_0 dx$ is regular in the sense of Definition 13.2.1. Then, for any supercritical speed $t>T$, there exists a $c_t$-concave map $\phi$ such that the map $F$ defined by $\pi_0^t=(Id,F)_\sharp\mu_0$ is defined almost everywhere on $p_x(\{x-y\in t\overset{\circ}{\mathbf{C}}\}\cap\Gamma_0^t)$ by the formula*

$$F(x)=x-t\nabla h^*(\tilde{\nabla}\phi(x)).$$

This section is divided into three parts. In the first one, we establish a technical result on the volume growth of set enlargements. This result is a key result in the study of special maps related to the transport problem (namely $c_t$-concave maps) that is performed in the second part. In the last part, we prove the theorem above.

### 13.5.1 Volume growth estimate for set enlargements

The goal of this part is to establish the technical proposition below.

**Proposition 13.5.2.** *Let $\Omega$ be a domain and $t>0$. For any constants $0<M_1<M_2<|\Omega|$, there exists a positive constant $\epsilon=\epsilon(\Omega,t,M_1,M_2)$ depending on $\Omega$, $t$, $M_1$, and $M_2$ such that*

$$\inf_{\mathcal{S}}\left(|(E+t\mathbf{C})\cap\Omega|-|E\cap\Omega|\right)\geq\epsilon$$

*where $\mathcal{S}=\{E$ compact subset of $\mathbf{R}^d;M_1\leq|E\cap\Omega|\leq M_2\}$.*

*Moreover, there exists a constant $0<c<|\Omega|$ depending on $t$ and $\Omega$ such that for any compact subset $E$ of $\mathbf{R}^d$ satisfying $|E\cap\Omega|>c$, the following equality holds*

$$|(E+t\mathbf{C})\cap\Omega|=|\Omega|.$$

This result is proved by contradiction. The first step of the proof is to reduce the result to a similar statement involving the De Giorgi perimeter. Then, we use a compactness result for functions of bounded variation.

### Notions of boundary area for a nonsmooth set

The first notion is the (relative) Minkowski content of a set. This is the notion we are mainly interested in. Its definition is the following.

**Definition 13.5.3** (Relative Minkowski content)**.** Let $E$ be a compact subset of $\mathbf{R}^d$. The (relative) Minkowski content of $E$ is defined as

$$Mink_{\mathcal{C}}(E\,|\,\Omega) = \liminf_{r\downarrow 0} \frac{|(E + r\mathbf{C}) \cap \Omega| - |E \cap \Omega|}{r}.$$

The second notion is the De Giorgi perimeter. We refer to the book [2] for a detailed exposition.

**Definition 13.5.4** (Relative perimeter)**.** Let $E$ be a mesurable subset of $\mathbf{R}^d$. The perimeter of $E$ relative to $\Omega$ is defined as

$$Per(E\,|\,\Omega) = \sup\left\{ \int_E \text{div } \varphi\, dx; \varphi \in \mathbf{C}^1_c(\Omega, B) \right\}$$

where $B = B(0,1)$ is the unit Euclidean ball centered at the origin in $\mathbf{R}^d$.

This definition is not the only one but is certainly the most elementary. Note that when $E \subset \mathbf{R}^d$ is a bounded open set with smooth boundary, the divergence theorem implies $Per(E\,|\,\Omega) = \mathcal{H}^{d-1}(\partial E \cap \Omega)$. Moreover, with this definition, it is easy to see that the perimeter is a lower semi-continuous functional with respect to the $L^1$-topology (also called topology of sets convergence) whose definition is recalled below.

**Definition 13.5.5** (Set of finite perimeters and convergence)**.** We set $\mathcal{M}_{<+\infty}$ the set of measurable subsets $E$ of $\mathbf{R}^d$ whose relative perimeter $Per(E\,|\,\Omega)$ is finite. The set $\mathcal{M}_{<+\infty}$ is endowed with the $L^1$-topology. Namely, a sequence $(E_n)_{n\in\mathbf{N}}$ converges to $E$ in $\mathcal{M}_{<+\infty}$ if, by definition, the (restriction to $\Omega$ of the) characteristic functions $\chi_{E_n}$ converges to $\chi_E$ in $L^1(\Omega)$.

**Lemma 13.5.6.** *Given a domain $\Omega$ in $\mathbf{R}^d$, the relative perimeter is a lower semi-continuous map*

$$Per(\cdot\,|\,\Omega) : \mathcal{M}_{<+\infty} \longrightarrow \mathbf{R}^+$$

*when the set $\mathcal{M}_{<+\infty}$ is endowed with the $L^1$-topology.*

*Démonstration.* Given $\varphi \in \mathbf{C}^1_c(\Omega, B)$, the map $E \mapsto \int_E \text{div } \varphi\, dx$ is clearly continuous for the $L^1$-topology. Therefore, $Per(\cdot\,|\,\Omega)$ is lower semi-continuous as the supremum of continuous functions. □

Our goal is to compare the relative Minkowski content to the De Giorgi perimeter. To this aim, we first reduce this comparison result to the case when $\mathbf{C} = B$. Recall that, by definition of $\mathbf{C}$, there exists a constant $r_m > 0$ such that the open ball $B(0, r_m)$ satisfies

$$B(0, r_m) \subset \mathbf{C}. \tag{13.14}$$

Thus, we get the inequality

$$Mink_{\mathcal{C}}(E\,|\,\Omega) \geq r_m \liminf_{r\downarrow 0} \frac{|(E + rB(1)) \cap \Omega| - |E \cap \Omega|}{r} = r_m Mink_B(E\,|\,\Omega). \tag{13.15}$$

Complementary to this result is the following proposition.

**Proposition 13.5.7.** *Let $E$ be a compact subset of $\mathbf{R}^d$. Then, the following inequality holds*

$$Mink_B(E\,|\,\Omega) \geq Per(E\,|\,\Omega). \tag{13.16}$$

*Démonstration.* We first rewrite

$$\frac{|(E+rB)\cap\Omega|-|E\cap\Omega|}{r}=\frac{1}{r}\int_{E_r\setminus E}\chi_\Omega$$

where $\chi_\Omega$ denotes the characteristic function of $\Omega$ and $E_r=E+rB$. Now, we set $d_E$ the Euclidean distance funtion to the set $E$. More precisely for any $x\in\mathbf{R}^d$,

$$d_E(x)=\min_{y\in E}|x-y|.$$

We have the following equalities

$$\begin{aligned}\frac{1}{r}\int_{E_r\setminus E}\chi_\Omega &= \frac{1}{r}\int_{E_r\setminus E}\chi_\Omega|\nabla d_E|\\ &= \frac{1}{r}\int_0^r\mathcal{H}^{d-1}(\partial E_s\cap\Omega)\,ds\\ &= \int_0^1\mathcal{H}^{d-1}(\partial E_{sr}\cap\Omega)\,ds.\end{aligned}$$

where the second equality follows from the coarea formula. Now, thanks to [2, Proposition 3.62], we get for any measurable subset $A$ of $\mathbf{R}^d$

$$\mathcal{H}^{d-1}(\partial A\cap\Omega)\geq Per(A\,|\,\Omega).$$

Therefore, this yields

$$\frac{|(E+rB(1))\cap\Omega|-|E\cap\Omega|}{r}\geq\int_0^1 Per(E_{sr}\,|\,\Omega)\,ds.$$

To conclude, note that $E_{sr}$ converges to $E_0=E$ with respect to the $L^1$-convergence. Thus, we get the result by combining Fatou's lemma together with Lemma 13.5.6. □

*Remark* 13.5.8. The proof above is inspired by [7].

### An intermediate result

Thanks to the properties of De Giorgi perimeter, we can prove the following result.

**Proposition 13.5.9.** *Let $\Omega$ be a domain of $\mathbf{R}^d$. For any constants $0<M_1<M_2<|\Omega|$, there exists a positive constant $\epsilon=\epsilon(\Omega,M_1,M_2)$ depending on $\Omega$, $M_1$, and $M_2$ such that*

$$\inf_{\mathcal{M}}Per(E\,|\,\Omega)\geq\epsilon$$

*where $\mathcal{M}=\{E$ measurable subset of $\Omega; M_1\leq|E|\leq M_2\}$*

*Démonstration.* We prove the result by contradiction by assuming that the infimum above is 0. Let $(E_i)_{i\in\mathbf{N}}$ be a minimizing subsequence. Thanks to [2, Theorem 3.39 p.145], there exists a subsequence still denoted by $(E_i)_{i\in\mathbf{N}}$ which converges to a measurable subset $E_\infty$ of $\Omega$ in $L^1(\Omega)$. By definition of this convergence, $E_\infty$ belongs to $\mathcal{M}$ and Lemma 13.5.6 yields

$$Per(E_\infty\,|\,\Omega)=0.$$

Consequently, up to modifying $E_\infty$ on a negligible subset, $\chi_{E_\infty}$ is constant on the connected set $\Omega$ (see [2, Proposition 3.2 p118] for a proof). This contradicts the fact that $E_\infty\in\mathcal{M}$ and the proof is complete. □

By combining (13.15), Proposition 13.5.7 together with the proposition above, we get

**Proposition 13.5.10.** *Let $\Omega$ be a domain of $\mathbf{R}^d$. For any constants $0<M_1<M_2<|\Omega|$, there exists $\varepsilon=\varepsilon(\Omega,M_1,M_2)>0$ depending on $\Omega$, $M_1$, and $M_2$ such that*

$$\inf_{\mathcal{S}}Mink_{\mathcal{C}}(E\,|\,\Omega)\geq\varepsilon.$$

**Proof of Proposition 13.5.2**

Both statements are proved by contradiction. Let us remark that the quantity $|(E+t\mathbf{C})\cap\Omega|-|E\cap\Omega|$ is unchanged if we modify $E$ outside a large ball $B(R)$ (whose radius $R$ depends on the diameters of $\Omega$ and $\mathbf{C}$). Thus in this proof, we only consider compact sets into $B(R)$. Let $(E_n)_{n\in\mathbf{N}}$ be a sequence of elements in $\mathcal{S}$ such that

$$|(E_n + t\mathbf{C}) \cap \Omega| \leq |E_n \cap \Omega| + \frac{1}{n} \tag{13.17}$$

and $(E_n)_{n\in\mathbf{N}}$ converges to the compact set $E_\infty \subset B(R)$ with respect to the Hausdorff metric (see [37] for a definition). By definition of this metric, we get

$$\limsup_{n\to+\infty} |E_n \cap \Omega| \leq |E_\infty \cap \Omega|. \tag{13.18}$$

Now, recall that $B(0, r_m) \subset \mathbf{C}$ (13.14). For large $n$, we have

$$E_\infty \subset E_n + \frac{r_m t}{2} B$$

by definition of the Hausdorff metric, therefore

$$E_\infty + \frac{t}{2}\mathbf{C} \subset E_n + t\mathbf{C}$$

and

$$|E_\infty \cap \Omega| \leq |(E_\infty + \frac{t}{2}\mathbf{C}) \cap \Omega| \leq |(E_n + t\mathbf{C}) \cap \Omega| \tag{13.19}$$

for large $n$.

Consequently thanks to (13.17) we get $\liminf_{n\to+\infty} |E_n \cap \Omega| \geq |E_\infty \cap \Omega|$ and, thanks to (13.18), $E_\infty \in \mathcal{S}$. Moreover, (13.19) also yields

$$|(E_\infty + \frac{t}{2}\mathbf{C}) \cap \Omega| - |E_\infty \cap \Omega| = 0. \tag{13.20}$$

Let us prove that the equality above leads to a contradiction. We set $\theta$ the non decreasing and right continuous map defined by

$$\theta(s) = |(E_\infty + s\mathbf{C}) \cap \Omega|.$$

In particular, this map is differentiable for almost every $s_0$ and the first derivative equals $Mink_{\mathcal{C}}(E_\infty + s_0\mathbf{C} \,|\, \Omega)$. Moreover, since $\theta$ is right continuous and non decreasing, the set $(E_\infty + s_0\mathbf{C})$ is in $\mathcal{S}$ when $s_0$ is small (up to slightly increase $M_2$). Therefore, Proposition 13.5.10 implies that for almost every small $s$, $\theta'(s) \geq \varepsilon$ and this contradicts (13.20). The first part of the proposition is proved.

The proof of the second part is very similar so we only sketch it. Consider a sequence $(E_n)_{n\in\mathbf{N}}$ of compact subsets of $B(R)$ such that $|E_n \cap \Omega| \to |\Omega|$ and $|(E_n + t\mathbf{C}) \cap \Omega| < |\Omega|$. Since $B(R)$ is compact, we can assume that $E_n$ converges to a compact set $E_\infty$ with respect to the Hausdorff metric. Thanks to (13.18), we get $|E_\infty \cap \Omega| = |\Omega|$. Now, using (13.14), we get

$$E_n + t\mathbf{C} \supset E_n + t r_m B \supset E_\infty$$

for large $n$. Thus,

$$|(E_n + t\mathbf{C}) \cap \Omega| \geq |E_\infty \cap \Omega| = |\Omega|$$

for large $n$, hence a contradiction.

## 13.5.2 Regularity of Kantorovich potentials

Throughout this section, we fix $t > T$ a supercritical speed.

Let us recall the notion of $c$-concave function.

**Definition 13.5.11** ($c$-concave function and $c$-subdifferential)**.** Let $\phi : \Omega_0 \to \mathbb{R} \cup \{-\infty\}$ be a function. We define $\phi^c : \operatorname{supp} \mu_1 \to \mathbb{R} \cup \{-\infty, +\infty\}$, the $c$-transform of $\phi$ by the formula

$$\phi^c(y) = \inf_{x \in \Omega_0} c(x,y) - \phi(x).$$

Such a function $\phi$ is said to be $c$-concave if for all $y \in \operatorname{supp} \mu_1$, $\phi^c(y) < +\infty$ (so that $(\phi^c)^c$ is well-defined) and $(\phi^c)^c = \phi$ (where usually, the role of $x$ and $y$ is unchanged because of the possible asymmetry of the cost function ; note that in the rest of the paper, we will write $\phi^{cc}$ for simplicity). The $c$-subdifferential of a $c$-concave function $\phi$ is defined by the formula $\partial_c \phi(\Omega_0) = \cup_{x \in \Omega_0} \partial_c \phi(x)$ where

$$\partial_c \phi(x) = \{y \in \operatorname{supp} \mu_1;\ \phi(x) + \phi^c(y) = c(x,y)\}$$

*Remark* 13.5.12. The $c$-subdifferential of a $c$-convave function is a $c$-monotone set, see for instance [40].

In order to motivate the study of (certain) $c_t$-concave maps, we start with a Rockafellar-type theorem.

**Theorem 13.5.13.** *Let $\mu_0$ and $\mu_1$ be two probability measures on $\mathbf{R}^d$ with compact support. We also assume that $\mu_0 = \rho_0 dx$ is regular in the sense of Definition 13.2.1. Then, there exists a $c_t$-concave map $\phi$ such that*

$$\Pi_0^t((Id, \partial_{c_t}\phi)) = 1$$

*where $(Id, \partial_{c_t}\phi) = \{(x,y) \in \Omega_0 \times \operatorname{supp} \mu_1; \phi(x) + \phi^c(y) = c_t(x,y)\}$. Such a map is called a Kantorovich potential.*

*Démonstration.* Let $\Gamma_0^t$ be the set introduced in the proof of Theorem 13.4.3. We shall prove

$$\Gamma_0^t \subset (Id, \partial_{c_t}\phi).$$

We fix $(x_0, y_0) \in \Gamma_0^t$ and define

$$\phi(x) = \inf_{m \in \mathbf{N}} \inf\{c_t(x_1,y_0) + \cdots + c_t(x_m, y_{m-1}) + c_t(x,y_m) - (c_t(x_0,y_0) + \cdots + c_t(x_m,y_m)); \forall i \in \{1, \cdots, m\}\ (x_i,y_i) \in \Gamma_0^t\}.$$

Since $t$ is a supercritical speed, $\operatorname{supp} \pi_0^t \subset \{c_t < +\infty\}$. Thus, for $(x_1,y_1) \cdots (x_m,y_m) \in \Gamma_0^t$, the term inside the brackets above is in $\mathbf{R} \cup \{+\infty\}$ and the map $\phi$ is well-defined. Moreover, $\phi(\Omega_0) \subset \mathbf{R} \cup \{\pm\infty\}$. Now we shall prove that for all $x \in \Omega_0$, $\phi(x) < +\infty$. To this aim, let us recall that $\Gamma_0^t$ is a $c_t$-monotone set. Taking $n = 0$ in the definition above leads to the inequality $\phi(x_0) \leq 0$, the $c_t$-monotonicity of $\Gamma_0^t$ gives us the reverse inequality, thus $\phi(x_0) = 0$. As a particular case, consider $x \in \{y_0\} + t\mathbf{C}$. By definition of the cost function, it is clear that $\phi(x) < +\infty$ for such a $x$. For arbitrary $x$, we shall show that there exists a finite chain of points linking $x_0$ to $x$ such that, loosely speaking, the previous condition is satisfied. More precisely, we shall prove that there exists $k \in \mathbf{N} \setminus \{0\}$ such that for any $x \in \Omega_0$, there exists $(x_i, y_i)_{i=1}^k \in \Gamma_0^t$ such that

$$\begin{array}{rcl} c_t(x_1,y_0) & < & +\infty, \\ \forall i \in \{1, \cdots, k-1\}\ c_t(x_{i+1},y_i) & < & +\infty, \\ c_t(x,y_k) & < & +\infty. \end{array} \tag{13.21}$$

To this aim, let us notice that by Proposition 13.3.1, we have the property $C(T) < +\infty$ which implies that for all closed set $A$,

$$\begin{array}{rcl} \mu_0(A + T\mathbf{C}) & \geq & \mu_1(A), \\ \mu_1(A + T\mathbf{C}) & \geq & \mu_0(A). \end{array} \tag{13.22}$$

Now, we fix $t'$ such that $T < t' < t$ and define by induction the following sequence of sets :

$$\begin{array}{rcl} A_0 & = & \{y_0\}, \\ A_{i+1} & = & \overline{p_y(p_x^{-1}(A_i + t'\mathbf{C}) \cap \Gamma_0^t)}. \end{array}$$

where $p_x$ and $p_y$ stand for the projections on the $x$ and $y$ coordinates respectively, and $\overline{A}$ is the closure of $A$ (in $\mathbf{R}^d$). Formally speaking, Property (13.21) is equivalent to the existence of an integer $k$ such that $A_k + t'\mathbf{C} \supset \Omega_0$ with $t' = t$. However, since the sets $p_y(p_x^{-1}(A_i + t\mathbf{C}) \cap \Gamma_0^t)$ are not necessarily closed sets, we have to require a little more to get the claim. Note that the existence of such a $k$ also implies

$$\forall x \in \Omega_0, \quad \phi(x) \leq ||c||_\infty (k+1). \tag{13.23}$$

Let us prove that $A_k + t'\mathbf{C} \supset \Omega_0$ for large $k$.

By using the optimal transport plan $\pi_0^t$, we obtain for any integer $i$

$$\mu_1(A_{i+1}) \geq \mu_0(A_i + t'\mathbf{C}).$$

Property (13.22) above gives us

$$\begin{aligned} \mu_1(A_{i+1}) &\geq \mu_1(A_i) + \mu_0((A_i + t'\mathbf{C}) \setminus (A_i + T\mathbf{C})) \\ &\geq \mu_1(A_i) + \operatorname{essinf}(\rho_0)\,(|(A_i + t'\mathbf{C}) \cap \Omega_0| - |(A_i + T\mathbf{C}) \cap \Omega_0|). \end{aligned} \tag{13.24}$$

Now, we claim that there exists $M_1 > 0$ such that for any integer $i \geq 1$,

$$|(A_i + T\mathbf{C}) \cap \Omega_0| \geq M_1. \tag{13.25}$$

To this aim, first notice that

$$\mu_0(A_0 + t'\mathbf{C}) = \mu_0(\{y_0\} + t'\mathbf{C}) > 0. \tag{13.26}$$

This follows from the existence of a small $\varepsilon$ such that

$$\{y_0\} + t'\mathbf{C} \supset \{y_0\} + T\mathbf{C} + \overline{B}(\varepsilon)$$

together with an application of (13.22) :

$$\mu_0(\overline{B}(y_0, \varepsilon) + T\mathbf{C}) \geq \mu_1(\overline{B}(y_0, \varepsilon)) > 0$$

since $y_0$ belongs to $\operatorname{supp}\mu_1$. Consequently, we get

$$\begin{aligned} \mu_0(A_1 + T\mathbf{C}) &\geq \mu_1(A_1) \\ &\geq \mu_0(A_0 + t'\mathbf{C}) \end{aligned}$$

where the first inequality follows from (13.22) and the second one from the definition of $A_1$. Since $(A_i)_{i\in\mathbb{N}}$ is a non decreasing sequence with respect to the inclusion and by assumption on $\mu_0$, we get (13.25). Now, we claim that the non decreasing sequence $|(A_i + T\mathbf{C}) \cap \Omega_0| \to |\Omega_0|$ when $i$ goes to infinity. Otherwise, the sequence $|(A_i + T\mathbf{C}) \cap \Omega_0|$ would be bounded away from 0 and $|\Omega_0|$, and using the first statement of Proposition 13.5.2, we would get the existence of a positive number $\varepsilon'$ such that for all $i \geq 1$,

$$|(A_i + T\mathbf{C} + (t' - T)\mathbf{C}) \cap \Omega_0| - |(A_i + T\mathbf{C}) \cap \Omega_0| \geq \varepsilon'.$$

This property combined with (13.24) clearly contradicts the fact that $\mu_1$ is a probability measure and the claim is proved. With this property in hand, we can now apply the second statement of Proposition 13.5.2 and obtain that for large $i$, $|(A_i + t'\mathbf{C}) \cap \Omega_0| = |\Omega_0|$. This implies that for large $i$, $\mu_0(A_i + t'\mathbf{C}) = 1$ which in return gives $\mu_1(A_{i+1}) = 1$ by definition of $A_{i+1}$. As a consequence,

$$\operatorname{supp}\mu_1 \subset A_{i+1}.$$

Since $t'$ is a supercritical speed, $C(t') < +\infty$. Consequently, the following inclusion holds

$$\Omega_0 \subset \operatorname{supp}\mu_1 + t'\mathbf{C}.$$

By combining these last two properties, we get (13.21).

Using the same approach, we shall show that $\phi > -\infty$ on $p_x(\Gamma_0^t)$ hence $\mu_0-$almost everywhere. Let us mention that this last property entails that $\phi^{c_t}(\operatorname{supp}\mu_1) \subset \mathbf{R} \cup \{-\infty\}$ as required in the definition of $c_t$-concave map. Once again, consider as a particular case a pair $(x, y) \in \Gamma_0^t$ such that $c_t(x_0, y) < +\infty$. By definition of $\phi$, we have for any $z$

$$\phi(z) \leq \phi(x) + c_t(z,y) - c_t(x,y). \tag{13.27}$$

Choosing $z = x_0$ in (13.27) together with the fact that $\phi(x_0) = 0$ gives the result. To prove the general case, let us define $C_0 = \overline{p_y(p_x^{-1}(\{x\}) \cap \Gamma_0^t)}$ (which is a non empty set by assumption on $(x,y)$) and $C_{i+1} = \overline{p_y(p_x^{-1}(C_i + t'\mathbf{C}) \cap \Gamma_0^t)}$. The same computations as above give $\mu_1(C_k) = 1$ for $k$ large enough, and consequently, $x_0 \in C_k + t'\mathbf{C}$. This entails $\phi(x) > -\infty$, and more precisely

$$\forall x \in p_x(\Gamma_0^t), \;\; \phi(x) \geq -||c||_\infty (k+1). \tag{13.28}$$

It remains to check that $\phi^{c_t c_t} = \phi$ and that $\Gamma_0^t \subset (Id, \partial_{c_t}\phi)$. The latter property follows easily from (13.27). Let us prove the other one. We start by renaming $y_m$ as $y$,

$$\phi(x) = \inf_{y \in \operatorname{supp}\mu_1} c_t(x,y) - u(y)$$

where $u(y) = -\infty$ if $y \notin p_y(\Gamma_0^t)$ and

$$u(y) = \sup_m \sup\{c_t(x_0,y_0) + \cdots + c_t(x_m,y) - (c_t(x_1,y_0) + \cdots + c_t(x_m,y_{m-1})); \, (x_1,y_1), \cdots (x_m,y) \in \Gamma_0^t\}$$

otherwise.

By (13.28), $u(p_y(\Gamma_0^t)) \subset \mathbf{R}$, that is to say $\phi = u^c$ with $u : \operatorname{supp}\mu_1 \to \mathbb{R} \cup \{-\infty\}$. Now, for arbitrary admissible map $\alpha$, we have

$$\alpha \leq \alpha^{cc} \tag{13.29}$$

by definition of the $c$-transform. Applying the $c$-transform again, we get

$$\alpha^c \geq \alpha^{ccc}.$$

On the other hand, (13.29) applied to $\alpha = u^c$ gives the reverse inequality. Therefore, $u^{ccc} = u^c$ which exactly means $\phi = \phi^{cc}$ as required. □

To state our result concerning the regularity of the Kantorovich potential, we need to recall the notion of approximate differential.

**Definition 13.5.14.** We say that $f : X \mapsto \mathbb{R} \cup \{-\infty\}$ has an approximate differential at a point $x$ if $f(x) \in \mathbf{R}$ and there exists a map $g : X \mapsto \mathbb{R}$ differentiable at $x$ such that the set $\{f \neq g\}$ has density 0 at $x$ and $f(x) = g(x)$. The approximate gradient of $f$ at $x$ is denoted by $\tilde{\nabla} f(x)$.

**Theorem 13.5.15.** *Under the assumptions of Theorem 13.5.13, the map* $\phi_{|p_x(\{x-y \in t\overset{\circ}{\mathbf{C}}\} \cap \Gamma_0^t)}$ *is measurable (up to a modification on a negligible set) and belongs to* $\mathcal{L}^\infty(\Omega_0)$. *Moreover,* $\phi$ *is approximately differentiable Lebesgue almost everywhere on* $p_x(\{x - y \in t\overset{\circ}{\mathbf{C}}\} \cap \Gamma_0^t)$.

*Démonstration.* First, notice that $|\phi|$ is bounded apart from a negligible set thanks to (13.23) and (13.28). For any positive integer $n$, we set

$$\Theta_n = \left\{x \in \overline{\Omega_0}; \exists\, y\,; (x,y) \in \Gamma_0^t \text{ and } x - y \in (1 - 1/n)t\mathbf{C}\right\}.$$

Thus, the following equality holds

$$\Theta := p_x(\{x - y \in t\overset{\circ}{\mathbf{C}}\} \cap \Gamma_0^t) = \bigcup_{n \in \mathbf{N}\setminus\{0\}} \Theta_n. \tag{13.30}$$

We claim that for any $n$, $\phi|_{\Theta_n}$ is a Lipschitz map. Indeed, for any $x \in \Theta_n$ and any $z \in B(x, \frac{tr_m}{2n}) \cap \Theta_n$ (see (13.14) for the definition of $r_m$), let us denote by $y$ a point such that $(x,y) \in \Gamma_0^t$ and $x - y \in (1 - \frac{1}{n})t\mathbf{C}$. We get by (13.27),

$$\phi(z) - \phi(x) \leq c_t(z,y) - c_t(x,y)$$

which implies the existence of a positive constant $k(n)$

$$\phi(z) - \phi(x) \leq k(n)|x - z|$$

by assumption on the cost function. Thus $\phi$ is locally Lipschitz and since it is bounded, it is a Lipschitz map on $\Theta_n$.

Now using Kirszbraun's theorem, we set $\phi_n$ a Lipschitz function defined on $\Theta$ and which coincide with $\phi$ on $\Theta_n$. Using (13.30), we get that $\phi_n$ converges to $\phi$ on $\Theta$. This implies the measurability of $\phi_{|\Theta}$. To conclude the proof, let us remark that if $n \leq m$, then $\Theta_n \subset \Theta_m$, and consequently $\phi_n|_{\Theta_n} = \phi_m|_{\Theta_n}$. Using Rademacher's Theorem together with Lebegue's density Theorem, we get

$$|A_n| = 0 \text{ and } |\tilde{A}_n| = 0$$

where

$$A_n = \{x \in \Theta; \phi_n \text{ is not differentiable at } x\}$$

and

$$\tilde{A}_n = \{x \in \Theta_n; x \text{ is not a point of density 1 of } \Theta_n\}.$$

Note that since $\Theta_n \subset \Theta_{n+1}$ for all $n$, a point of density 1 of $\Theta_n$ is a point of density 1 of $\Theta$ as well. We claim that $\phi$ is approximately differentiable on $Z = \Theta \setminus \left(\cup_{n\in\mathbf{N}\setminus\{0\}}(A_n \cup \tilde{A}_n)\right)$.

Indeed, given $x \in Z$, there exists $n_0$ such that $x \in \Theta_{n_0}$ and $\phi_{n_0}$ is differentiable at $x$. Now since by definition of $Z$, $x$ is a point of density 1 of $\Theta_{n_0}$ and $\Theta_{n_0} \subset \{\phi = \phi_{n_0}\}$, $\phi$ is approximatively differentiable at $x$.
□

### 13.5.3 Proof of Theorem 13.5.1

*Démonstration.* In this short section, we combine previous results in order to prove Theorem 13.5.1. Let $(x, y) \in \{x - y \in t\overset{\circ}{\mathbf{C}}\} \cap \Gamma_0^t \cap Z \times \mathbf{R}^d$. In particular, we have

$$\phi(x) + \phi^c(y) = c_t(x, y).$$

Therefore by definition of the $c$-transform and since $c_t(x, y) = h(\frac{x-y}{t})$ with $h$ a strictly convex function, we get for $z$ close to $x$

$$\phi(z) - \phi(x) \leq h(\frac{z-y}{t}) - h(\frac{x-y}{t}) < +\infty. \tag{13.31}$$

Let $g$ be a differentiable map at $x$ such that the set $\{\phi = g\}$ has density one at $x$ and $g(x) = \phi(x)$. We get that for any $z \in \{\phi = g\}$,

$$h(\frac{z-y}{t}) - h(\frac{x-y}{t}) \geq \tilde{\nabla}\phi(x) \cdot (z - x) + o(|z - x|)$$

and since $y - x \in t\overset{\circ}{\mathbf{C}}$, $h$ is continuous at $\frac{x-y}{t}$. Therefore, the inequality above yields

$$\tilde{\nabla}\phi(x) \in \partial h(\frac{x-y}{t})$$

which gives the result since $h$ is stricly convex. □


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# Chapitre 14

# Constructing a relativistic heat flow by transport time steps

Robert J. McCann [1], Marjolaine Puel [2]

**abstract**

An alternative construction to Andreu, Caselles, and Mazon [12] is given for $L^1_w([0,T], BV(\Omega))$ solutions to the relativistic heat equation (14.1) (see [14], [37], [40], [20], [19]) under the assumption of initial data bounded from below and from above. For that purpose, we introduce a time discretized scheme in the style of Jordan, Kinderlehrer, and Otto [30] [38] involving an optimal transportation problem with a discontinuous hemispherical cost function. The limiting process is based on a monotonicity argument and on a bound of the Fisher information by an entropy balance characteristic of the minimization problem.

**Résumé**

Nous présentons ici une construction alternative à celle d'Andreu, Caselles et Maẓon [12] de solution $L^1_w([0,T], BV(\Omega))$ de l'équation de la chaleur relativiste (14.1) (see [14], [37], [40], [20], [19]) dans le cas de conditions initiales bornées inférieurement et supérieurement. Pour cela, nous introduisons un schéma discret en temps dans le style de Jordan, Kinderlehrer et Otto [30] [38] basé sur un problème de transport optimal faisant intervenir une fonction de coût hémisphérique et discontinue. Le passage à la limite lorsque le pas de temps tend vers zéro repose sur un argument de monotonie et une borne de l'information de Fisher par la variation de l'entropie, inégalité caractéristique du problème de transport optimal.

## 14.1 Introduction

We consider in this work a relativistic heat equation which has been introduced for example in the paper of Rosenau [40] or Mihalas–Mihalas [37] ; it fills in a gap in the Fokker-Planck theory by imposing an upper bound for the propagation velocity. This equation can be written

$$\partial_t \rho = \operatorname{div}(\rho \frac{\nabla \rho}{\sqrt{\rho^2 + |\nabla \rho|^2}}) = \operatorname{div}(\rho \frac{\nabla \log \rho}{\sqrt{1 + |\nabla \log \rho|^2}}). \tag{14.1}$$

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Viewing the ordinary heat equation as a transport equation by velocity $\nabla \log \rho$, then equation (14.1) involving the velocity $\dfrac{\nabla \rho}{\sqrt{\rho^2 + |\nabla \rho|^2}}$ motivates the appellation *relativistic*.

The aim of this present work is to construct solutions to equation (14.1) following a strategy introduced by Jordan, Kinderlehrer, Otto [30] and Otto [38], subsequently developed by many authors, M. Agueh [1] and Ambrosio, Gigli, Savare [3] in particular, and suggested in the relativistic context by Brenier [14]. This strategy applies to finding a solution to general transport equations given by

$$\partial_t \rho = \operatorname{div}(\rho \nabla c^*(\nabla(F'(\rho)))) \tag{14.2}$$

where $c^*$ is a convex mobility function on $\mathbf{R}^d$ and $F$ is a convex function on $[0, \infty[$ representing the entropy. It is based on a new point of view on (14.2) saying that the transport of the density $\rho$ is seen as the gradient flow of the convex function $F$ with respect to a distance induced by the cost function $c$, the Legendre transform of $c^*$ ( see [2], [3], [5], [43] for the notion of gradient flow). The solution is obtained as a limit of a solution of a time discretized scheme and the peculiarity of the method is that this discrete scheme involves an optimal mass transport problem.

This work comes after a series of papers. In [30], the case of the Wasserstein distance $c(z) = \frac{|z|^2}{2}$ and entropy $F(\rho) = \rho \log \rho - V\rho$ were addressed by Jordan, Kinderlehrer and Otto, who obtain the Fokker-Planck equation

$$\partial_t \rho = \operatorname{div}(\rho \nabla V) + \beta^{-1} \Delta \rho \text{ where } V \text{ is a given smooth function.}$$

In [38], Otto treated the case of $c(z) = \frac{|z|^q}{q}$ with $F(\rho) = \frac{n\rho^b}{b(b-1)}$ and $b = n + (p-2)/(p-1)$ which leads to the doubly degenerate equation

$$\partial_t \rho = \operatorname{div}(|\nabla \rho^n|^{p-2} \nabla \rho^n) \quad \text{with } \frac{1}{p} + \frac{1}{q} = 1, \quad p \geq 2.$$

And in [1], Agueh considered the case of cost functions satisfying

$$\beta |z|^q \leq c(z) \leq \alpha(|z|^q + 1) \text{ for all } z \in \mathbf{R}^d, \text{ where } \alpha, \beta > 0 \text{ and } q > 1,$$

which includes a very large class of equations such as the Fokker Planck equation, the porous medium and fast diffusion equation, the p-Laplacian and the doubly degenerate diffusion equation.

Note that for the relativistic heat equation, the entropy is $F(\rho) = \rho \log \rho - \rho$ and

$$\nabla c^*(z) = \frac{z}{\sqrt{1 + |z|^2}}.$$

which gives

$$c^*(z) = \sqrt{1 + |z|^2} - 1.$$

The corresponding cost function

$$c(z) = \begin{cases} 1 - \sqrt{1 - |z|^2} & \text{if } |z| \leq 1 \\ +\infty & \text{if } |z| > 1 \end{cases}$$

is hemispherical and discontinuous, hence does not belong to the set of cost function for which this strategy has been successful.

The aim of this paper is to apply this "Optimal Transportation Strategy" to the Cauchy problem for equation (14.1), and more precisely for any cost functions satisfying

$$c(z) = \begin{cases} \tilde{c}(|z|) \geq 0 & \text{if } |z| \leq 1 \\ \\ +\infty & \text{if } |z| > 1 \end{cases}$$

where $\tilde{c}$ is a continuous strictly convex function on $[0, 1]$ belonging to $C^2([0, 1[)$ and with $|\nabla c(z)| \to \infty$ when $|z| \to 1$ (and hence $c^* \in C^2(\mathbf{R}^d)$) and, as in the work of Agueh [1], for any strictly convex Entropy functions satisfying $F \in C^2(\mathbf{R})$ such that $\frac{F(\lambda)}{\lambda} \to \infty$ when $\lambda \to \infty$ and $\lambda^d F(\lambda^{-d})$ is convex.

The Cauchy problem for such an equation has been recently studied by Andreu, Caselles and Mazon [11] [12] and the speed of propagation for the support of its solutions has been studied by Andreu, Caselles, Mazon and Moll in [13]. The convergence of the relativistic equation toward the heat equation as the light speed goes to infinity has been also investigated very recently by Caselles [19]. Those works belong to a large program (see the long series of paper [7] [9] [10] [11] [12]) where the Cauchy problem is examined for degenerate elliptic and parabolic equations. As in the strategy of Jordan, Kinderlehrer and Otto, the proof of the existence of solution to this equation by Andreu, Caselles and Mazon involves a discrete in time equation. In their work, the time discrete density $\rho^h$ is obtained by solving the elliptic equation on each time interval $[ih; (i+1)h]$

$$\rho_i^h - \rho_{i-1}^h = h \ \mathrm{div}(\rho_i^h \nabla c^*(D \log \rho_i^h)).$$

We will assume in this work that the initial density is bounded from below and from above while in their work, Andreu, Caselles and Mazon assume only that the initial data is non-negative.

Nevertheless, the point of view of this present work is interesting since the method employed — in particular the construction of the discrete scheme — involves the study of a mass transport problem with a discontinuous convex cost function. The limiting process remains a delicate step because of the weak regularity of the discrete sequence constructed via the minimization process, and presents a real difficulty. As far as we know, the construction of optimal transport maps for discontinuous cost functions has not been completed before but is a necessary condition to obtain relativistic phenomena for the corresponding transport equation. However, the Kantorovich duality for discontinuous cost functions has been investigated in the paper of Ambrosio and Pratelli [4], and the use of approximate differentiability as in Ambrosio, Gigli, Savare [3] also proves a crucial tool.

### 14.1.1 Description of the "Optimal transportation strategy"

Fix $\Omega \subset\subset \mathbf{R}^d$ bounded throughout. The method consists in constructing a time discrete scheme as follows :

*Let $P(\Omega)$ be the set of Borel probability measures on $\Omega$, $\rho_0 \in P(\Omega)$ given, find $\rho^h(t,y) \in P([0,T] \times \Omega)$ defined by*

$$\begin{cases} \rho^h(0,y) = \rho_0(y) \\ \\ \rho^h(t,y) = \rho_i^h(y) \quad \textit{for } t \in ]ih; (i+1)h], \qquad h \textit{ (being the time step)} \end{cases} \tag{14.3}$$

*where $\rho_i^h(y)$ is a solution of the minimization problem $\mathcal{P}_i^h$ defined by*

$$I(\rho_{i-1}^h, \rho_i^h) = \inf_{\rho \in P(\Omega)} I(\rho_{i-1}^h, \rho)$$

*with*

$$I(\rho_{i-1}^h, \rho) = \int_\Omega F(\rho(y))dy + h \inf_{\gamma \in \Gamma_i^h(\rho_{i-1}^h, \rho)} \int_{\Omega \times \Omega} c(\frac{x-y}{h}) d\gamma(x,y),$$

*$\Gamma_i^h$ denoting the set of probability measures that have $\rho_{i-1}^h$ and $\rho$ as marginals and $c$ is the convex cost function, the Legendre transform of $c^*$ defined as in [15]* by

$$c^*(z) = \sup_{w \in \mathbf{R}^d} (w \cdot z - c(w)). \tag{14.4}$$

This process follows the ideas presented in the work of Otto [38] which extends the notion of gradient flow to more general cost functions (cf also Villani's book [43]). This point of view is particulary simple to explain when the cost function is the quadratic function, minimizing $E(\rho) + h|\frac{\rho_0 - \rho}{h}|^2$ gives formally $\frac{\rho_0 - \rho}{h} = E'(\rho)$ which is a discrete version of $\partial_t \rho = E'(\rho)$, meaning that $\rho$ is a gradient flow of $E$. In a more general setting, we minimize the Entropy among all the densities reachable at time $T$ by moving along geodesics induced by the cost function $c$.

Let us now describe the different steps of the construction of a solution to (14.2) :

**Step 1** Prove that the minimization problem has a unique solution, the optimal transport plan $\gamma$.

**Step 2** Define an optimal map corresponding to this optimal measure and derive the Euler-Lagrange equation of this minimization problem which gives formally

$$\nabla c\left(\frac{S_i^h(y)-y}{h}\right) = \nabla(F'(\rho_i^h))$$

where $S_i^h$ denotes the optimal map pushing $\rho_i^h$ forward to $\rho_{i-1}^h$.

Using $\nabla c^*(\nabla c(y)) = y$, this leads to

$$S_i^h(y) - y = h\nabla c^*(\nabla(F'(\rho_i^h))). \tag{14.5}$$

**Step 3** Obtain an approximate time discrete equation and pass to the limit when the time step goes to zero :

By multiplying equation (14.5) by $\rho\nabla\xi$ where $\xi$ is a smooth test function, we obtain in the sense of distributions

$$\rho_i^h - \rho_{i-1}^h = h\ \mathrm{div}(\rho_i^h \nabla c^*(\nabla(F'(\rho_i^h)))) + \text{Correction terms.} \tag{14.6}$$

Solutions to (14.2) are obtained by passing to the limit in (14.6) when $h$ goes to zero. The identification of $\overline{A}$, the limit of $\rho_i^h\nabla c^*(\nabla(F'(\rho_i^h))$ involves a Minty Browder argument (see for example Evans's book [25]) based on a monotonicity property of the gradient of the mobility function $c^*$. Indeed, we prove for any test function $\xi(t,x) \geq 0$ and $\zeta(t,x) \in C^\infty(\mathbf{R}^d, \mathbf{R}^d)$

$$\int \xi(\overline{A} - \overline{\rho}\nabla c^*(\zeta))(D^{ac}(F'(\overline{\rho})) - \zeta) \geq 0 \tag{14.7}$$

which yields $\overline{A} = \overline{\rho}\nabla c^*(D^{ac}(F'(\overline{\rho})))$ by taking $\zeta = D^{ac}(F'(\overline{\rho})) + \delta w$, and by passing successively to the limit $\delta \searrow 0$ and $\delta \nearrow 0$.

Formally, to obtain (14.7), we pass to the $\varepsilon < h \to 0$ limit in

$$\int_0^T \int_\Omega \xi(t,y)\rho^{\varepsilon h}(t,y)(\nabla c^*(\nabla(F'(\rho^{\varepsilon h}(t,y)))) - \nabla c^*(\zeta(t,y))) \cdot (\nabla(F'(\rho^{\varepsilon h}(t,y))) - \zeta(t,y))dydt \geq 0.$$

This limiting process is strongly based :

1- on the displacement convexity [35] of the Entropy function (ensured by the monotonicity of $F$ and the convexity of $\lambda^d F(\lambda^{-d})$ that yields formally the Fisher information-entropy inequality satisfied by the minimizer $\rho^h$ of problem $\mathcal{P}_i^h$

$$\int_0^T \int_\Omega \rho^h(t,y)\nabla c^*(\nabla(F'(\rho^h(t,y)))) \cdot \nabla(F'(\rho^h(t,y)))dydt \leq \int_\Omega (F(\rho^h(0,y)) - F(\rho^h(T,y)))dy \tag{14.8}$$

or its localized version

$$\begin{aligned}\int_0^T \int_\Omega \xi(t,y)\rho^h(t,y)\nabla c^*(\nabla(F'(\rho^h(t,y)))) \cdot \nabla(F'(\rho^h(t,y)))dydt \\ \leq -\int_0^T \int_\Omega \nabla\xi(t,y)\rho^h(t,y) \cdot \nabla c^*(\nabla(F'(\rho^h(t,y))))F'(\rho^h(t,y))dydt \\ -\int_0^T \int_\Omega \xi(t,y)\partial_t F(\rho^h(t,y))dydt\end{aligned}$$

2- on the corresponding equality satisfied formally by any smooth solution $\tilde{\rho}$ of (14.1) and obtained by multiplying the equation by $F'(\tilde{\rho}(t,y))$ and integrating by parts :

$$\int_\Omega \tilde{\rho}(t,y)\nabla F'(\tilde{\rho}(t,y)) \cdot \nabla c^*(\nabla F'(\tilde{\rho}(t,y)))dydt = \int_\Omega [(F(\rho_0(y)) - F(\tilde{\rho}(T,y))]dy. \tag{14.9}$$

These relations avoid the problem of nonlinearities in $\rho^h$ since we will prove a strong convergence which allows us to pass to the limit in the right hand side of (14.8).

### 14.1.2 Assumptions on the cost function and on the Entropy function

We will here give some direct consequences of the assumptions on the cost function that justify its relation with relativistic phenomena.

Recall that we deal with cost functions satisfying

$$c(z) = \begin{cases} \tilde{c}(|z|) \geq 0 & \text{if } |z| \leq 1 \\ \\ +\infty & \text{if } |z| > 1 \end{cases}$$

where $\tilde{c} \in C^2([0,1[)$ is bounded and strictly convex with $\tilde{c}''(\lambda) > 0$ and $\tilde{c}'(\lambda) \to \infty$ as $\lambda \to 1$. Assume also that $\tilde{c}(0) = 0 = \tilde{c}'(0)$ ; then $c^* \in C^2(\mathbf{R}^d)$.

Recall that we have the two relations $\nabla c\,(\nabla c^*(z)) = z$ and $z \cdot \nabla c^*(z) = c(\nabla c^*(z)) + c^*(z)$ and note that since $c^*$ is radial, $\nabla c^*(z) = \frac{z}{|z|}\omega(|z|)$ which implies that $z \cdot \nabla c^*(z) = |z||\nabla c^*(z)|$.

Remark then that the discontinuity of $c$ — implying its infinite part — is strongly linked with the bound on $\nabla c^*(z)$. Indeed, since $c$ is defined from $c^*$ by

$$c(z) = \sup_{w \in \mathbf{R}^d} (z \cdot w) - c^*(w),$$

the fact that $c$ is not finite when $|z| > 1$ means that the supremum is not attained and then the relation $z = \nabla c^*(w)$ cannot be matched by any $w \in \mathbf{R}^d$. This means that $|\nabla c^*| \leq 1$ (and reciprocally) so we recover that the discontinuity of the cost function is equivalent to the relativistic aspect of the transport. It also implies — since $|\nabla c^*(z)| = \omega(|z|)$ is a non decreasing function of $|z|$ — that $\lim_{|z| \to \infty} |\nabla c^*(z)| = 1$.

As we said, we consider any strictly convex Entropy functions satisfying $F \in C^2(\mathbf{R})$ such that $\frac{F(\lambda)}{\lambda} \to \infty$ when $\lambda \to \infty$ and $\lambda^d F(\lambda^{-d})$ is convex, the last condition being necessary for displacement convexity of the Entropy [35].

### 14.1.3 Notations and definitions

#### Optimal transport theory

We now recall two ways to link pairs of probability measures.

*Definition* 14.1.1. Let $\rho_1 \in P(\Omega)$ and $S$ a Borel map $S : \Omega \to \Omega$. We say that $\rho_0$ is the *push-forward* of $\rho_1$ through $S$ if for any bounded Borel function $\varphi$

$$\int_\Omega \varphi(S(x)) d\rho_1(x) = \int_\Omega \varphi(x) d\rho_0(x).$$

*Definition* 14.1.2. Given two probability measures $\rho_0$ and $\rho_1$, the set of *transport plans* between them refers to joint probability measures on $\Omega \times \Omega$ with $\rho_0$ and $\rho_1$ as marginals :

$$\Gamma(\rho_0, \rho_1) = \{\gamma(x,y) \in P(\Omega \times \Omega) \quad \text{s.t.} \int_{\Omega\times\Omega} \varphi(x) d\gamma(x,y) = \int_\Omega \varphi(x) d\rho_0(x)$$

$$\text{and} \int_{\Omega\times\Omega} \varphi(y) d\gamma(x,y) = \int_\Omega \varphi(y) d\rho_1(y)\}$$

for all Borel test functions $\varphi : \Omega \longrightarrow \mathbf{R}$.

Finally, we need to define the $c-$superdifferential of a function

*Definition* 14.1.3. $x \in \partial^c \varphi(y)$ if $|x - y| \leq 1$ and for all $v \in \Omega$

$$\varphi(v) \leq c(x - v) - c(x - y) + \varphi(y).$$

## Definition of BV Functions and of functions of BV functions

In this section, we recall definitions that can be found in the series of paper of Andreu, Caselles and Mazon, that we will use throughout the second part of this paper and that are fundamental for the understanding of the notion of solution to equation (14.1) which we construct.

### Some functional spaces

Throughout this paper, we will deal with $BV$ functions, the set of $\rho \in L^1(\Omega)$ functions such that the gradient of $\rho$ defined as a distribution is a vector valued Radon measure whose total variation, i.e.

$$||\rho||_{BV} = ||D\rho||_{TV} = \sup\{\int_\Omega \rho \mathrm{div}(\xi)dx \text{ where } \xi \in C_0^\infty(\Omega; \mathbf{R}^d) \text{ s.t. } |\xi(x)| \leq 1\}$$

is finite.

A sequence $\rho_i$ of $BV$ functions is said to converge $w^*BV(\Omega)$ toward $\rho$ if $\rho_i \to \rho$ in $L^1_{loc}(\Omega)$ and its gradient $D\rho_i$ converges toward $D\rho$, weak-$*$ as measures, i.e. against any compactly supported continuous test function.

We also need to introduce the space $L^1([0,T], BV_2(\Omega))$ where $BV_2(\Omega) = BV(\Omega) \cap L^2(\Omega)$ with the norm $||\rho||_{BV_2(\Omega)} = ||\rho||_{L^2(\Omega)} + ||D\rho||_{TV}$ contained in the space where the solution will live, $L^1_w([0,T], BV(\Omega))$. The difference between those two spaces comes from the fact that $BV(\Omega)$ is not separable.

*Definition* 14.1.4. A function $\rho$ belongs to $L^1([0,T], BV(\Omega))$ if it is a limit almost everywhere in time of a sequence of simple functions, i.e. defined by $\sum \mathbf{1}_{[\mathbf{t_i}, \mathbf{t_{i+1}}]} \rho_\mathbf{i}$ where $\rho_i \in BV(\Omega)$, hence Bochner integrable.

In particular $C([0,T], BV(\Omega)) \subset L^1([0,T], BV(\Omega))$.

*Definition* 14.1.5. A function $\rho$ belongs to $L^1([0,T], BV_2(\Omega))$ if it is a limit almost everywhere in time of a sequence of simple functions, i.e. defined by $\sum \mathbf{1}_{[\mathbf{t_i}, \mathbf{t_{i+1}}]} \rho_\mathbf{i}$ where $\rho_i \in BV_2(\Omega)$.

Whereas

*Definition* 14.1.6. $L^1_w([0,T], BV(\Omega))$ is the space of weakly measurable functions $\rho : [0,T] \to BV(\Omega)$ (i.e., $t : [0,T] \to \langle \rho(t); \zeta \rangle$ is measurable for every $\zeta \in BV(\Omega)'$), such that $\int_0^T ||\rho(t)||_{BV(\Omega)} dt < \infty$.

Note that, since $BV(\Omega)$ has a separable predual, it follows for $\rho \in L^1_w([0,T], BV(\Omega))$, that $t : [0,T] \to ||\rho(t)||_{BV(\Omega)}$ is measurable.

The first space is useful because we know its dual $L^1([0,T], BV_2(\Omega))'$ equals (see Dunford Schwartz [24])

$$L^\infty([0,T], BV_2(\Omega)') = \{\rho \text{ weak}^* \text{ measurable functions}, \rho : [0,T] \to BV(\Omega)_2' \quad \text{such that} \quad \operatorname*{ess\,sup}_{[0,T]} \sup_{\|\xi\|_{BV_2(\Omega)} \leq 1} |\int_\Omega \rho\xi| < \infty \Big\}$$

and we define the duality bracket for $\rho \in L^1([0,T], BV_2(\Omega))$ and $\tilde{\rho} \in L^\infty([0,T], BV_2(\Omega)')$ by

$$\langle \rho; \tilde{\rho} \rangle = \int_0^T \langle \rho(t); \tilde{\rho}(t) \rangle dt.$$

As a matter of fact, the solution will live in $L^\infty([0,T] \times \Omega) \cap L^1_w([0,T], BV(\Omega))$ and the equation will hold in the sense of distributions, which leads to one of the principal difficulties of the third step of the proof, namely that the solution cannot be taken as a test function to obtain (14.9).

### Functions defined on BV

As we will see, $D\rho$, the gradient of the $BV$ function $\rho$ can be decomposed as the sum of its absolute continuous part $D^{ac}\rho$ ( with respect to the Lebesgue measure), also called the Radon Nikodym

derivative of $D\rho$ (which coincides with the approximate derivative) of $\rho$, and its singular part $D^s\rho$, divided into a jump part $D^j\rho$ and a Cantor Part $D^c\rho$. So we write

$$D\rho = D^{ac}\rho + D^s\rho = D^{ac}\rho + D^j\rho + D^c\rho$$

Moreover, we will need to define the composition of certain functions with $BV$ functions and their derivatives. For example, if $f(x,\lambda,\xi)$ is an integrand depending on the space variable $x$, on $\lambda \in \mathbf{R}$ and on the vector field $\xi$, we will define the functional $\mathcal{F}(\rho, D\rho)$ of a $BV$ function $\rho$ as in the paper of Dal Maso [22] by

$$\mathcal{F}(\rho, D\rho) = \int_0^T \int_\Omega f(x,\rho,D^{ac}\rho)dtdx + \int_\Omega f^0(x,\rho,\frac{D\rho}{|D\rho|})|D^c\rho| + \int_{J_\rho}(\int_{\rho_-(x)}^{\rho_+(x)} f^0(x,s,\nu_\rho(x))ds)d\mathcal{H}^{d-1}(x) \tag{14.10}$$

where $f^0$ is the recession function equal to $\lim_{t\to 0} tf(x,\lambda,\frac{\xi}{t})$, $J_\rho$ is the set of approximate jump points of $\rho$, $\nu_\rho = D\rho/|D\rho|$ the Radon Nikodym derivative of $D\rho$ with respect to its total variation $|D\rho|$. Indeed with those definition, the jump part of the singular part of $D\rho$ can be written

$$D^j\rho = (\rho^+ - \rho^-)\nu_\rho \mathcal{H}^{d-1}_{|J_\rho}$$

where $\mathcal{H}^{d-1}$ is the $d-1$ Hausdorff measure on $\mathbf{R}^d$.

For example, in the theorem cited below, there is in fact no ambiguity in the right hand side term of equation (14.14) since the corresponding recession function is zero.

We can also compute $g^0$ in the case where $f = g(\lambda,\xi) = \lambda\xi\cdot\nabla c^*(\xi)$. We obtain easily using the properties of the cost function that

$$\lim_{t\to 0} tf(x,\lambda,\frac{\xi}{t}) = \lim_{t\to 0}\lambda|\xi||\nabla c^*(\frac{\xi}{t})| = \lambda|\xi|. \tag{14.11}$$

Finally, we also need the result of De Cicco, Fusco and Verde about the $L^1$-lower semicontinuity in $BV$

*Theorem* 14.1.7 ( $L^1$-lower semicontinuity in $BV$ [23]). Let $\Omega$ be an open set of $\mathbf{R}^d$ and $h : \Omega\times\mathbf{R}\times\mathbf{R}^d \to [0,\infty)$ a locally bounded Caratheodory function (that is , measurable with respect to $x$ in $\Omega$ for every $(\lambda,\xi)\in\mathbf{R}\times\mathbf{R}^d$, and continuous with respect to $(\lambda,\xi)$ for almost every $x$ in $\Omega$) such that for every $(\lambda,\xi)\in\mathbf{R}\times\mathbf{R}^d$, the function $h(\cdot,\lambda,\xi)$ is of class $C^1(\Omega)$. Let us assume that
(i) $h(x,\lambda,\cdot)$ is convex in $\mathbf{R}^d$ for every $(x,\lambda)\in\Omega\times\mathbf{R}$
(ii) $h(x,\cdot,\xi)$ is continuous in $\mathbf{R}$ for every $(x,\xi)\in\Omega\times\mathbf{R}^d$.

Then the functional $\mathcal{H}$ defined by

$$\mathcal{H}(\rho, D\rho) = \int_0^T \int_\Omega h(x,\rho,D^{ac}\rho)dtdx + \int_\Omega h^0(x,\rho,\frac{D\rho}{|D\rho|})|D^c\rho| + \int_{J_\rho}(\int_{\rho_-(x)}^{\rho_+(x)} h^0(x,s,\nu_\rho(x))ds)d\mathcal{H}^{d-1}(x)$$

is lower semi continuous with respect to $L^1(\Omega)$ convergence.

This extension of the functional $\mathcal{F}$ to BV functions allows us to extend the definition of

$$\int_0^T\int_\Omega DF'(\rho)\cdot\nabla c^*(D(F'(\rho)))$$

to $\rho\in L^1([0,T],BV(\Omega))$ by decomposing $\xi\cdot\nabla c^*(\xi) = c(\nabla c^*(\xi)) + c^*(\xi)$ and by applying Theorem 14.1.7 to $c^*(\xi)$.

**Definition of the measure** $(z, D\rho)$

There is another way of defining $(DF'(\rho);\nabla c^*(D(F'(\rho))))$, for $\rho\in L^\infty([0,T]\times\Omega)$ and $\mathrm{div}(\nabla c^*(D(F'(\rho))))\in L^1([0,T]\times\Omega)$ which is inspired by [6]. But in this present work, this is not relevant since the last property is not satisfied even at the time discrete level. This make the notion of

entropic solution more difficult to introduce in this framework and this property and the uniqueness of solution coming from it is not addressed here.

However, we define $(z; D\rho)$ as a distribution when $z \in L^\infty([0,T]\times\Omega; \mathbf{R}^d)$, $\rho \in L^1([0,T], BV_2(\Omega)) \cap L^\infty([0,T]\times\Omega)$, when $\mathrm{div} z = B$ in the sense of distributions, $B$ is the weak$^*$ limit of a bounded sequence $B^h$ in the dual of $L^1([0,T], BV_2(\Omega))$ and $z$ is the weak$^*$ limit of a bounded sequence $z^h$. Indeed, for any test function $\xi$, we write

$$< \xi; (z; D\rho) >= -\int_0^T < B; \rho\xi >_{L^1([0,T],BV_2(\Omega))'; L^1([0,T],BV_2(\Omega))} dt - \int_0^T \int_\Omega z\rho\nabla\xi dtdy.$$

But if $z$ is the weak$^*$ limit of a bounded sequence $z^h$, we still have to prove that

$$< \xi; (z; D\rho) >= \lim_{h\to 0} < \xi; (z^h; D\rho) >= \lim_{h\to 0} \left( -\int_0^T \int_\Omega \mathrm{div} z^h \rho\xi dtdy - \int_0^T \int_\Omega z^h \rho\nabla\xi dtdy. \right)$$

For that to be true, we need that $\mathrm{div} z^h = B^h$ for $h$ and $t$ fixed in $L^1(\Omega)$. Indeed, in this case, we can write

$$\begin{aligned}
< \xi; (z; D\rho) > &= -\int_0^T < B; \rho\xi >_{L^1([0,T],BV_2(\Omega))'; L^1([0,T],BV_2(\Omega))} dtdy - \int_0^T \int_\Omega z\rho\nabla\xi dtdy \\
&= -\lim_{h\to 0} \int_0^T \int_\Omega B^h \rho\xi dtdy - \int_0^T \int_\Omega z\rho\nabla\xi dtdy \\
&= -\lim_{h\to 0} \int_0^T \int_\Omega \mathrm{div} z^h \rho\xi dtdy - \int_0^T \int_\Omega z\rho\nabla\xi dtdy \\
&= -\lim_{h\to 0} (\int_0^T \int_\Omega \mathrm{div} z^h \rho\xi dtdy + \int_0^T \int_\Omega z^h \rho\nabla\xi dtdy).
\end{aligned}$$

Moreover, $(z; D\rho)$ is a Radon measure since by Theorem 1.5 in [6], we have

$$\begin{aligned}
< \xi; (z; D\rho) > &= \lim_{h\to 0} \left[ -\int_0^T \int_\Omega \mathrm{div}\ z^h \rho\xi dtdy - \int_0^T \int_\Omega z^h \rho\nabla\xi dtdy \right] \\
&\leq ||z||_{L^\infty([0,T]\times\Omega)} ||\xi||_{L^\infty([0,T]\times\Omega)} \int_0^T \int_\Omega |D\rho| dydt
\end{aligned}$$

We define the measure $(z; D^s\rho)$ as the subtraction

$$(z; D^s\rho) = (z; D\rho) - zD^{ac}\rho\mathcal{L} = \lim_{h\to 0} (z^h; D^s\rho)$$

where $\mathcal{L}$ denotes the Lebesgue measure. It is proven in [31] that $(z^h; D^s\rho)$ is a singular measure and that

$$|(z^h; D^s\rho)| \leq ||z^h||_{L^\infty(\Omega)} |D^s\rho|.$$

Then we also have when $\xi \geq 0$

$$< \xi; (z; D^s\rho) > \leq ||z||_{L^\infty([0,T]\times\Omega)} \int_0^T \int_\Omega \xi |D^s\rho| dydt \tag{14.12}$$

### 14.1.4 Statement of the result

Throughout this paper, we assume $\Omega \subset\subset \mathbf{R}^d$ to be a bounded convex domain, and $0 < m < \rho_0 < M$ which implies $\int_\Omega F(\rho_0) < \infty$. The support $\mathrm{supp}\,\mu$ of a measure $\mu$ on $\mathbf{R}^d$ refers to the smallest closed set of full mass. Let us now state our main result.

*Theorem* 14.1.8. Let $\Omega \subset\subset \mathbf{R}^d$ be a bounded convex domain, and $\rho_0 \in P(\Omega)$ satisfy $0 < m < \rho_0 < M$. Let $c : \mathbf{R}^d \longrightarrow [0,\infty]$ be a cost function on $\mathbf{R}^d$ given by $c(z) = \tilde{c}(|z|)$, where $\tilde{c} \in C^0([0,1]) \cap C^2([0,1[)$ with $|\nabla c(z)| \to \infty$ as $|z| \to 1$, $\tilde{c}(0) = 0 = \tilde{c}'(0)$ and $\tilde{c}''(\lambda) > 0$ on $[0,1[$. Let $F \in C^2(\mathbf{R})$ be a convex function such that $\frac{F(\lambda)}{\lambda} \to \infty$ when $\lambda \to \infty$ and $\lambda^d F(\lambda^{-d})$ is convex.

(i) **Characterization of the support of the optimal measure : Finite speed of propagation**

If $\gamma_i^h$ is the optimal measure for the minimization problem $(\mathcal{P}_i^h)$ with second marginal $\rho_i^h$, then $\rho^{ih} \in BV(\Omega)$ and

$$\operatorname{supp}\gamma_i^h \subset \{(x,y) \quad | \quad \frac{|x-y|}{h} < 1\} \cup Z_i^h \text{ with } \gamma_i^h(Z_i^h) = 0.$$

(ii) **Euler-Lagrange equation : a discrete scheme**

There exists a one-to-one map $S_i^h \in L^\infty(\Omega;\Omega)$ defined Lebesgue-a.e. by

$$S_i^h(y) = y + h\nabla c^*(D^{ac}(F'(\rho_i^h(y)))) \tag{14.13}$$

such that $\gamma_i^h = (S_i^h \times id)_\# \rho_i^h$ and $D^{ac}(F'(\rho_i^h(y))) = F''(\rho_i^h(y)) D^{ac}\rho_i^h$.

(iii) **Convergence of the measure** $\rho^h$. Let $\rho^h$ be the piecewise constant function defined from $\rho_i^h$ by (14.3). As $h \to 0$, a subsequence of $\rho^h$ converges strongly in $L^1([0,T]\times\Omega)$ towards $\overline{\rho} \in L^\infty([0,T]\times\Omega) \cap L^1_w([0,T], BV(\Omega))$ and $D\rho^h \rightharpoonup D\overline{\rho}$ in the sense of measures.

(iv) **Limiting equation**

Up to a subsequence, the limit of $\rho^h$, $\overline{\rho}$ belongs to $L^1_w([0,T], BV(\Omega)) \cap L^\infty([0,T]\times\Omega)$ and satisfies in the sense of distributions

$$\partial_t \overline{\rho} = \operatorname{div}(\overline{\rho}\nabla c^*(D^{ac}(F'(\overline{\rho})))). \tag{14.14}$$

Let us now make some remarks about this theorem.

- The first point of the theorem gives the finite speed of propagation, indeed, since in a time interval of length $h$, the displacement is bounded by $h$, the speed of propagation is bounded by 1. This property characterizes a relativistic transport.
- In (14.13), since $F \in C^2(]0,1[)$ and $\rho^h$ is bounded from below, $F'(\rho^h)$ belongs to $BV(\Omega)$ and the chain rule for $BV$ functions gives $D^{ac}(F'(\rho_i^h(y))) = F''(\rho_i^h(y)) D^{ac}\rho_i^h$.
- The second point of this theorem is the most important one. First, it claims the existence of an optimal map for the minimization problem $\mathcal{P}_i^h$ and it gives the Euler-Lagrange equation. The existence of an optimal map for a discontinuous cost function cannot be obtained by the same argument as for a smooth cost function. Indeed, the proof of the existence of a map for a smooth test function by the Kantorovich duality (see Gangbo McCann [28], Villani [43]) is based on a uniform lipschitz bound on the potential function given by the module of continuity of the cost function. More precisely, let $\gamma_{\text{opt}}$ be the optimal measure for the minimization problem, the Kantorovich duality gives

$$\int_{\Omega\times\Omega} hc(\frac{x-y}{h}) d\gamma_{\text{opt}}(x,y) = \int_\Omega \phi(x) d\rho_{i-1}(x) + \int_\Omega \psi(y) d\rho_i(y),$$

where the potential functions $\psi$ and $\phi$ are linked by the following relation

$$hc(\frac{x-y}{h}) - \phi(x) - \psi(y) \geq 0 \quad \forall (x,y) \in \Omega\times\Omega \tag{14.15}$$

which becomes

$$hc(\frac{x-y}{h}) - \phi(x) - \psi(y) = 0 \quad \forall (x,y) \in \operatorname{supp}\gamma_{\text{opt}}. \tag{14.16}$$

When the cost function $c$ is smooth, the relations (14.15)–(14.16) implies that the potential function $\psi$ is Lipschitz — and then differentiable — and then it implies also the equality between the gradients

$$\nabla\psi(y) = \nabla c(\frac{x-y}{h}) \tag{14.17}$$

that gives directly the shape of the optimal map $x = S(y) = y + h\nabla c^*(\nabla\psi(y))$ since $\nabla c^* = (\nabla c)^{-1}$. In this present work, the potential function will not be Lipschitz anymore and then we have to find another argument to prove its almost everywhere differentiability. Moreover, to write an equality like (14.17), we need that the support of the optimal measure $\gamma$ is — up to a negligible set — included in $\{(x, y)$ such that $\frac{|x-y|}{h} < 1\}$ to be able to define the gradient of $c$. So we introduce a mollified problem and pass to the limit. The sequence $\rho^h$ will not be Lipschitz but will be in $BV(\Omega)$ which gives only the almost everywhere approximate differentiability. As Ambrosio, Gigli and Savare showed in Theorem 6.4.2 [3], this is sufficient to define an optimal map. The strategy of cost mollification has often been used, for example in proof of the existence of a map for (non strictly) convex cost via decomposition on one dimensional rays by Ambrosio and Pratelli [4] (see also [26]).
Note that the regularity results for the optimal map of Ma, Trudinger and Wang [34] and Loeper [33] do not apply for this sign of cost function.
Note that the shape of the Euler-Lagrange equation involves $\nabla c^*$ and then we recover the finite speed of propagation (the relativistic effect) since $\nabla c^*$ is bounded.

– The proof is strongly based on a Fisher information-entropy inequality which thanks to the lower bound on the density gives the $L^1([0,T], BV(\Omega))$ bound on the solution.
– To pass to the limit when $h$ goes to zero, we want to use a monotonicity argument (see Otto [38], Lions's book [32] or Evan's book [25]) to identify the non linear term. The identification of the weak limit of the flux is based on (14.8) and on the entropy equation (14.9) where in fact we would like to use the solution as a test function in the limiting equation. Then we have to define all terms of those relations very carefully and find the most precise framework in which the equation holds. For that purpose, we need the generalization of functionals to $BV$ introduced in the previous paragraph and the introduction of the distribution $(z; D\rho)$ because in one sense, it replaces the product $\mathrm{div} z\rho$ when $\rho \in BV(\Omega)$ and $z \in L^\infty(\Omega)$.
– Note that the correction terms in (14.6) prevent the divergence term from being $L^1$ which means that the time discrete function is less regular than in Andreu Caselles Mazon [12] even if we recover the expected regularity in the continuous time limit. To avoid this difficulty, we introduce a different approximation of the right hand side which will be in $L^1$ which is a crucial point in the definition of the distribution $(z; D\rho)$.
– The notion of entropic solution and the uniqueness [12] of solution in the class of $L^1_w([0,T], BV_2(\Omega))$ is not available here because of the approximation in the discrete in time equation and is not addressed in this paper.
– Particular case : Dimension 1.
In dimension 1, this problem becomes radically simpler (see [18]). Let $\rho \in P(\Omega)$, we can extend it to $\mathbf{R}$ and following Villani's notations in [43], we introduce Cumulative distribution functions

$$l(x) = \int_{-\infty}^{x} d\rho = d\rho[(-\infty, x]].$$

The function $l$ is right-continuous, non decreasing, and has limits $l(-\infty) = 0$ and $l(+\infty) = 1$. Then we define the generalized inverse of $l$ on $[0,1]$

$$l^{-1}(t) = \inf\{x \in \mathbf{R} \text{ s. t. } l(x) > t\}.$$

In the particular case of dimension 1, the optimal transport plan and the optimal map does not depend on the convex cost function as soon as $c(x,y) = c(|x-y|)$ (cf [36]). The optimal transport plan is given by

$$\gamma_i = (l_{i-1}^{-1} \times l_i^{-1})_{\#} d\mathcal{H}^1$$

where $l_{i-1}$ is the Cumulative distribution function for $\rho_{i-1}$ and $l_i$ is the Cumulative distribution function for $\rho_i$. The optimal map is given — when $\rho_{i-1}$ does not give mass to points — by

$$\rho_{i-1} = (l_{i-1}^{-1} \circ l_i)_{\#}\rho_i.$$

Then the Euler-Lagrange equation given by (14.5) gives directly

$$l_{i-1}^{-1} \circ l_i(y) - y = h\nabla c^*(\nabla(F'(l_i'(y))))$$

or

$$\begin{aligned}\frac{l_{i-1}^{-1}(x)-l_i^{-1}(x)}{h} &= \nabla c^*(\nabla(F'(l_i'(l_i^{-1}(x)))))\\ &= \nabla c^*(\nabla(F'(\frac{1}{(l_i^{-1})'(l_i\circ l_i^{-1}(x))}))) = \nabla c^*(\nabla(F'(\frac{1}{(l_i^{-1})'(x)})))\end{aligned}$$

which corresponds to an implicit Euler scheme for $l^{-1}$.

In the following section, we prove the existence of the infimum for problem $(\mathcal{P}_i(h))$. In section 3, we prove some optimal transport theory (construction of an optimal map corresponding to the optimal measure and study of its properties) and we derive the Euler-Lagrange equation of the minimization problem (14.5). Then, in section 4, we construct the piecewise constant function $\rho^h$ which satisfies a time discrete version of (14.1). Finally, in the last section, we pass to the limit when the time step goes to zero, we use an argument of monotonicity to identify the limiting equation with equation (14.1).

## 14.2 Existence of solution to the minimization problem

Let us first study the minimization problem $(\mathcal{P}) = (\mathcal{P}_1^1)$ for the first step when $h = 1$ (we drop the superscript $h = 1$ and subscript $i = 1$ in this case) :

$\rho_0 \in P(\Omega)$ *given, find* $\rho_1 \in P(\Omega)$ *such that*

$$I(\rho_0,\rho_1) = \inf_{\rho\in P(\Omega)} I(\rho_0,\rho)$$

*with*

$$I(\rho_0,\rho) = \int_\Omega F'(\rho(y))dy + \inf_{\gamma\in\Gamma(\rho_0,\rho)}\int_{\Omega\times\Omega} c(x-y)d\gamma(x,y),$$

$\Gamma(\rho_0,\rho)$ *denoting the set of measures that have* $\rho_0$ *and* $\rho$ *as marginals.*

**Notations** We will use the following notations : given $\gamma\in\Gamma(\rho_0,\rho)$,

$$\begin{aligned}E(\rho) &= \int_\Omega F(\rho(y))dy =: \tilde{E}(\gamma), \quad \text{when seen as a function of } \gamma,\\ \tilde{W}(\gamma) &= \int c(x-y)d\gamma(x,y), \text{ and}\\ W_c(\rho_0,\rho) &= \inf_{\gamma\in\Gamma(\rho_0,\rho)}\tilde{W}(\gamma).\end{aligned}$$

Following Agueh [1], we can prove that

*Proposition* 14.2.1. Recall that we assume $m < \rho_0 < M$.

(i) There exists a unique minimum $\rho_1^R$ satisfying

$$I(\rho_0,\rho_1^R) = \inf_{\rho<R} I(\rho_0,\rho).$$

(ii) There is a maximum principle that insures that $m < \rho_1^R < M$ when $R$ is chosen such that $2M < R$.

*Démonstration.* Let us denote $I_R = \inf_{\rho<R} I(\rho_0,\rho)$.

(i) To prove the existence of a minimizer, we use that $I_R \le I(\rho_0,\rho_0) = E(\rho_0) < \infty$ and thanks to Jensen's inequality, since $\int_\Omega \rho(y)dy = 1$, we can write that $I_R \ge \inf_\rho E(\rho) \ge |\Omega|F(\frac{1}{|\Omega|})$. Thus $I_R$ is finite. Let $\rho^{(n)}$ be a minimizing sequence. Since $\rho^{(n)} < R$, the sequence is bounded in $L^\infty(\Omega)$ and

since $\Omega$ is bounded, up to a subsequence, $\rho^{(n)}$ converges in $L^1(\Omega)'$ towards $\rho_1^R \in L^\infty(\Omega)$ satisfying $\rho_1^R \leq R$. Because of the lower semi-continuity of $I(\rho)$, we obtain

$$I(\rho_0, \rho_1^R) \leq \lim_{n\to\infty} \inf I(\rho_0, \rho^{(n)}) \leq I_R \leq I(\rho_0, \rho_1^R)$$

The uniqueness comes from the strict convexity of $I$. Note that

$$E(\rho_1^R) \leq I(\rho_0, \rho_1^R) \leq I(\rho_0, \rho_0) = E(\rho_0).$$

(ii) To prove the maximum principle, we argue by contradiction to prove that $m < \rho_1^R < M$. Note that it implies also that $\rho_1^R$ does not depend on $R$. Let us do the argument for the upper bound, the same proof leads to the lower bound. For that, assume that $H = \{x \mid \text{ such that } \rho_1^R > M\}$ has a strictly positive Lebesgue measure; we can then construct a new measure that would be a better measure, which gives us a contradiction. Starting from $\gamma_{\rm opt}$, the optimal measure associated to $\rho_1^R$, let us construct a sequence of measures $\gamma^\eta$ with second marginal $\rho^\eta$ depending on $\eta$, that will make the cost decrease proportionately to $\eta$ while the entropy increases proportionately to $\eta^2$. Then for $\eta$ small enough, the total $I(\rho_0, \rho^\eta)$ will be smaller than $I(\rho_0, \rho_1^R)$.

To be precise, we recall the construction of this sequence which follows the argument made by Martial Agueh in [1].

Note that $\gamma_{\rm opt}(H^c \times H) > 0$ where $H^c = \mathbf{R}^d \setminus H$ ; otherwise

$$M|H| < \int_H \rho_1^R(y)dy = \gamma_{\rm opt}(\mathbf{R}^d \times H) = \gamma_{\rm opt}(H \times H) \leq \gamma_{\rm opt}(H \times \mathbf{R}^d) = \int_H \rho_0(x)dx \leq M|H|.$$

On $H^c \times H$, for a part of the measure depending on $\eta$, we will leave $x$ in place instead of sending $x$ to $y$, i.e. we define the action of the sequence $\gamma^\eta$ against a test function $\xi$ by

$$\int_{\mathbf{R}^d\times\mathbf{R}^d} \xi(x,y)d\gamma^\eta(x,y) = \int_{\mathbf{R}^d\times\mathbf{R}^d} \xi(x,y)d\gamma_{\rm opt}(x,y) + \eta \int_{H^c\times H} (\xi(x,x) - \xi(x,y))d\gamma_{\rm opt}(x,y).$$

The corresponding marginal $\rho^\eta$ can also be defined as $\rho_1^R + \eta(v_0 - v_1)$ where $v_0$ is the first marginal of $\nu = \gamma_{\rm opt}\mathbf{1}_{H^c\times H}$, the restriction of $\gamma_{\rm opt}$ to the set $H^c \times H$ (respectively, $v_1$ is the second marginal of $\nu$). Since $\nu << \gamma_{\rm opt}$, we have

$$\begin{aligned} &\bullet \quad 0 \leq v_0 \leq M \text{ a.e., and } 0 \leq v_1 \leq R \text{ a.e.,} \\ &\bullet \quad v_0 = 0 \text{ on } H \text{ and } v_1 = 0 \text{ on } H^c. \end{aligned} \tag{14.18}$$

Then $0 \leq \rho^\eta \leq M + \eta M \leq R$ on $H^c$ and $M - \eta v_1 \leq \rho^\eta \leq R$ on $H$, so we obtain $0 \leq \rho^\eta \leq R$ ($\eta$ is chosen such that $\eta R < M$ and $2M \leq R$). Moreover,

$$\int_\Omega \rho^\eta(y)dy = 1 + \eta(\gamma_{\rm opt}(H^c \times H) - \gamma_{\rm opt}(H^c \times H)) = 1.$$

So we effectively have constructed a new measure $\gamma^\eta$ belonging to $\Gamma(\rho_0, \rho^\eta)$ with $\rho^\eta \leq R$. We still have to prove that $I(\rho_0, \rho^\eta) < I(\rho_0, \rho_1^R)$. For that, we will compute

$$\begin{aligned} W_c(\rho_0, \rho^\eta) - W_c(\rho_0, \rho_1^R) &\leq \int_{\mathbf{R}^d\times\mathbf{R}^d} c(x-y)d\gamma^\eta(x,y) - \int_{\mathbf{R}^d\times\mathbf{R}^d} c(x-y)d\gamma_{\rm opt}(x,y) \\ &= -\eta \int_{H^c\times H} c(x-y)d\gamma_{\rm opt}(x,y). \end{aligned}$$

This means that with the new measure, we have managed to lower the cost by a quantity of order $\eta$. But the entropy has to increase : let us check how much by computing

$$\int_\Omega (F(\rho^\eta(y)) - F(\rho_1^R(y)))dy.$$

Using the convex property of $F$ and (14.18), we decompose

$$\begin{aligned}\int_\Omega (F(\rho^\eta(y)) - F(\rho_1^R(y)))dy &= \int_{H^c} (F(\rho^\eta(y)) - F(\rho_1^R(y)))dy + \int_H (F(\rho^\eta(y)) - F(\rho_1^R(y)))dy \\ &\leq \eta[\int_{H^c} F'(\rho_1^R(y) + \eta v_0(y))v_0(y)dy - \int_H F'(\rho_1^R(y) - \eta v_1(y))v_1(y)dy] \\ &\leq \eta[\int_{H^c} F'(M + \eta v_0(y))v_0(y)dy - \int_H F'(M - \eta v_1(y))v_1(y)dy]\end{aligned}$$

since $\rho_1^R(y) \leq M$ if $y \in H^c$ and $\rho_1^R(y) \geq M$ if $y \in H$, and $F'$ is non-decreasing. But also

$$\begin{aligned}\int_{H^c} F'(M + \eta v_0(y))v_0(y)dy &- \int_H F'(M - \eta v_1(y))v_1(y)dy \\ &= \int_{H^c \times H} (F'(M + \eta v_0(x)) - F'(M - \eta v_1(y)))d\gamma_{\text{opt}}(x,y) \\ &\leq \eta \sup|F''| \sup_\Omega |v_0 - v_1|.\end{aligned}$$

Since $F$ is $C^2(\mathbf{R}^+)$, the entropy increases by a quantity of order $\eta^2$, which means that for $\eta$ small enough, $I(\rho_0, \rho^\eta) < I(\rho_0, \rho_1^R)$. This contradiction establishes Proposition 14.2.1 □

## 14.3 Optimal Transport Theory : a time discrete equation

In this work we deal with a convex cost function which may be discontinuous and takes infinity as a value. As we said in the introduction, for this kind of cost function, we cannot apply the classical result of Gangbo McCann [28], [29] or Caffarelli [17] to define an optimal map associated to the optimal measure involved in initial time step $\mathcal{P}_0^1$. To construct a map in the present case, we will use the properties of the optimal map for a mollified case using strongly the double minimization process (combining the cost minimization with respect to $\gamma$ for a fixed $\rho$ with the entropy minimization with respect to $\rho$). Indeed, note that for given $\rho_0$, it is easy to construct $\rho_1$ such that the value of

$$W_c(\rho_0, \rho_1) = \inf_{\gamma \in \Gamma(\rho_0, \rho_1)} \int c(x-y)d\gamma(x,y),$$

the double minimization always produces $\rho_1$ for which the infimum is finite.

### 14.3.1 Previous results

We recall in this section Proposition 2.6 and Proposition 2.7 of the paper of M. Agueh [1] (see also [35]) and a result due to D. Cordero [21] and F. Otto [38] also quoted in [1] that we will use for the mollified approximation of $\mathcal{P}$. L. Ambrosio, Gigli, and G. Savare's Theorem 6.2.7 [3] could also be used.

*Proposition* 14.3.1 ([21],[38]). Let $\rho_0, \rho_1 \in P(\Omega)$ and assume that $c : \mathbf{R}^d \to [0, \infty[$ is strictly convex and satisfies $c, c^* \in C^2(\mathbf{R}^d)$. Denote by $S$, the c-optimal map that pushes $\rho_1$ forward to $\rho_0$, and define the interpolant map $S_t$, and the interpolant measure $\mu_{1-t}$, by

$$S_t = (1-t)\text{Id} + tS \quad \text{and} \quad \mu_{1-t} = (S_t)_\# \rho_1,$$

for $t \in [0,1]$. Then

(i) $S_t$ is injective for $t < 1$, and $\mu_{1-t}$ is absolutely continuous with respect to Lebesgue.

Moreover, there exists a subset $B$ of $\Omega$, of full measure for $\mu_1 = \rho_1(y)dy$, such that, for $y \in B$ and $t \in [0,1]$,

(ii) $\nabla S(y)$ is diagonalizable with positive eigenvalues.

(iii) the pointwise Jacobian $\det \nabla S$ satisfies

$$0 \neq \rho_1(y) = \rho_{1-t}(S_t(y)) \det[(1-t)\mathrm{Id} + t\nabla S(y)], \tag{14.19}$$

where $\rho_{1-t}$ is the density function of $\mu_{1-t}$.

In addition, if $\rho_1 > 0$ a.e., then

(iv) the pointwise divergence $\mathrm{div} S$ is integrable on $\Omega$, and

$$\int_\Omega \mathrm{div}(S(y)-y)\xi(y)dy \leq -\int_\Omega < S(y)-y; \nabla\xi > dy,$$

for $\xi \geq 0$ in $C_c^\infty(\mathbf{R}^d)$.

*Proposition* 14.3.2 (Optimal map for smooth cost function [1]). Let $\rho_0 \in P(\Omega)$ be such that $m \leq \rho_0 \leq M$ a.e.. Assume that $F : [0,\infty[ \to \mathbf{R}$ is strictly convex, and satisfies $F \in C^2((0,\infty))$, and $c : \mathbf{R}^d \to [0,\infty[$ is strictly convex, of class $C^1$ such that $0 = c(0) < c(|z|)$ for $|z| \neq 0$ and $c$ coercive, i.e. $\lim_{|z|\to\infty} c(z) = \infty$. If $\rho_1$ denotes the minimizer for $(\mathcal{P}_0^1)$, then the following hold :

$$\int_{\Omega\times\Omega} \nabla c(x-y)\cdot\xi(y)d\gamma(x,y) + \int_\Omega P_F(\rho_1(y))\mathrm{div}\xi(y)dy = 0,$$

for $\xi \in C_c^\infty(\Omega;\mathbf{R}^d)$ ; here $P_F(\lambda) = \lambda F'(\lambda) - F(\lambda)$ for $\lambda \in (0,\infty)$, and $\gamma$ is the c-optimal measure in $\Gamma(\rho_0,\rho_1)$. Moreover,

(i) $P_F(\rho_1) \in W^{1,\infty}(\Omega)$.

(ii) If $S$ is the c-optimal map that pushes $\rho_1$ forward to $\rho_0$, then

$$S(y) - y = \nabla c^*[\nabla(F'(\rho_1(y)))], \tag{14.20}$$

for a.e. $y \in \Omega$, and for $\xi \in C^2(\overline{\Omega})$,

$$\left|\int_\Omega (\rho_1(y)-\rho_0(y))\xi(y)dy + \int_\Omega \rho_1(y)\nabla c^*[\nabla(F'(\rho_1(y)))]\cdot\nabla\xi(y)dy\right|$$

$$\leq \frac{1}{2}\sup_{x\in\overline{\Omega}}|D^2\xi(x)| \int_{\Omega\times\Omega}|x-y|^2 d\gamma(x,y).$$

*Proposition* 14.3.3 ( [1] (Displacement convexity of energy (above-tangent form))). Let $\rho_0,\rho_1 \in P(\Omega)$ be density functions of two Borel probability measures $\mu_0$ and $\mu_1$ on $\mathbf{R}^d$, respectively. Let $c : \mathbf{R}^d \to [0,\infty[$ be strictly convex, such that $c, c^* \in C^2(\mathbf{R}^d)$. Let $F : [0,\infty) \to \mathbf{R}$ be differentiable on $]0,\infty[$, such that $F(0) = 0$, and $\lambda \to \lambda^d F(\lambda^{-d})$ be convex non-increasing on $]0,\infty[$. Then, the internal energy inequality holds, i.e.

$$\int_\Omega F(\rho_0(y))dy - \int_\Omega F(\rho_1(y))dy \geq -\int_\Omega P_F(\rho_1(y))\mathrm{div}(S(y)-y)dy.$$

In addition, if $P_F(\rho_1) \in W^{1,\infty}(\Omega)$ and $\rho_1 > 0$ a.e., then

$$\int_\Omega F(\rho_0(y))dy - \int_\Omega F(\rho_1(y))dy \geq \int_\Omega \nabla[F'(\rho_1(y))]\cdot(S(y)-y)\rho_1(y)dy. \tag{14.21}$$

### 14.3.2 Construction of an optimal map for $(\mathcal{P}_0^1)$ : a mollification process

**Definition of the mollified problem**

Recall that the hemispherical cost

$$c(z) = \begin{cases} 1-\sqrt{1-|z|^2} & \text{where } |z| \leq 1, \\ +\infty & \text{where } |z| > 1, \end{cases}$$

is the motivating example, whose Legendre transform (14.4) is given by the hyperboloid

$$c^*(z) = \sqrt{1+|z|^2} - 1.$$

In a more general setting, we deal with radial cost functions $c : \mathbf{R}^d \longrightarrow [0,\infty]$ satisfying $c(z) = \tilde{c}(|z|)$ where $\tilde{c} \in C^2([0,1[) \cap C^0([0,1])$ satisfies $\tilde{c}''(\lambda) > 0$ on $[0,1[$ and $\tilde{c}(0) = 0 = \tilde{c}'(0)$ and $|\nabla c(z)| \to \infty$ as $|z| \to 1$. Let mollify those general cost functions $c$ by the Yosida regularization [16]

$$c^\epsilon(z) = \inf_{w \in \mathbf{R}^d} (c(z-w) + \frac{|w|^2}{2\varepsilon}).$$

Note that from $w = 0$ we obtain $c^\varepsilon(z) \leq c(z)$. In fact, the Legendre transform $c^{\varepsilon *}$ of this mollified cost is a strict convexification of $c^*$, namely

$$c^{\varepsilon *}(z) = c^*(z) + \frac{\varepsilon}{2}|z|^2.$$

Note that the mollified $c^\varepsilon$ is finite and convex, hence continuous.

Here, we recall an argument of [1] to justify the regularity of $c^\varepsilon$. The function $c^{\varepsilon *}(z)$ is a $C^2(\mathbf{R}^d)$ function non-negative and strictly convex and $\nabla c^\varepsilon = (\nabla c^{\varepsilon *})^{-1}$ and then $c^\varepsilon$ belongs to $C^1(\mathbf{R}^d)$ and the function

$$z \to \frac{\mathrm{cof}(D^2 c^{\varepsilon *}[\nabla c^\varepsilon(z)])}{\det(D^2 c^{\varepsilon *}[\nabla c^\varepsilon(z)])} := D^2 c^\varepsilon(z)$$

is well-defined and is continuous on $\mathbf{R}^d$. Then $c^\varepsilon \in C^2(\mathbf{R}^d)$.

Consider the sequence of approximate minimization problems $(\mathcal{P}^\varepsilon)$

*Find $\gamma^\varepsilon$ with first marginal $\rho_0$ and second marginal $\rho^\varepsilon$ such that*

$$I^\varepsilon(\rho_0, \rho^\varepsilon) = \tilde{I}^\varepsilon(\gamma^\varepsilon) = \min_{\gamma \in P_0(\Omega \times \Omega)} (\tilde{E}(\gamma) + \tilde{W}_{c^\varepsilon}(\gamma))$$

where $P_0(\Omega \times \Omega) = \{\gamma \text{ such that } \int_{\Omega \times \Omega} \varphi(x) d\gamma(x,y) = \int_\Omega \varphi(x) d\rho_0(x)\}$ i.e. the probabilities with first marginal $\rho_0$.

Here $\rho_0$ represents any $\rho_i$ and is assumed to satisfy $m \leq \rho_0 \leq M$.

The mollification allows us to apply the previous results quoted above to this sequence of problems since $c^\varepsilon$ has the regularity required.

*Lemma* 14.3.1. **The Kantorovich duality for the mollified problem**

For $\varepsilon > 0$ fixed, we have

$$\int_{\Omega \times \Omega} c^\varepsilon(x-y) d\gamma^\varepsilon(x,y) = \int_\Omega \phi^\varepsilon(x) d\rho_0(x) + \int_\Omega \psi^\varepsilon(y) d\rho^\varepsilon(y) \tag{14.22}$$

where the potential $\psi^\varepsilon = -F'(\rho^\varepsilon)$ and $\phi^\varepsilon$ is the corresponding $c$-transform given by

$$\phi^\varepsilon(x) = \inf_{y \in \overline{\Omega}} (c^\varepsilon(x-y) - \psi^\varepsilon(y)). \tag{14.23}$$

*Démonstration.* First recall the Kantorovich duality theory for smooth cost described in Rachev and Rüschendorf [39] which gives that

$$\begin{aligned} \inf_{\gamma \in \Gamma(\rho_0^\varepsilon, \rho^\varepsilon)} \int_{\Omega \times \Omega} c^\varepsilon(x-y) d\gamma(x,y) &= \sup_{\phi, \psi \in T_c} \int_\Omega \phi(x) d\rho_0(x) + \int_\Omega \psi(y) d\rho^\varepsilon(y) \\ &= \sup_{\psi, \psi^c} \int_\Omega \psi^c(x) d\rho_0(x) + \int_\Omega \psi(y) d\rho^\varepsilon(y) \end{aligned}$$

where $\phi, \psi \in T_c$ means that $\psi(y) + \phi(x) \leq c^\varepsilon(x-y)$ for all $x, y \in \Omega$, and $\psi^c$ is given by (14.23). Moreover the supremum is attained. In our study, since $\rho^\varepsilon$ is the second marginal corresponding to the optimal measure of the complete minimization problem $\mathcal{P}^\varepsilon$, the Euler-Lagrange equation (14.20) for this problem is (see [1])

$$S^\varepsilon(y) - y = \nabla c^{\varepsilon *}(\nabla(F'(\rho^\varepsilon(y))))$$

where $S^\varepsilon(y)$ is the optimal map associated to the potential $\psi^\varepsilon$ by

$$S^\varepsilon(y) - y = -\nabla c^{\varepsilon *}(\nabla \psi^\varepsilon(y)).$$

This means that up to a constant (that we can fix to be zero without loss of generality since $\int_\Omega \rho^\varepsilon(y)dy = 1$), we have

$$\psi^\varepsilon(y) = -F'(\rho^\varepsilon(y)).$$

□

*Lemma* 14.3.2. **Existence of a limiting measure**

(i) Up to a subsequence, $\gamma^\varepsilon$ converges, as $\varepsilon \to 0$, towards the probability $\gamma^\infty$ in $C^0(\Omega)'$.

(ii) The support of the limiting measure $\gamma^\infty$ is included in $\Delta_1 = \{(x,y) \text{ such that } |x-y| \leq 1\}$.

(iii) Identification of the limiting measure :

The limiting measure is the optimal measure for the initial problem, i.e.

$$\gamma^\infty = \gamma_{\text{opt}} \quad \text{or} \quad \tilde{I}(\gamma^\infty) = \min_{\gamma \in P_0(\Omega\times\Omega)} (\tilde{E}(\gamma) + \tilde{W}_c(\gamma)).$$

*Démonstration.* (i) Since $\gamma^\varepsilon$ is a sequence of probability measures, it is relatively compact and converges up to a subsequence in $C^0(\Omega)'$.

(ii) Let us define $N_\delta = \{(x,y) \in \Omega\times\Omega \text{ such that } |x-y| \geq 1+\delta\}$, we claim that $\gamma^\varepsilon(N_\delta) \to 0$ when $\varepsilon \to 0$. Then we obtain that $\gamma^\infty(N_\delta) = 0$ for any $\delta > 0$, hence $\operatorname{supp}\gamma^\infty \subset \Delta_1$.

Note that,

$$C \geq E(\rho_0) \geq \tilde{I}^\varepsilon(\gamma^\varepsilon) \geq \int_{N_\delta} c^\varepsilon(x-y)d\gamma^\varepsilon(x,y) \geq \tilde{c}^\varepsilon(1+\delta)\gamma^\varepsilon(N_\delta).$$

Since $\tilde{c}^\varepsilon(1+\delta) \to \infty$ when $\varepsilon \to 0$, the previous inequality implies that $\gamma^\varepsilon(N_\delta) \to 0$ when $\varepsilon \to 0$ and then (ii) holds.

(iii) First of all, we prove that $\tilde{I}^\varepsilon(\gamma^\varepsilon) \to \tilde{I}(\gamma^\infty)$.

Let us first prove that

$$\limsup_{\varepsilon\to 0} (\tilde{I}(\gamma^\infty) - \tilde{I}^\varepsilon(\gamma^\varepsilon)) \leq 0 \tag{14.24}$$

The lower semi-continuity of the entropy leads to the first part of the inequality

$$\tilde{E}(\gamma^\infty) \leq \liminf_{\varepsilon\to 0} \tilde{E}(\gamma^\varepsilon).$$

For the part involving the cost function, we use the fact that $c^\varepsilon \geq 0$ and that the support of the limiting measure $\gamma^\infty$ is a subset of $\Delta_1$. Indeed, it leads to

$$\begin{aligned}
\int_{\Omega\times\Omega} c(x-y)d\gamma^\infty(x,y) \quad - \quad & \int_{\Omega\times\Omega} c^\varepsilon(x-y)d\gamma^\varepsilon(x,y) \\
\leq \quad & \int_{\Delta_1} [c(x-y)d\gamma^\infty(x,y) - c^\varepsilon(x-y)d\gamma^\varepsilon(x,y)] \\
= \quad & \int_{\Delta_1} (c(x-y) - c^\varepsilon(x-y))d\gamma^\varepsilon(x,y) + \int_{\Delta_1} c(x-y)(d\gamma^\infty(x,y) - d\gamma^\varepsilon(x,y)).
\end{aligned}$$

Up to the introduction of a $C^0(\Omega)$ extension of the restriction of $c$ to the unit ball, since the sequence $\gamma^\varepsilon \to \gamma^\infty$ in the sense of measures (or in $C^0(\Omega)'$), the second term goes to zero.

To prove that the first term goes to zero, we will prove that

$$||c(z) - c^\varepsilon(z)||_{L^\infty(\{|z|\leq 1\})} \leq M(\varepsilon) \tag{14.25}$$

where $M(\varepsilon) \to 0$ when $\varepsilon \to 0$.

Indeed, for any $t \in [0,1]$ and $t > \theta > 0$

$$\tilde{c}(t) - \tilde{c}(t-\theta) \leq \tilde{c}(1) - \tilde{c}(1-\theta)$$

since for $\theta$ fixed, $t \to \tilde{c}(t) - \tilde{c}(t-\theta)$ is a non-decreasing function. Note also that

$$c(z) - c^\varepsilon(z) = \sup_{w \in B(z,1)} (c(z) - \tilde{c}(|z-w|) - \frac{|w|^2}{2\varepsilon}) = \sup_{w \in B(z,1)} (\tilde{c}(|z|) - \tilde{c}(|z| - |w|) - \frac{|w|^2}{2\varepsilon}).$$

Since for any $w \in B(z,1)$,

$$c(z) - \tilde{c}(|z| - |w|) - \frac{|w|^2}{2\varepsilon} \leq \tilde{c}(1) - \tilde{c}(1 - |w|) - \frac{|w|^2}{2\varepsilon} \leq \tilde{c}(1) - \tilde{c}^\varepsilon(1),$$

we finally obtain

$$c(z) - c^\varepsilon(z) \leq \tilde{c}(1) - \tilde{c}^\varepsilon(1).$$

By taking $M(\varepsilon) = \tilde{c}(1) - \tilde{c}^\varepsilon(1)$, (14.25) holds.

Since $\gamma^\varepsilon(\Delta_1) \leq 1$, we have

$$\int_{\Delta_1} (c(x-y) - c^\varepsilon(x-y)) d\gamma^\varepsilon(x,y) \leq M(\varepsilon)$$

and then (14.24) holds.

Let us now prove the other inequality

$$\tilde{I}(\gamma^\infty) \geq \limsup_{\varepsilon \to 0} \tilde{I}(\gamma^\varepsilon).$$

Recall that we have the following inequalities,

$$\tilde{I}^\varepsilon(\gamma^\varepsilon) \leq \tilde{I}^\varepsilon(\gamma) \leq \tilde{I}(\gamma)$$

for any $\gamma$ which has $\rho^0$ as first marginal, including $\gamma = \gamma^\infty$.

Thus, we obtain the expected inequality since

$$\limsup_{\varepsilon \to 0} \tilde{I}^\varepsilon(\gamma^\varepsilon) \leq \tilde{I}(\gamma)$$

for any measure $\gamma$ which has $\rho_0$ as first marginal and in particular for $\gamma^\infty$.

To conclude the proof of (iii), we use the last inequality to obtain that

$$\tilde{E}(\gamma^\infty) + \int_{\Omega \times \Omega} c(x-y) d\gamma^\infty(x,y) \leq \inf_\gamma [\tilde{E}(\gamma) + \int_{\Omega \times \Omega} c(x-y) d\gamma(x,y)]$$

and then $\gamma^\infty$ is equal to $\gamma_{opt}$, the minimizer for the initial problem $(\mathcal{P})$ and $\rho^\infty = \rho_1$.

□

*Lemma* 14.3.3. **The Kantorovich duality for the limiting problem**

(i) There exist $\phi^\infty \in L^\infty(\Omega)$ and $\psi^\infty \in L^\infty(\Omega) \cap BV(\Omega)$ such that, up to a subsequence, the potentials $(\psi^\varepsilon, \phi^\varepsilon)$ from (14.26)-(14.27) satisfy

$$\psi^\varepsilon \rightharpoonup \psi^\infty \text{ weak in BV}(\Omega), \text{ strong in } L^1(\Omega)$$
$$\rho^\varepsilon \rightharpoonup \rho^\infty = \rho_1 = (F^*)'(-\psi^\infty) \text{ weak in BV}(\Omega), \text{ strong in } L^1(\Omega),$$
$$\phi^\varepsilon \rightharpoonup \phi^\infty \text{ weak}^* in L^\infty(\Omega) = L^1(\Omega)'.$$

(ii) The Kantorovich duality holds for the limiting problem

$$\int_{\Omega \times \Omega} c(x-y) d\gamma_{\text{opt}}(x,y) = \int_\Omega \phi^\infty(x) d\rho_0(x) + \int_\Omega \psi^\infty(y) d\rho_1(y). \tag{14.26}$$

Moreover,

$$c(x-y) - \phi^\infty(x) - \psi^\infty(y) \geq 0 \tag{14.27}$$

almost everywhere on $\Omega \times \Omega$ and the equality holds $\gamma_{\text{opt}}$-almost everywhere.

*Démonstration.* (i) Since for all $\varepsilon$, $\psi^\varepsilon(y) = -F'(\rho^\varepsilon(y))$ with $m < \rho^\varepsilon < M$, up to a subsequence, $\psi^\varepsilon$ converges weak$^*$ in $L^\infty(\Omega)$ towards $\psi^\infty \in L^\infty(\Omega)$. Moreover, using (14.21) and (14.20), applied for $c^\varepsilon$, we obtain

$$\int_\Omega \nabla(F'(\rho^\varepsilon(y))) \cdot \rho^\varepsilon(y) \nabla c^*(\nabla(F'(\rho^\varepsilon(y))) + \varepsilon\rho^\varepsilon(y)|\nabla(F'(\rho^\varepsilon(y)))|^2 dy \leq \int_\Omega (F(\rho_0(y)) - F(\rho^\varepsilon(y)))dy$$

which gives in particular

$$\int_\Omega \rho^\varepsilon(y)\nabla(F'(\rho^\varepsilon(y)) \cdot \nabla c^*(\nabla(F'(\rho^\varepsilon(y)))dy \leq \int_\Omega (F(\rho_0(y)) - F(\rho^\varepsilon(y)))dy \tag{14.28}$$

and then, since $m < \rho^\varepsilon < M$ (see [1]), $\nabla\rho^\varepsilon$ is a bounded sequence in $L^1(\Omega)$. Indeed,

$$\int_\Omega |\nabla\rho^\varepsilon(y)|dy = \int_{|\nabla\rho^\varepsilon|<1} |\nabla\rho^\varepsilon(y)|dy + \int_{|\nabla\rho^\varepsilon|>1} |\nabla\rho^\varepsilon(y)|dy \leq |\Omega| + \int_{|\nabla\rho^\varepsilon|>1} |\nabla\rho^\varepsilon(y)|dy.$$

Since we can write $|\nabla(F'(\rho^\varepsilon))| = |\nabla\rho^\varepsilon||F''(\rho^\varepsilon)|$, we have

$$\begin{aligned}\int_{|\nabla\rho^\varepsilon|>1} |\nabla\rho^\varepsilon(y)|dy &= \int_{|\nabla\rho^\varepsilon|>1} \frac{|\nabla(F'(\rho^\varepsilon(y))|}{|F''(\rho^\varepsilon(y))|}dy \\ &\leq \frac{1}{K}\int_\Omega |\nabla(F'(\rho^\varepsilon(y)))|dy\end{aligned}$$

where $K = \inf_{[m,M]} F''$.

Recall that we have the relation $|z||\nabla c^*(z)| = |z \cdot \nabla c^*(z)|$ which implies $|z| \leq \frac{|z\cdot\nabla c^*(z)|}{w(s)}$ when $|z| > s$ since $|\nabla c^*(z)| = \omega(|z|)$ is a nondecreasing function of $|z|$.

Moreover, when $|\nabla\rho^\varepsilon| > 1$, $|\nabla(F'(\rho^\varepsilon))| \geq K$ and then

$$\begin{aligned}\int_{|\nabla\rho^\varepsilon|>1} |\nabla\rho^\varepsilon(y)|dy &\leq \frac{1}{Km|\nabla c^*(K)|}\int_\Omega \rho^\varepsilon(y)\nabla(F'(\rho^\varepsilon(y))) \cdot \nabla c^*(\nabla(F'(\rho^\varepsilon(y)))) \\ &\leq C\int_\Omega [F(\rho_0(y)) - F(\rho^\varepsilon(y))]dy\end{aligned}$$

and then

$$||\nabla\rho^\varepsilon||_{L^1(\Omega)} \leq C \tag{14.29}$$

where the constant $C$ does not depend on $\varepsilon$.

Then, $\rho_1 \in BV(\Omega)$ and consequently, $\psi^\infty$, the limit of $\psi^\varepsilon$ belongs also to $BV(\Omega)$ and the convergence is strong in $L^1(\Omega)$ (cf.[27], Theorem 4, section 5.2.3).

Finally, since $\phi^\varepsilon$ is defined from $\psi^\varepsilon$ by (14.23), $\phi^\varepsilon$ is a bounded sequence in $L^\infty(\Omega)$. Indeed by (14.23)

$$-\psi^\varepsilon(y) \leq -\psi^\varepsilon(y) + c^\varepsilon(x-y) = \phi^\varepsilon(x) \leq -\psi^\varepsilon(x) \Rightarrow \forall x \in \Omega, \ -\sup_{[m,M]} F' \leq \phi^\varepsilon(x) \leq -\inf_{[m,M]} F'.$$

Then, up to the extraction of a subsequence $\phi^\varepsilon$ converges in weak$^*$ in $L^\infty(\Omega)$.

(ii) In the previous Lemma, we proved that

$$\lim_{\varepsilon\to 0} \tilde{E}(\gamma^\varepsilon) + \int_{\Omega\times\Omega} c^\varepsilon(x-y)d\gamma^\varepsilon(x,y) = \tilde{E}(\gamma_{\rm opt}) + \int_{\Omega\times\Omega} c(x-y)d\gamma_{\rm opt}(x,y). \tag{14.30}$$

On the other hand, (14.22) says that

$$\tilde{E}(\gamma^\varepsilon) + \int_{\Omega\times\Omega} c^\varepsilon(x-y)d\gamma^\varepsilon(x,y) = \tilde{E}(\gamma^\varepsilon) + \int_\Omega \phi^\varepsilon(x)d\rho_0 + \int_\Omega \psi^\varepsilon(y)d\rho^\varepsilon.$$

Since $\phi^\varepsilon$ converges toward $\phi^\infty$ in weak$^*$ in $L^\infty(\Omega) = L^1(\Omega)'$, we have

$$\lim_{\varepsilon\to 0}\int_\Omega \phi^\varepsilon(x)d\rho_0(x) = \int_\Omega \phi^\infty(x)d\rho_0(x).$$

Moreover $\rho^\varepsilon \to \rho_1$ strongly in $L^1(\Omega)$ and $\rho^\varepsilon$ is bounded in $L^\infty(\Omega)$, so the dominated convergence theorem yields

$$\begin{aligned}\tilde{E}(\gamma^\varepsilon) + \int_\Omega \psi^\varepsilon(y)d\rho^\varepsilon(y) &= \int_\Omega F(\rho^\varepsilon(y))dy - \int_\Omega \rho^\varepsilon(y)F'(\rho^\varepsilon(y))dy \\ &\to \int_\Omega F(\rho_1(y))dy - \int_\Omega \rho_1(y)F'(\rho_1(y))dy \\ &= E(\gamma_{\rm opt}) + \int_\Omega \psi^\infty(y)d\rho_1(y).\end{aligned}$$

Combined with (14.30), this gives

$$\tilde{E}(\gamma_{\rm opt}) + \int_{\Omega\times\Omega} c(x-y)d\gamma_{\rm opt}(x,y) = \tilde{E}(\gamma_{\rm opt}) + \int_\Omega \phi^\infty(x)d\rho_0(x) + \int_\Omega \psi^\infty(y)d\rho_1(y)$$

which leads to (14.26).

Finally, since for all $\varepsilon$,

$$c^\varepsilon(x-y) - \phi^\varepsilon(x) - \psi^\varepsilon(y) \geq 0$$

using the fact that $c^\varepsilon(x-y) \leq c(x-y)$ we have

$$c(x-y) - \phi^\varepsilon(x) - \psi^\varepsilon(y) \geq 0$$

which gives (14.27) at the limit.

Since (14.26) holds, the equality holds in the support of $\gamma_{\rm opt}$.

□

*Lemma* 14.3.4. **Existence of an optimal map**

(i) The support of $\gamma_{\rm opt}$ is included, up to a negligible set, in the $c$-superdifferential of the potential function $\psi^\infty$, i.e. if $(x,y) \in \text{supp}\,(\gamma_{\rm opt})$, $x \in \partial^c\psi^\infty(y)$.

(ii) The set $\{y \in \Omega;\ \exists x_1, \neq x_2$ with $|x_1 - y| < 1$ and $(x_1,y) \in \text{supp}\,(\gamma_{\rm opt})$ and $(x_2,y) \in \text{supp}\,(\gamma_{\rm opt})\}$ is a negligible set.

(iii) Let $\gamma_{\rm opt}$ be the optimal measure for the minimization problem $(\mathcal{P})$,

$$\text{supp}\,\gamma_{\rm opt} \subset \{(x,y) \text{ with } |x-y| < 1\} \cup Z_0 \text{ with } \gamma_{\rm opt}(Z_0) = 0.$$

More precisely, for each $\delta \in ]0,1[$, we obtain an estimate of the measure of the set $Z_\delta = \{(x,y) \in \text{supp}\,\gamma_{\rm opt} \text{ with } |x-y| > 1-\delta\}$,

$$\gamma_{\rm opt}(Z_\delta) \leq \frac{E(\rho_0) - E(\rho_1)}{(1-\delta)|\nabla c(1-\delta)|}. \tag{14.31}$$

(iv) We define Lebesgue almost everywhere a one to one map by

$$S(y) = y - \nabla c^*(\nabla\psi^\infty(y)) = y + \nabla c^*(\nabla(F'(\rho_1(y)))) \text{ satisfying } \gamma_{\rm opt} = (S\times id)_\#\rho_1 \tag{14.32}$$

where, when $\rho \in BV(\Omega)$, $\nabla\rho \in L^1(\Omega)$ denotes the approximate derivative (or Radon Nikodym derivative of $D\rho$ with respect to the Lebesgue measure also called the absolute continuous part $D^{ac}\rho$).

*Remarque* 14.3.1. As we said before in the setting of the result, there is no ambiguity in the formulation of equation (14.32) since the recession function associated to $\nabla c^*$ is zero and then, we could also write $\nabla c^*(D\rho_1)$.

Before proving this Lemma, let us recall the following result.

*Lemma* 14.3.5. If both $\psi_1(z)$ and $\psi_2(z)$ are approximately differentiable at $z=y$, if $\psi_1(y)=\psi_2(y)$ and $\psi_1(z)\geq\psi_2(z)$ in a neighborhood of $y$, then

$$\nabla\psi_1(y)=\nabla\psi_2(y).$$

*Démonstration.* **Step 1** Let us first consider the case where $\psi_1=0$. Assume $\tilde{\psi}(z)$ approximately differentiable at $z=y$, $\tilde{\psi}(y)=0$ and $\tilde{\psi}(z)\leq 0$ near $z=y$. Let us prove that $\nabla\tilde{\psi}(y)=0$.

For a contradiction, suppose $\nabla\tilde{\psi}(y)=\lambda e_n$ where $e_n$ is a given direction and $\lambda>0$.

For any $a>0$, the definition of approximate differentiability [27] asserts that

$$\mathcal{D}=\{z\in\mathbf{R}^n \text{ such that } \frac{|\tilde{\psi}(z)-\lambda(z_n-y_n)|}{|z-y|}<a\}$$

has full Lebesgue density at $z=y$.

In particular, it must intersect the cone

$$\mathcal{C}=\{z\in\mathbf{R}^n \text{ such that } z_n-y_n>\sqrt{|z_1-y_1|^2+...+|z_{n-1}-y_{n-1}|^2}\}$$

inside each ball $B(y,\frac{1}{k})=\{z\in\mathbf{R}^n \text{ such that } |z-y|\leq\frac{1}{k}\}$.

Fix $a<\frac{\lambda}{\sqrt{2}}$ and choose a sequence $z^k\in\mathcal{C}\cap B(y,\frac{1}{k})\cap\mathcal{D}$. Since $z^k\in\mathcal{C}$, $2|z_n^k-y_n|^2>|z^k-y|^2$ and then we find

$$-\lambda|z_n^k-y_n|<-a|z^k-y|<\tilde{\psi}(z^k)-\lambda(z_n^k-y_n)$$

which means that $\tilde{\psi}(z^k)>0$ which leads to a contradiction.

**Step 2** Apply step 1 to $\tilde{\psi}(z)=\psi_2(z)-\psi_1(z)$.

□

**Proof of Lemma (14.3.4)**

(i) On the support of $\gamma_{\text{opt}}$, we have almost everywhere

$$c(x-y)=\phi^{\infty}(x)+\psi^{\infty}(y)$$

and then, since

$$\phi^{\infty}(x)=\inf_{v\in\overline{\Omega}}(c(x-v)-\psi^{\infty}(v)),$$

we obtain for all $(x,y)\in\text{supp}\,(\gamma_{\text{opt}})$ and for all $v\in\Omega$

$$\psi^{\infty}(v)\leq c(x-v)-c(x-y)+\psi^{\infty}(y) \tag{14.33}$$

which means that the support of $\gamma_{\text{opt}}$ is included in the $c$-superdifferential of the potential function $\psi^{\infty}$.

(ii) The proof of this point is based on the approximate differentiability of $\psi^{\infty}$. Indeed, since $\psi^{\infty}\in BV(\Omega)$, by Theorem 4 of section 6.1.3 in [27], $\psi^{\infty}$ is approximately differentiable almost everywhere. Let us denote $\nabla\psi^{\infty}$ its approximate derivative which belongs to $L^1(\Omega)$. (As we said before, the absolute continuous part of $D\psi^{\infty}$ coincides with the approximate derivative of $\psi^{\infty}$ so in this section, we will keep this notation to insist on the fact that we talk about $L^1(\Omega)$ functions). We will prove that this avoids the two possible problems,

Case 1

$\exists(x^1,x^2)$ such that $(x^1,y)\in\text{supp}\,(\gamma_{\text{opt}})$; $(x^2,y)\in\text{supp}\,(\gamma_{\text{opt}})$, $|x^1-y|<1$ and $|x^2-y|<1$

Case 2

$\exists(x^1,x^2)$ such that $(x^1,y)\in\text{supp}\,(\gamma_{\text{opt}})$, $(x^2,y)\in\text{supp}\,(\gamma_{\text{opt}})$, $|x^1-y|<1$ and $|x^2-y|=1$.

To prove a contradiction in the first case, apply Lemma (14.3.5) to the functions $\psi^1(v)=\psi^{\infty}(y)-c(x^1-y)+c(x^1-v)$ and $\psi^2(v)=\psi^{\infty}(v)$ from (14.33) which leads to $\nabla\psi^{\infty}(y)=-\nabla c(x^1-y)$, and then apply Lemma (14.3.5) a second time to $\psi^1(v)=\psi^{\infty}(y)-c(x^2-y)+c(x^2-v)$ and $\psi^2(v)=\psi^{\infty}(v)$ which leads to $\nabla\psi^{\infty}(y)=-\nabla c(x^2-y)$. Which leads to a contradiction since $\nabla c(x^2-y)=\nabla c(x^1-y)$ implies that $x^1=x^2$.

In the second case, we assume that $\exists x^1$ and $x^2$ such that $|x^1-y|<1$ and $|x^2-y|=1$ with

$$\psi^\infty(z)-\psi^\infty(y)\leq c(x^i-z)-c(x^i-y) \text{ for } i=1,2.$$

Note that the previous point says that if $\psi^\infty$ is approximately differentiable at $y$, $\nabla\psi^\infty(y)=-\nabla c(x^1-y)$. The problem is that $\nabla c(x^2-y)$ cannot be defined since $|x^2-y|=1$.

To get a contradiction, denote $B(y,r)=\{|z-y|\leq r\}$, we will prove that exists a set $\mathcal{S}$ such that

$$\frac{|\mathcal{S}\cap B(y,r)|}{|B(y,r)|}=C \text{ where } C \text{ is a constant} \tag{14.34}$$

satisfying for any $z\in\mathcal{S}$, for any $a>0$,

$$\psi^\infty(z)-\psi^\infty(y)-\nabla\psi^\infty(y)\cdot(z-y)<-a|z-y|$$

which contradicts the fact that $\psi^\infty$ is approximately differentiable at the point $y$.

Without loss of generality, we assume that $x^2=y+e_n$ (or $x_i^2=y_i+\delta_{i,n}$).

Take $\mathcal{S}=\{z \text{ such that } z_n-y_n>\sqrt{|z_1-y_1|^2+...+|z_{n-1}-y_{n-1}|^2}\}$. $\mathcal{S}\cap B(y,r)$ satisfies (14.34).

For any $z\in\mathcal{S}$, we claim that

(a) Since $z\in\mathcal{S}$, then $|z-y|<\sqrt{2}|z_n-y_n|$

(b) $|z-x^2|<1$

Indeed, we have

$$\begin{aligned}|z-x^2|^2 &= |z_1-y_1|^2+...+|z_{n-1}-y_{n-1}|^2+|z_n-(1+y_n)|^2\\ &= |z-y|^2+1-2(z_n-y_n)\\ &\leq 1+2|z_n-y_n|^2-2(z_n-y_n).\end{aligned}$$

Then, if $0<z_n-y_n<1$ —which can be ensured by taking $r<1$— we get $|z-x^2|<1$.

(c) The inequality

$$-a|z-y|-\nabla\psi^\infty(y).(z-y)\leq\psi^\infty(z)-\psi^\infty(y)\leq c(x^i-z)-c(x^i-y) \tag{14.35}$$

is not possible.

Indeed, we can write

$$\begin{aligned}-a|z-y|-\nabla\psi^\infty(y)\cdot(z-y) &= -a|z-y|-\lambda\cdot(z-y)\\ &\geq \sqrt{2}[-a-n\max_i(|\lambda_i|)]|z_n-y_n|=C|z_n-y_n|\end{aligned}$$

and

$$\begin{aligned}c(x^2-z)-c(x^2-y) &= c(x^2-z)-\tilde{c}(1)\\ &\leq \tilde{c}(\sqrt{1-2(z_n-y_n)+|z-y|^2})-\tilde{c}(1)\\ &\leq \nabla\tilde{c}(\sqrt{1-2(z_n-y_n)+|z-y|^2})(\sqrt{1+|z-y|^2-2(z_n-y_n)}-1)\end{aligned}$$

since $\tilde{c}(1)-\tilde{c}(1-\theta)\geq\nabla\tilde{c}(1-\theta)\theta$. And then since $|z-y|<\sqrt{2}|z_n-y_n|$, we obtain

$$c(x^2-z)-c(x^2-y)\leq\nabla\tilde{c}(\sqrt{1-2(z_n-y_n)+2|z_n-y_n|^2})(|z_n-y_n|^2-(z_n-y_n)+o(|z_n-y_n|))$$

So if (14.35) for $i=2$ holds, we would have for $\delta=(z_n-y_n)>0$

$$C\frac{1}{\nabla\tilde{c}(\sqrt{1-2\delta+2\delta^2})}\leq-1+\delta+o(1)$$

which is not possible for any $C$ since passing to the limit $\delta\to0$, it would give $0\leq-1$.

(iii) We will first prove (14.31) from where it is easy to deduce the first claim of (iii).

Define $\gamma_{1-\eta}(x,y) = (x, \eta x + (1-\eta)y)_{\#}\gamma_{\text{opt}}$ and $\rho_{1-\eta}$ its second marginal (the use of this interpolation map has been introduced in McCann's paper [35]).

We will use the fact that

$$I(\rho_0, \rho_1) \leq I(\rho_0, \rho_{1-\eta})$$

for any $\eta$ to obtain an estimate on $\gamma_{\text{opt}}(Z_\delta)$.

On the first hand, we use the convexity of the entropy. Indeed, the entropy satisfies

$$E(\rho_{1-\eta}) \leq \eta E(\rho_0) + (1-\eta)E(\rho_1)$$

The convexity of $t \to E(\rho_{1-t})$ is classical (see Agueh [1] and McCann [35]) for $t \in [0,1[$ and $t \in ]0,1]$ then for $t \in [0,1]$.

On the other hand, we compute the difference of both terms involving the cost.

Let us compute

$$\begin{aligned} W_c(\rho_0,\rho_1) - W_c(\rho_0,\rho_{1-\eta}) &\geq \int_{\Omega\times\Omega} c(x-y)d\gamma_{\text{opt}}(x,y) - \int_{\Omega\times\Omega} c(x-y)d\gamma_{1-\eta}(x,y) \\ &\geq \int_{Z_\delta} (c(x-y) - c((1-\eta)(x-y)))d\gamma_{\text{opt}}(x,y) \end{aligned}$$

since $c((1-\eta)(x-y))) \leq c(x-y)$ yields

$$\int_{\Omega\times\Omega/Z^\delta} (c(x-y) - c((1-\eta)(x-y)))d\gamma_{\text{opt}}(x,y) \geq 0.$$

And then, using the fundamental theorem of calculus, we obtain

$$W_c(\rho_0,\rho_1) - W_c(\rho_0,\rho_{1-\eta}) \geq \int_{Z_\delta} \eta|x-y||\nabla c((1-\theta)(x-y))|d\gamma_{\text{opt}}(x,y)$$

where $\theta \in [0,\eta]$. Since $|\nabla c(z)| = \nabla\tilde{c}(|z|)$ is increasing with respect to $|z|$ and

$$|1-\theta||x-y| \geq |1-\eta| \inf_{(x,y)\in Z^\delta} |x-y|,$$

it leads to

$$\begin{aligned} W_c(\rho_0,\rho_1) - W_c(\rho_0,\rho_{1-\eta}) &\geq \eta\gamma_{\text{opt}}(Z_\delta) \inf_{Z_\delta}(|x-y|)|\nabla\tilde{c}((1-\eta)\inf_{Z^\delta}|x-y|)| \\ &\geq \eta\gamma_{\text{opt}}(Z_\delta)(1-\delta)\nabla\tilde{c}((1-\eta)(1-\delta)). \end{aligned}$$

Finally, since

$$W_c(\rho_0,\rho_1) + E(\rho_1) \leq W_c(\rho_0,\rho_{1-\eta}) + E(\rho_{1-\eta})$$

we have

$$W_c(\rho_0,\rho_1) - W_c(\rho_0,\rho_{1-\eta}) \leq E(\rho_{1-\eta}) - E(\rho_1) \leq \eta(E(\rho_0) - E(\rho_1))$$

and then, dividing by $\eta$ and letting $\eta \to 0$, we obtain

$$\gamma_{\text{opt}}(Z_\delta) \leq \frac{E(\rho_0) - E(\rho_1)}{(1-\delta)|\nabla\tilde{c}(1-\delta)|}$$

which goes to zero when $\delta \to 0$.

(iv) Point (ii) and (iii) imply that almost everywhere in the support of $\gamma_{\text{opt}}$, $|x-y| < 1$ and $-\nabla c(x-y) = \nabla\psi^\infty(y)$ which means that

$$x = S(y) = y - \nabla c^*(\nabla\psi^\infty(y)).$$

It means that since $\operatorname{supp}\gamma \subset \Delta_1$, it is in fact supported almost everywhere on the graph of a one to one function which is the optimal map.

□

*Remarque* 14.3.2. **The optimal map in dimension 1** In dimension 1, the whole argument is much simpler since the map is already known and does not depend on the cost function and then does not depend on $h$ or $\varepsilon$.

### 14.3.3 The optimal map for an arbitrary $h$

From now on, we deal with the complete minimization problem when $h > 0$ and for any problem $\mathcal{P}_i^h$. We use the notation $c_h(x-y) = c(\frac{x-y}{h})$.

*Proposition* 14.3.4. (i) Let $\gamma_i^h$ be the optimal measure for the minimization problem $(\mathcal{P}_i^h)$,

$$\operatorname{supp}\gamma_i^h \in \{(x,y) \quad | \quad \frac{|x-y|}{h} < 1\} \cup Z_i^h \text{ with } \gamma_{\text{opt}}^h(Z_i^h) = 0.$$

(ii) More precisely, we obtain an estimate of the measure of the set $Z_{i\delta}^h = \{(x,y) \in \operatorname{supp}\gamma_i^h \quad | \quad \frac{|x-y|}{h} > 1-\delta\}$,

$$\gamma_i^h(Z_{i\delta}^h) \leq \frac{E(\rho_{i-1}) - E(\rho_i^h)}{h(1-\delta)|\nabla c(1-\delta)|}$$

(iii) We define Lebesgue almost everywhere a one to one map $S_i^h(y) \in L^\infty(\Omega,\Omega)$ by

$$S_i^h(y) = y - h\nabla c^*(\nabla\psi_i^h(y)) = y + h\nabla c^*(\nabla(F'(\rho_i^h(y))) \text{ satisfying } \gamma_i^h = (S_i^h \times id)_{\#}\rho_i^h \tag{14.36}$$

where $\rho_i^h$ and $\psi_i^h$ belong to $BV(\Omega)$.

*Démonstration.* To prove (1) to (iii), apply the previous result replacing $x-y$ by $\frac{x-y}{h}$ and $E(\rho_0) - E(\rho_1^h)$ by $\frac{E(\rho_{i-1}) - E(\rho_i^h)}{h}$.

Note also that since $\Omega$ is a bounded domain, $S_i^h(y) \in L^\infty(\Omega,\Omega)$.

□

## 14.4 From the discrete equation to the continuous equation

In this section, we have to pass to the limit when the time step goes to zero.

For this purpose, we use a monotonicity argument quoted in Evans [25], Lions [32] or in Otto [38] and since we deal with BV functions, we use also very delicate concepts defined in Andreu Caselles Mazon [8], [12].

### 14.4.1 Construction, compactness and convergence of the measure $\rho^h$

In the sequel, we will assume without loss of generality that $h$ is chosen such that $\frac{T}{h}$ is an integer.

**Notation** In the sequel, the gradient of $\rho^h$ will involve both its absolute continuous part and its singular part and then, we will now use the notation $D^{ac}\rho^h$ instead of $\nabla\rho^h$ for the absolute continuous part of $D\rho^h$. We shall also need the following space of distributions $u(t,y)$ on $[0,T]\times\Omega$ :

$$L^1([0,T];(W^{2,\infty}(\Omega))') = \left\{ u \mid \int_0^T \int_\Omega u\xi dtdy \leq C_u||\xi||_{W^{2,\infty}(\Omega)} \text{ for any test function } \xi \in C_c^\infty(\Omega) \right\}.$$

*Proposition* 14.4.1 (Compactness and convergence of $\rho^h$ ). Let $\rho^h$ be the piecewise constant in time function defined by (14.3). Then

(i) the sequence $\rho^h$ is bounded in $L_w^1([0,T];BV(\Omega))$ and its time derivative $\partial_t\rho^h$ is a bounded sequence in $L^1([0,T];(W^{2,\infty}(\Omega))')$.

(ii) a subsequence of $\rho^h$ converges strongly in $L^1([0,T]\times\Omega)$ and weakly$^*$ in $L_w^1([0,T],BV(\Omega))$ to a limit $\overline{\rho} \in L^\infty([0,T]\times\Omega) \cap L_w^1([0,T],BV_2(\Omega))$ as $h \to 0$.

*Démonstration.* (i) Recall that for a fixed $h$, $\rho_i^h(y)$ is the $L^1(\Omega)$ strong and w$^*BV(\Omega)$ limit of $\rho_i^{\varepsilon h}$ solution of the mollified problem $\mathcal{P}_i^\varepsilon$. Then we have

$$||D(\rho^h)||([0,T]\times\Omega) \leq \liminf_{\varepsilon\to 0} ||\nabla\rho^{\varepsilon h}||_{L^1([0,T]\times\Omega)} = \liminf_{\varepsilon\to 0} \sum_i h||\nabla\rho_i^{\varepsilon h}||_{L^1(\Omega)} \leq C$$

where the constant $C$ depends neither on $h$ nor on $\varepsilon$ (cf 14.29) and then $\rho^h$ is a bounded sequence in $L^1_w([0,T],BV(\Omega))$.

Concerning the time derivative, we have

$$\begin{aligned}
\int_0^T |\int_\Omega \partial_t\rho^h\xi(x)dx|dt| &= \sum_{i=1}^{\frac{T}{h}} |h\int_\Omega \frac{(\rho_i^h-\rho_{i-1}^h)}{h}\xi(x)dx| \\
&= \sum_{i=1}^{\frac{T}{h}} |h\int_\Omega \frac{(\xi(y)-\xi(x))}{h}d\gamma_i^h(x,y)|dt \\
&\leq \sum_{i=1}^{\frac{T}{h}} |\int_\Omega (x-y)\cdot\nabla\xi(y)d\gamma_i^h(x,y)|dt + Ch||D^2\xi||_{L^\infty(\Omega)} \\
&\leq \sum_{i=1}^{\frac{T}{h}} h\int_\Omega \nabla c^*(D^{ac}(F'(\rho_i^h(y))))\rho_i^h(y)\cdot\nabla\xi(y)dydt + Ch||D^2\xi||_{L^\infty(\Omega)} \\
&\leq C||\nabla\xi||_{L^\infty(\Omega)} + Ch||D^2\xi||_{L^\infty(\Omega)}.
\end{aligned}$$

(ii) The sequence $\rho^h$ is bounded in $L^1_w([0,T],BV(\Omega))$, and $BV(\Omega)$ is compactly imbedded in $L^1(\Omega)$. On the other hand $\partial_t\rho^h$ is bounded in $L^1([0,T],(W^{2,\infty}(\Omega))')$ and $L^1(\Omega)\subset(W^{2,\infty}(\Omega))'$. The Aubin Lemma (see [32] [41] [42]) then implies that $\rho^h$ is relatively compact in $L^1([0,T]\times\Omega)$.
□

### 14.4.2 Properties of the sequence $\rho^{\varepsilon h}$

We first recall a result proved by Agueh in [1] presented here for $C^2(\mathbf{R}^d)$ cost functions but proved in fact for a larger set of cost functions.

*Proposition* 14.4.2 (Displacement convexity of the $L^\infty$ norm [1]). Let $\rho_0$, $\rho_1\in P(\Omega)$ be such that $\rho_0$, $\rho_1\leq M$ a.e., and assume that $0\leq c\in C^2(\mathbf{R}^d)$ strictly convex satisfies $c(0)=0$ with Legendre transform $c^*\in C^2(\mathbf{R}^d)$. Denote by $S$ the c-optimal map that pushes $\rho_1$ forward $\rho_0$, and define the interpolant map

$$S_t=(1-t)id+tS,\ \text{for } t\in[0,T].$$

Then $\|(S_t)_\#\rho_1\|_{L^\infty(\Omega)}\leq M$, meaning for nonnegative functions $\xi\in C_c(\mathbf{R}^d)$ we have

$$\int_\Omega \xi(S_t(y))\rho_1(y)dy\leq M\int_\Omega\xi(x)dx. \tag{14.37}$$

The proof of this proposition for regular cost function consists in introducing $\rho_{1-t}=(S_t)_\#\rho_1$ and prove that $\rho_{1-t}\leq M$.

The next proposition establishes local and global inequalities (14.38) and (14.40) relating the generalized Fisher information integrated along the curve $t\in[0,T]\longrightarrow\rho^{\epsilon h}$ to the net change in entropy. We hereafter refer to such bounds as *entropy-information inequalities*.

*Proposition* 14.4.3 (Localized entropy-information inequality).

(i) At each instant in time, $\rho^{\varepsilon h}$ belongs to $W^{1,\infty}(\Omega)$ and lies in a ball in $L^\infty(\Omega)\cap W^{1,1}(\Omega)$ whose radius is independent of $t\in[0,T]$ and $\varepsilon>0$. Moreover the entropy-information inequality

$$\int_0^T\int_\Omega \rho^{\varepsilon h}(t,y)\nabla F'(\rho^{\varepsilon h}(t,y))\cdot\nabla c^{\varepsilon *}(\nabla(F'(\rho^{\varepsilon h}(t,y))))dy\leq\int_\Omega(F(\rho^h(0,y))-F(\rho^{\varepsilon h}(T,y)))dy+\sum_1^{\frac{T}{h}}o_\varepsilon(1) \tag{14.38}$$

is satisfied where $o_\varepsilon(1)\to 0$ when $\varepsilon\to 0$. This integration in time yields

$$||\nabla\rho^{\varepsilon h}||_{L^1([0,T]\times\Omega)}+\sqrt{\varepsilon}||\nabla\rho^{\varepsilon h}||_{L^2([0,T]\times\Omega)}\leq C+\sum_1^{\frac{T}{h}}o_\varepsilon(1) \tag{14.39}$$

where the constant $C$ is now independent of $h>0$ as well as of $\varepsilon>0$.

(ii) Space compactness : $\rho^{\varepsilon h}$ satisfies

$$\sum_1^{\frac{T}{h}} h\int_\Omega |\rho_{i-1}^{\varepsilon h}(y)-\rho_{i-1}^{\varepsilon h}(S_i^{\varepsilon h}(y))|dy = O(h)+\sum_1^{\frac{T}{h}} o_\varepsilon(1)$$

(iii) Time compactness : $\rho^{\varepsilon h}$ satisfies

$$\sum_1^{\frac{T}{h}} h\int_\Omega |\rho_i^{\varepsilon h}(y)-\rho_{i-1}^{h}(y)|dydt = \delta(\varepsilon,h)$$

with $0=\lim_{h\to 0}\lim_{\varepsilon\to 0}\delta(\varepsilon,h)$.

(iv) The sequence $\rho^{\varepsilon h}(t,y)$ satisfies a localized entropy-information inequality for any test function $\xi(t,y)\geq 0$

$$\begin{aligned}
&\int_0^T\int_\Omega \xi(t,y)\rho^{\varepsilon h}(t,y)\nabla c^*(\nabla(F'(\rho^{\varepsilon h}(t,y))))\cdot\nabla(F'(\rho^{\varepsilon h}(t,y)))dydt\\
&\qquad\leq -\int_0^T\int_\Omega \nabla\xi(t,y)\rho^{\varepsilon h}(t,y)\cdot\nabla c^*(\nabla(F'(\rho^{\varepsilon h}(t,y))))F'(\rho^{\varepsilon h}(t,y))dtdy \qquad (14.40)\\
&\qquad -\int_0^T\int_\Omega [\xi(t,y)\partial_t(F(\rho^{\varepsilon h}(t,y)))]dtdy+\tilde\delta(\varepsilon,h)
\end{aligned}$$

where $0=\lim_{h\to 0}\lim_{\varepsilon\to 0}\tilde\delta(\varepsilon,h)$

*Démonstration.* (i) Each term $\rho^{\varepsilon h}\leq M$ and equation (14.21) implies that $||h\nabla\rho_i^{\varepsilon h}||_{L^1(\Omega)}$ is bounded independently of $\varepsilon$ and by summing

$$h\int_\Omega \rho_i^{\varepsilon h}(y)\nabla F'(\rho_i^{\varepsilon h}(y))\cdot\nabla c^{\varepsilon *}(\nabla(F'(\rho_i^{\varepsilon h}(y))))dy\leq\int_\Omega (F(\rho_{i-1}^h(y))-F(\rho_i^{\varepsilon h}(y)))dy$$

on each time interval, we obtain (14.38) since $\lim_{\varepsilon\to 0}\int_\Omega (F(\rho_i^{\varepsilon h}(y)-F(\rho_i^h(y)))dy=0$.

Then (14.39) holds. Moreover, for $\varepsilon$ fixed, we can use an argument of [1] to prove that $\nabla\rho^{\varepsilon h}\in L^\infty(\Omega)$. Indeed since

$$\nabla\rho_i^{\varepsilon h}F''(\rho_i^{\varepsilon h})=\nabla(F'(\rho_i^{\varepsilon h}(y)))=\frac{\rho_i^{\varepsilon h}(y)}{\rho_i^{\varepsilon h}(y)}\nabla c^\varepsilon(\frac{S_i^{\varepsilon h}(y)-y}{h})$$

we obtain for any test function $\xi$

$$\begin{aligned}
|\int_\Omega \nabla\rho_i^{\varepsilon h}(y)\xi(y)dy| &\leq \frac{1}{m\inf_{[m,M]}F''}|\int_\Omega \rho_i^{\varepsilon h}(y)\nabla c^\varepsilon(\frac{S_i^{\varepsilon h}(y)-y}{h})\xi(y)dy|\\
&= \frac{1}{m\inf_{[m,M]}F''}|\int_{\Omega\times\Omega}\nabla c^\varepsilon(\frac{x-y}{h})\xi(y)d\gamma_i^{\varepsilon h}(x,y)|\\
&\leq \frac{1}{m\inf_{[m,M]}F''}||\xi||_{L^1(\Omega)}\sup_{x,y\in\Omega\times\Omega}|\nabla c^\varepsilon(\frac{x-y}{h})|
\end{aligned}$$

and since $c^\varepsilon\in C^1(\mathbf{R}^d)$ we deduce that $\rho_i^{\varepsilon h}\in W^{1,\infty}(\Omega)$.

(ii) Since $\rho_i^{\varepsilon h}\in W^{1,\infty}(\Omega)$ (it may be approximated by a sequence of $C^1(\Omega)$ function) , the Fundamental Theorem Of Calculus (FTOC) yields

$$I_a = \sum_1^{\frac{T}{h}} h\int_\Omega |\rho_{i-1}^{\varepsilon h}(y) - \rho_{i-1}^{\varepsilon h}(S_i^{\varepsilon h}(y))|dy$$

$$= \sum_1^{\frac{T}{h}} h \int_\Omega | \int_0^1 \nabla(\rho_{i-1}^{\varepsilon h})(y + s(S_i^{\varepsilon h}(y) - y)) \cdot (S_i^{\varepsilon h}(y) - y)ds|dy.$$

Since $S_i^{\varepsilon h}(y) - y = h\nabla c^*(\nabla(F'(\rho_i^{\varepsilon h}(y))) + h\varepsilon\nabla(F'(\rho_i^{\varepsilon h}(y))$, it leads to

$$I_a \leq \sum_1^{\frac{T}{h}} h^2 \int_\Omega \int_0^1 |\nabla(\rho_{i-1}^{\varepsilon h})(y + s(S_i^{\varepsilon h}(y) - y))|dsdy$$

$$+ \sum_1^{\frac{T}{h}} h^2\varepsilon \int_\Omega \int_0^1 |\nabla(\rho_{i-1}^{\varepsilon h})(y + s(S_i^{\varepsilon h}(y) - y))||\nabla(F'(\rho_{i-1}^{\varepsilon h}))(y)|dyds$$

and then

$$I_a \leq \frac{1}{m} \int_0^1 \sum_1^{\frac{T}{h}} h^2 \int_\Omega |\nabla\rho_{i-1}^{\varepsilon h}(y + s(S_i^{\varepsilon h}(y) - y))|\rho_i^{\varepsilon h}(y)dyds$$

$$+ \frac{h\varepsilon}{\sqrt{m}} \sup_{[m,M]} |F''| \int_0^1 (\sum_1^{\frac{T}{h}} h \int_\Omega |\nabla(\rho_{i-1}^{\varepsilon h})(y + s(S_i^{\varepsilon h}(y) - y))|^2 \rho_i^{\varepsilon h}(y)dy)^{\frac{1}{2}} (\sum_1^{\frac{T}{h}} h \int_\Omega |\nabla(\rho_{i-1}^{\varepsilon h})(y)|^2 dy)^{\frac{1}{2}} ds$$

which becomes using (14.37) ( by approaching $|\nabla\rho_{i-1}^{\varepsilon h}|$ by $C^\infty(\Omega)$ non-negative function) in both terms

$$I_a \leq \frac{h}{m} M \int_0^1 \sum_1^{\frac{T}{h}} h \int_\Omega |\nabla\rho_{i-1}^{\varepsilon h}(y)|dyds$$

$$+ \frac{h}{\sqrt{m}} \sqrt{M} \sup_{[m,M]} |F''| \int_0^1 (\sum_1^{\frac{T}{h}} h \int_\Omega \varepsilon |\nabla(\rho_{i-1}^{\varepsilon h})(y)|^2 dy)^{\frac{1}{2}} (\sum_1^{\frac{T}{h}} h \int_\Omega \varepsilon |\nabla(\rho_{i-1}^{\varepsilon h})(y)|^2 dy)^{\frac{1}{2}} ds$$

$$\leq (C + \sum_1^{\frac{T}{h}} o_\varepsilon(1))h$$

by the $L^1([0,T] \times \Omega)$ bound on the sequence $\nabla\rho^{\varepsilon h}$ and the $L^2([0,T] \times \Omega)$ bound on $\sqrt{\varepsilon}\nabla\rho^{\varepsilon h}$ written in (14.38).

(iii) Once again, we adapt arguments involved in [1]. We will in fact prove that

$$\int_h^T \int_\Omega (F'(\rho^{\varepsilon h}(t,y)) - F'(\rho^h(t-h,y)))(\rho^{\varepsilon h}(t,y) - \rho^h(t-h,y))dydt$$

$$= \sum_1^{\frac{T}{h}} h \int_\Omega (F'(\rho_i^{\varepsilon h}(y)) - F'(\rho_{i-1}^h(y)))(\rho_i^{\varepsilon h}(y) - \rho_{i-1}^h(y))dy \leq Ch$$

which means when $\varepsilon < h \to 0$ by denoting $\rho^+$ the $L^1([0,T] \times \Omega)$ limit of $\rho^{\varepsilon h}(t,y)$ and $\rho^-$ the $L^1([0,T] \times \Omega)$ limit of $\rho^h(t-h,y)$ that $\rho^+ = \rho^-$ since at the limit

$$0 \leq (F'(\rho^+) - F'(\rho^-))(\rho^+ - \rho^-) \leq 0.$$

And then $\rho^+ = \rho^-$ or

$$\lim_{\varepsilon < h \to 0} \int_h^T \int_\Omega |\rho^{\varepsilon h}(t,y) - \rho^h(t-h,y)|dydt = 0.$$

But by denoting $\Phi_i(y) = F'(\rho_i^{\varepsilon h}(y)) - F'(\rho_{i-1}^{\varepsilon h}(y))$ we can rewrite

$$\begin{aligned} I_b &= \sum_1^{\frac{T}{h}} h \int_\Omega (F'(\rho_i^{\varepsilon h}(y)) - F'(\rho_{i-1}^{\varepsilon h}(y)))(\rho_i^{\varepsilon h}(y) - \rho_{i-1}^h(y))dy \\ &\quad + \sum_1^{\frac{T}{h}} h \int_\Omega (F'(\rho_{i-1}^{\varepsilon h}(y)) - F'(\rho_{i-1}^h(y)))(\rho_i^{\varepsilon h}(y) - \rho_{i-1}^h(y))dy \\ &= \sum_1^{\frac{T}{h}} h \int_\Omega (\Phi_i(y) - \Phi_i(S_i^{\varepsilon h}(y))\rho_i^{\varepsilon h}(y)dy + h\sum_1^{\frac{T}{h}} o_\varepsilon(1). \end{aligned}$$

And then by the FTOC, we obtain

$$\begin{aligned} I_b &= \sum_1^{\frac{T}{h}} h \int_\Omega \int_0^1 (S_i^{\varepsilon h}(y) - y) \cdot \nabla \Phi_i(y + s(S_i^{\varepsilon h}(y) - y))\rho_i^{\varepsilon h}(y)dyds + h\sum_1^{\frac{T}{h}} o_\varepsilon(1) \\ &\leq (C + \sum_1^{\frac{T}{h}} o_\varepsilon(1))h \end{aligned}$$

using the same argument as above by applying (14.37) for both terms, the uniform $L^1([0,T]\times\Omega)$ bound on the sequence $\nabla\rho^{\varepsilon h}$ and the $L^2([0,T]\times\Omega)$ bound of $\sqrt{\varepsilon}\nabla\rho^{\varepsilon h}$.

iv) We now want to prove the localized inequality (14.40) for the mollified problem. For a fixed test function $\xi(t,x) \geq 0$, take $h$ small enough such that $\operatorname{supp}\xi \subset [h,T]\times\Omega$.

We want an estimate for the following quantity

$$\int_0^T \int_\Omega \xi(t,y)\nabla F'(\rho^{\varepsilon h}(t,y))\rho^{\varepsilon h}(t,y) \cdot \nabla c^*(\nabla(F'(\rho^{\varepsilon h}(t,y))))dydt$$

$$= \sum_1^{\frac{T}{h}} h \int_\Omega \xi(t_i,y)\nabla F'(\rho_i^{\varepsilon h}(y))\rho_i^{\varepsilon h}(y) \cdot \nabla c^*(\nabla(F'(\rho_i^{\varepsilon h}(y))))dy + O(h) + h\sum_1^{\frac{T}{h}} o_\varepsilon(1).$$

Indeed by inequality (14.21),

$$\begin{aligned} &|\int_0^T \int_\Omega \xi(t,y)\nabla(F'(\rho^{\varepsilon h}(t,y))) \cdot \nabla c^*(\nabla(F'(\rho^{\varepsilon h}(t,y))))dydt - \sum_1^{\frac{T}{h}} h \int_\Omega \xi(t_i,y)\nabla(F'(\rho_i^{\varepsilon h}(y))) \cdot \nabla c^*(\nabla(F'(\rho_i^{\varepsilon h}(y))))dy| \\ &\leq ||\partial_t \xi||_{L^\infty} \sum_1^{\frac{T}{h}} h \int_\Omega (F(\rho^{\varepsilon h}(0,y)) - F(\rho^{\varepsilon h}(T,y))dy) \\ &\leq (C + \sum_1^{\frac{T}{h}} o_\varepsilon(1))h \end{aligned}$$

Then by introducing $P_F(\lambda) = \lambda F'(\lambda) - F(\lambda)$, we can write

$$\sum_1^{\frac{T}{h}} h \int_\Omega \xi(t_i,y)\nabla F'(\rho_i^{\varepsilon h}(y))\rho_i^{\varepsilon h}(y) \cdot \nabla c^*(\nabla(F'(\rho_i^{\varepsilon h}(y))))dy$$

$$= \sum_1^{\frac{T}{h}} h \int_\Omega \xi(t_i,y)\nabla(P_F(\rho_i^{\varepsilon h}(y))) \cdot \nabla c^*(\nabla(F'(\rho_i^{\varepsilon h}(y))))dy.$$

Moreover since $\xi \geq 0$,

$$\sum_1^{\frac{T}{h}} h \int_\Omega \xi(t_i,y)\nabla(P_F(\rho_i^{\varepsilon h}(y))) \cdot \nabla c^*(\nabla(F'(\rho_i^{\varepsilon h}(y))))dy \leq \sum_1^{\frac{T}{h}} h \int_\Omega \xi(t_i,y)\nabla(P_F(\rho_i^{\varepsilon h}(y))) \cdot \nabla c^{\varepsilon *}(\nabla(F'(\rho_i^{\varepsilon h}(y))))dy.$$

So we will consider the quantity

$$\begin{aligned} I_0 &= \sum_1^{\frac{T}{h}} h \int_\Omega \xi(t_i,y)\nabla(P_F(\rho_i^{\varepsilon h}(y))) \cdot \nabla c^{\varepsilon *}(\nabla(F'(\rho_i^{\varepsilon h}(y))))dy \\ &= -\sum_1^{\frac{T}{h}} h \int_\Omega \nabla\xi(t_i,y)P_F(\rho_i^{\varepsilon h}(y)) \cdot \nabla c^{\varepsilon *}(\nabla(F'(\rho_i^{\varepsilon h}(y))))dy \\ &\quad -\sum_1^{\frac{T}{h}} h \int_\Omega \xi(t_i,y)P_F(\rho_i^{\varepsilon h}(y))\mathrm{div}\nabla c^{\varepsilon *}(\nabla(F'(\rho_i^{\varepsilon h}(y))))dy \\ &= I_1 + I_2 \end{aligned}$$

But since $S_i^{\varepsilon h}(y) - y = h\nabla c^{\varepsilon *}(\nabla(F'(\rho_i^{\varepsilon h}(y))))$ we obtain

$$I_2 = -\sum_1^{\frac{T}{h}} \int_\Omega \xi(t_i,y)P_F(\rho_i^{\varepsilon h}(y))\mathrm{div}(S_i^{\varepsilon h}(y) - y)dy.$$

We now use the intermediate result proved by Agueh in [1]

$$\begin{aligned} -P_F(\rho_i^{\varepsilon h}(y))\mathrm{div}(S_i^{\varepsilon h}(y) - y) &= -P_F(\rho_i^{\varepsilon h}(y))\mathrm{tr}(\nabla S_i^{\varepsilon h}(y) - Id) \\ &= \rho_i^{\varepsilon h}(y)[-d(\frac{1}{(\rho_i^{\varepsilon h}(y))^{1/d}})^{(d-1)}P_F((\frac{1}{(\rho_i^{\varepsilon h}(y))^{1/d}})^{-d})(\frac{\mathrm{tr}(\nabla S_i^{\varepsilon h}(y)}{d(\rho_i^{\varepsilon h}(y))^{1/d}} - \frac{1}{(\rho_i^{\varepsilon h}(y))^{1/d}})] \\ &\leq \rho_i^{\varepsilon h}(y)[F((\frac{\mathrm{tr}\nabla S_i^{\varepsilon h}(y)}{d(\rho_i^{\varepsilon h}(y))^{1/d}})^{-d})(\frac{\mathrm{tr}\nabla S_i^{\varepsilon h}(y)}{d(\rho_i^{\varepsilon h}(y))^{1/d}})^{-d} - F((\frac{1}{(\rho_i^{\varepsilon h}(y))^{1/d}})^{-d})(\frac{1}{(\rho_i^{\varepsilon h}(y))^{1/d}})^{-d}] \\ &\leq F(\frac{\rho_i^{\varepsilon h}(y)}{\det\nabla S_i^{\varepsilon h}(y)})\det\nabla S_i^{\varepsilon h}(y) - F(\rho_i^{\varepsilon h}(y)) \end{aligned}$$

since $\lambda^d F(\lambda^{-d})$ (which derivative is $-d\lambda^{d-1}P_F(\lambda^{-d})$ ) is convex and non decreasing and $\det\nabla S_i^{\varepsilon h}(y) \leq (\frac{\mathrm{tr}\nabla S_i^{\varepsilon h}(y)}{d})^d$ (see the displacement convexity in [35]).

Since $\xi(t_i,y) \geq 0$, we obtain

$$I_2 \leq \sum_1^{\frac{T}{h}} \int_\Omega \xi(t_i,y)(F(\frac{\rho_i^{\varepsilon h}(y)}{\det\nabla S_i^{\varepsilon h}(y)})\det\nabla S_i^{\varepsilon h}(y) - F(\rho_i^{\varepsilon h}(y)))dy$$

and then

$$\begin{aligned} I_2 &\leq \sum_1^{\frac{T}{h}} \int_\Omega \xi(t_i,S_i^{\varepsilon h}(y))F(\frac{\rho_i^{\varepsilon h}(y)}{\det\nabla S_i^{\varepsilon h}(y)})\det\nabla S_i^{\varepsilon h}(y) - \xi(t_i,y)F(\rho_i^{\varepsilon h}(y))dy \\ &\quad + \sum_1^{\frac{T}{h}} \int_\Omega (\xi(t_i,y) - \xi(t_i,S_i^{\varepsilon h}(y)))F(\frac{\rho_i^{\varepsilon h}(y)}{\det\nabla S_i^{\varepsilon h}(y)})\det\nabla S_i^{\varepsilon h}(y)dy. \end{aligned}$$

The two right hand terms $I_{21}+I_{22}$ can be treated as follow using the relation

$$\rho_{i-1}^h(S_i^{\varepsilon h}(y))\det\nabla S_i^{\varepsilon h}(y)=\rho_i^{\varepsilon h}(y)$$

and we obtain

$$\begin{aligned}
I_{21} &= \sum_1^{\frac{T}{h}}\int_\Omega \xi(t_i,S_i^{\varepsilon h}(y)F(\rho_{i-1}^h(S_i^{\varepsilon h}(y)))\frac{\rho_i^{\varepsilon h}(y)}{\rho_{i-1}^h(S_i^{\varepsilon h}(y))}-\xi(t_i,y)F(\rho_i^{\varepsilon h}(y))dy\\
&= \sum_1^{\frac{T}{h}}\int_\Omega \xi(t_i,y)(F(\rho_{i-1}^h(y))-F(\rho_i^{\varepsilon h}(y)))dy\\
&= \sum_1^{\frac{T}{h}}\int_\Omega \xi(t_i,y)(F(\rho_{i-1}^{\varepsilon h}(y))-F(\rho_i^{\varepsilon h}(y)))dy+\sum_1^{\frac{T}{h}}o_\varepsilon(1)\\
&= -\int_0^T\int_\Omega \partial_t F(\rho^{\varepsilon h}(t,y))\xi(t,y)dydt+\sum_1^{\frac{T}{h}}o_\varepsilon(1)
\end{aligned}$$

and

$$\begin{aligned}
I_{22} &= \sum_1^{\frac{T}{h}}\int_\Omega (\xi(t_i,y)-\xi(t_i,S_i^{\varepsilon h}(y)))F(\rho_{i-1}^h(S_i^{\varepsilon h}(y)))\det\nabla S_i^{\varepsilon h}(y)dy\\
&= -\sum_1^{\frac{T}{h}}\int_\Omega \nabla\xi(t_i,y)(S_i^{\varepsilon h}(y)-y)F(\rho_{i-1}^h(S_i^{\varepsilon h}(y)))\det\nabla S_i^{\varepsilon h}(y)dy+O(h)\\
&= -\sum_1^{\frac{T}{h}}\int_\Omega h\nabla\xi(t_i,y)\cdot\nabla c^{\varepsilon *}(\nabla(F'(\rho_i^{\varepsilon h}(y))))F(\rho_{i-1}^h(S_i^{\varepsilon h}(y)))\det\nabla S_i^{\varepsilon h}(y)dy+O(h)
\end{aligned}$$

We will use that fact that

$$\det\nabla S_i^{\varepsilon h}(y)=\frac{\rho_i^{\varepsilon h}(y)}{\rho_{i-1}^h(S_i^{\varepsilon h}(y))}=1+\frac{\rho_i^{\varepsilon h}(y)-\rho_{i-1}^h(S_i^{\varepsilon h}(y))}{\rho_{i-1}^h(S_i^{\varepsilon h}(y))}$$

which yields

$$\begin{aligned}
I_{22} &= -\sum_1^{\frac{T}{h}}\int_\Omega h\nabla\xi(t_i,y)\cdot\nabla c^{\varepsilon *}(\nabla(F'(\rho_i^{\varepsilon h}(y))))F(\rho_i^{\varepsilon h}(y))dy\\
&\quad +\sum_1^{\frac{T}{h}}\int_\Omega h\nabla\xi(t_i,y)\cdot\nabla c^{\varepsilon *}(\nabla(F'(\rho_i^{\varepsilon h}(y))))(F(\rho_i^{\varepsilon h}(y))-F(\rho_{i-1}^h(S_i^{\varepsilon h}(y))))\\
&\quad -\sum_1^{\frac{T}{h}}\int_\Omega h\nabla\xi(t_i,y)\cdot\nabla c^{\varepsilon *}(\nabla(F'(\rho_i^{\varepsilon h}(y))))F(\rho_{i-1}^h(S_i^{\varepsilon h}(y)))\frac{\rho_i^{\varepsilon h}(y)-\rho_{i-1}^h(S_i^{\varepsilon h}(y))}{\rho_{i-1}^h(S_i^{\varepsilon h}(y))}+O(h)
\end{aligned}$$

Finally, we write that

$$\begin{aligned}
&|-\sum_1^{\frac{T}{h}}\int_\Omega h\nabla\xi(t_i,y)\cdot\nabla c^*(\nabla(F'(\rho_i^{\varepsilon h}(y))))F(\rho_{i-1}^h(S_i^{\varepsilon h}(y)))\frac{\rho_i^{\varepsilon h}(y)-\rho_{i-1}^h(S_i^{\varepsilon h}(y))}{\rho_{i-1}^h(S_i^{\varepsilon h}(y))}|\\
&\qquad\qquad \leq C||\nabla\xi||_{L^\infty([O,T]\times\Omega)}||F||_{L^\infty([m,M])}\sum_1^{\frac{T}{h}}h\int_\Omega|\rho_{i-1}^{\varepsilon h}(S_i^{\varepsilon h}(y))-\rho_i^{\varepsilon h}(y)|dy
\end{aligned}$$

and

$$|-\sum_1^{\frac{T}{h}}\int_\Omega \varepsilon h\nabla\xi(t_i,y)\cdot\nabla(F'(\rho_i^{\varepsilon h}(y)))F(\rho_{i-1}^h(S_i^{\varepsilon h}(y)))\frac{\rho_i^{\varepsilon h}(y)-\rho_{i-1}^h(S_i^{\varepsilon h}(y))}{\rho_{i-1}^h(S_i^{\varepsilon h}(y))}|$$

$$\leq C\sqrt{\varepsilon}||\nabla\xi||_{L^\infty(\Omega_T)}||F||_{L^\infty([m,M])}||F''||_{L^\infty([m,M])}||\sqrt{\varepsilon}\nabla\rho^{\varepsilon h}||_{L^2(\Omega_T)}(\sum_1^{\frac{T}{h}}h\int_\Omega|\rho_{i-1}^{\varepsilon h}(S_i^{\varepsilon h}(y))-\rho_i^{\varepsilon h}(y)|dy)^{\frac{1}{2}}$$

Moreover, since

$$|F(\rho_{i-1}^h(S_i^{\varepsilon h}(y)))-F(\rho_i^{\varepsilon h}(y))|\leq||F'||_{L^\infty([m,M])}|\rho_{i-1}^h(S_i^{\varepsilon h}(y))-\rho_i^{\varepsilon h}(y)|,$$

we have

$$\sum_1^{\frac{T}{h}}\int_\Omega h\nabla\xi(t_i,y)\cdot\nabla c^*(\nabla(F'(\rho_i^{\varepsilon h}(y))))(F(\rho_i^{\varepsilon h}(y))-F(\rho_{i-1}^h(S_i^{\varepsilon h}(y))))$$

$$\leq C||\nabla\xi||_{L^\infty([O,T]\times\Omega)}||F'||_{L^\infty([m,M])}\sum_1^{\frac{T}{h}}h\int_\Omega|\rho_{i-1}^{\varepsilon h}(S_i^{\varepsilon h}(y))-\rho_i^{\varepsilon h}(y)|dy$$

and

$$\sum_1^{\frac{T}{h}}\int_\Omega h\varepsilon\nabla\xi(t_i,y)\cdot\nabla(F'(\rho_i^{\varepsilon h}(y)))(F(\rho_i^{\varepsilon h}(y))-F(\rho_{i-1}^h(S_i^{\varepsilon h}(y))))$$

$$\leq C\sqrt{\varepsilon}||\nabla\xi||_{L^\infty(\Omega_T)}||F'||_{L^\infty([m,M])}||F''||_{L^\infty([m,M])}||\sqrt{\varepsilon}\nabla\rho^{\varepsilon h}||_{L^2(\Omega_T)}(\sum_1^{\frac{T}{h}}h\int_\Omega|\rho_{i-1}^{\varepsilon h}(S_i^{\varepsilon h}(y))-\rho_i^{\varepsilon h}(y)|dy)^{\frac{1}{2}}$$

where $\Omega_T=[0,T]\times\Omega$.
And then, since

$$\begin{aligned}\sum_1^{\frac{T}{h}}h\int_\Omega|\rho_{i-1}^h(S_i^{\varepsilon h}(y))-\rho_i^{\varepsilon h}(y)|dy &\leq \sum_1^{\frac{T}{h}}h[\int_\Omega|\rho_{i-1}^h(S_i^{\varepsilon h}(y))-\rho_{i-1}^{\varepsilon h}(S_i^{\varepsilon h}(y))|dy+\int_\Omega|\rho_{i-1}^{\varepsilon h}(S_i^{\varepsilon h}(y))-\rho_{i-1}^{\varepsilon h}(y)|dy\\ &\quad+\int_\Omega|\rho_{i-1}^{\varepsilon h}(y)-\rho_{i-1}^h(y)|dy+\int_\Omega|\rho_{i-1}^h(y)-\rho_i^{\varepsilon h}(y)|dy]\\ &\leq \sum_1^{\frac{T}{h}}o_\varepsilon(1)+O(h)+\delta(\varepsilon,h)\end{aligned}$$

we obtain (14.40) using (ii) and (iii).
□

*Proposition* 14.4.4 ( Approximate equation and Entropy information). The piecewise constant in time function $\rho^h$ satisfies

(i) The approximate heat equation

$$\int_0^T\int_\Omega\partial_t\rho^h(y)\xi(t,y)dydt=O(h)+\int_0^T\int_\Omega\nabla c^*(D^{ac}(F'(\rho^h(y))))\rho^h(y)\cdot\nabla\xi(t,y)dydt \tag{14.41}$$

for any test function $\xi$, where $D^{ac}(F'(\rho^h(y)))=D^{ac}\rho^h(y)F''(\rho^h(y))$ and $D^{ac}\rho^h$ denotes the Radon-Nikodym derivative of the measure $D\rho^h$ with respect to the Lebesgue measure.

(ii) The entropy-information inequality

$$\mathcal{G}(F'(\rho^h); D(F'(\rho^h))) \leq \int_\Omega [F(\rho_0(y)) - F(\rho^h(T,y))]dy \tag{14.42}$$

where the left hand side represents the integral (14.46) of a function generalizing $g(z,\xi) = (F^*)' \circ (F(z))\xi\nabla c^*(\xi))$.

*Remarque* 14.4.1. Note however that the same kind of localized inequality as (14.40) is also valid for $\rho^h$.

*Démonstration.* (i) We will first prove the following discrete in time equation localized in time

$$\int_\Omega \frac{\rho_i^h(y) - \rho_{i-1}^h(y)}{h}\xi(y)dy = \int_\Omega \nabla c^*(D^{ac}(F'(\rho_i^h(y))))\rho_i^h(y)\nabla\xi(y)dy + O(h) \tag{14.43}$$

holds for any test function $\xi \in C^\infty(\Omega)$. Multiplying (14.36) by $\rho_i^h\nabla\xi$, we obtain for any test function $\xi \in C_c^\infty(\Omega)$

$$\int_\Omega (\frac{S_i^h(y) - y}{h})\nabla\xi(y)\rho_i^h(y)dy = \int_\Omega \nabla c^*(D^{ac}(F'(\rho_i^h(y))))\rho_i^h(y)\nabla\xi(y)dy.$$

To obtain properly the discrete in time equation, we have to compute for any test function $\xi$

$$\int_\Omega \frac{\rho_i^h(y) - \rho_{i-1}(y)}{h}\xi(y)dy = \frac{1}{h}\int_{\Omega\times\Omega} (\xi(y) - \xi(x))d\gamma_i^h(x,y).$$

By using the FTOC applied to the test function $\xi$, we have

$$|\int_\Omega \frac{1}{h}[(\xi(y) - \xi(x)) - (x-y)\nabla\xi(y)]d\gamma_i^h(x,y)| \leq \frac{1}{2h}\sup_{z\in\Omega}|D^2\xi(z)|\int_\Omega |x-y|^2 d\gamma_i^h(x,y) \leq C\frac{h}{2}$$

and since

$$\int_\Omega (\frac{S_i^h(y) - y}{h})\nabla\xi(y)\rho_i^h(y) = \int_{\Omega\times\Omega} \frac{x-y}{h}\nabla\xi(y)d\gamma_i^h(x,y),$$

we obtain the expected discrete in time equation

$$|\int_\Omega \frac{\rho_i^h(y) - \rho_{i-1}^h(y)}{h}\xi(y)dy - \int_\Omega \nabla c^*(D^{ac}(F'(\rho_i^h(y))))\rho_i^h(y)\nabla\xi(y)dy| \leq \frac{1}{2h}\sup_{z\in\Omega}|D^2\xi(z)|\int_\Omega |x-y|^2 d\gamma_i^h(x,y) \tag{14.44}$$

and then we have (14.43). Let us write (14.43) on each interval $]ih,(i+1)h]$ and sum it with respect to $i$. Since for any test function $\xi \in C^\infty([0,T]\times\Omega)$

$$|\sum_1^{\frac{T}{h}} h\int_\Omega \frac{\rho^h((i+1)h,y) - \rho^h(ih,y)}{h}\xi(t_i,y)dydt \quad - \quad \int_0^T\int_\Omega \nabla c^*(D^{ac}(F'(\rho^h(y))))\rho^h(y)\nabla\xi(t_i,y)dydt|$$

$$\leq \sum_1^{\frac{T}{h}} h\frac{1}{h}\sup_{[0,T]\times\Omega}|D_x^2\xi|\int_\Omega |x-y|^2 d\gamma_i^h(x,y)$$

we obtain (14.41) because $|x-y| < h$ and $\sum_1^{\frac{T}{h}} h = T$ and

$$|\sum_1^{\frac{T}{h}} h\int_\Omega \nabla c^*(D^{ac}(F'(\rho^h(y))))\rho^h(y)\nabla\xi(t_i,y)dydt - \int_0^T\int_\Omega \nabla c^*(D^{ac}(F'(\rho^h(y))))\rho^h(y)\nabla\xi(t,y)dydt| \leq C||\partial_t\nabla\xi||_{L^\infty}h.$$

(ii) We want to prove that

$$\begin{aligned}\int_0^T\int_\Omega \rho^h\nabla c^*(D(F'(\rho^h))\cdot D(F'(\rho^h)) &\leq \liminf_{\varepsilon\to 0}\int_0^T\int_\Omega \rho^{\varepsilon h}\nabla c^*(\nabla(F'(\rho^{\varepsilon h})))\cdot\nabla(F'(\rho^{\varepsilon h}))\\ &\leq \liminf_{\varepsilon\to 0}\int_\Omega [F(\rho_0(y))-F(\rho^{\varepsilon h}(T,y))]dy.\end{aligned} \tag{14.45}$$

but since $\rho^h$ is in $BV$, we have to give a sense to the left hand side.

*Remarque* 14.4.2. In the above expression, we define $\rho^{\varepsilon h}(t,y)$ the same way we defined $\rho^h(t,y)$ from $\rho_i^h(y)$ through (14.3).

Using the same notation as in Andreu Caselles Mazon [12], let us denote

$$g(F'(\rho),D(F'(\rho))) = (F^*)'\circ F'(\rho)D(F'(\rho))\cdot\nabla c^*(D(F'(\rho))).$$

We can then define the generalized version of the Fisher information for BV functions by defining

$$\begin{aligned}\mathcal{G}(\rho,D\rho) &= \int_0^T\int_\Omega g(\rho,D^{ac}\rho)dtdx+\int_\Omega g^0(\rho,\frac{D\rho}{|D\rho|})|D^c\rho|\\ &\quad+\int_{J_\rho}(\int_{\rho_-(x)}^{\rho_+(x)} g^0(s,\nu_\rho(x))ds)d\mathcal{H}^{d-1}(x)\end{aligned} \tag{14.46}$$

where $g^0$ is the recession function equal to $\lim_{t\to 0} tg(x,z,\frac{\xi}{t}) = |\xi|$ cf (14.11).

To get the entropy-information inequality, we want to prove a lower semicontinuity for $\mathcal{G}$ :

$$\mathcal{G}(F'(\rho^h),D(F'(\rho^h)))\leq\liminf_{\varepsilon\to 0}\int_0^T\int_\Omega g(F'(\rho^{\varepsilon h})),\nabla(F'(\rho^{\varepsilon h}))). \tag{14.47}$$

For this purpose, let us decompose this quantity in two parts, which will be semicontinuity. Indeed, we write when $\rho\in W^{1,1}(\Omega)$

$$\begin{aligned}g(F'(\rho),\nabla(F'(\rho))) &= (F^*)'\circ F'(\rho)\nabla(F'(\rho))\nabla c^*(\nabla F'(\rho)))\\ &= (F^*)'\circ F'(\rho)(c(\nabla c^*(\nabla(F'(\rho)))+c^*(\nabla(F'(\rho))))\end{aligned}$$

Using the result of De Cicco, Fusco and Verde on lower semi-continuity for $BV$ functions [23], since $c^*$ is convex and $(F^*)'$ is continuous, we obtain

$$\int_0^T\int_\Omega (F^*)'\circ(F'(\rho^h))c^*(D(F'(\rho^h)))\leq\liminf_{\varepsilon\to 0}\int_0^T\int_\Omega (F^*)'\circ(F'(\rho^{\varepsilon,h}))c^*(\nabla(F'(\rho^{\varepsilon,h}))).$$

In another hand, since $\nabla c^*(\nabla(F'(\rho^{\varepsilon h}))$ is bounded in $L^\infty([0,T]\times\Omega)$, it converges in $w^*L^\infty([0,T]\times\Omega)$. It still remains to identify this limit. For any test function $\xi\in\mathcal{D}(\Omega)$, we have, for a fixed $t$

$$\begin{aligned}&\int_\Omega \xi(y)\nabla c^*\nabla(F'(\rho^{\varepsilon h}(t,y)))\rho^{\varepsilon h}(t,y)dy\\ &=\int_\Omega \xi(y)\nabla c^{\varepsilon *}\nabla(F'(\rho^{\varepsilon h}(t,y)))\rho^{\varepsilon h}(t,y)dy-\varepsilon\int_\Omega\xi(y)|\nabla(F'(\rho^{\varepsilon h}(t,y)))|\rho^{\varepsilon h}(t,y)dy\\ &=\int_{\Omega\times\Omega}\xi(y)(x-y)d\gamma_i^{h,\varepsilon}(x,y)+O(\sqrt{\varepsilon})\to\int_{\Omega\times\Omega}\xi(y)(x-y)d\gamma_i^h(x,y)\\ &=\int_\Omega\xi(y)\nabla c^*(D^{ac}(F'(\rho^h(t,y))))\rho^h(t,y)dy\end{aligned}$$

and then the uniqueness of the limit says that $\nabla c^*(\nabla(F'(\rho^{\varepsilon h})) \to \nabla c^*(D^{ac}(F'(\rho^h))) \quad w^*L^\infty([0,T]\times\Omega)$ and then

$$|\nabla c^*(D^{ac}(F'(\rho^h(t,y))))| \le \liminf_{\varepsilon\to 0} |\nabla c^*(\nabla(F'(\rho^{\varepsilon h}))|.$$

Since $c$ is non decreasing on $\mathbf{R}^+$, we have

$$c(\nabla c^*(|D^{ac}(F'(\rho^h))|) \le c(|\nabla c^*(\nabla(F'(\rho^{\varepsilon h})|)$$

and then

$$\int_0^T \int_\Omega \rho^h c(\nabla c^*(D^{ac}(F'(\rho^h)))) \le \liminf_{\varepsilon\to 0} \int_\Omega \rho^h c(\nabla c^*(\nabla(F'(\rho^{\varepsilon h})))).$$

Moreover – using that $\rho^{\varepsilon h}$ converges strongly in $L^1([0,T]\times\Omega)$ towards $\rho^h$ – we have

$$\lim_{\varepsilon\to 0} \int_0^T \int_\Omega (\rho^h - \rho^{\varepsilon h}) c(\nabla c^*(\nabla(F'(\rho^{\varepsilon h})))) = 0.$$

Hence we recover the expected lower-semi-continuity property (14.47).

□

*Remarque* 14.4.3. In our case $g^0(x,z,\xi) = |\xi|$ and then

$$\int_\Omega g^0(\rho, \frac{D\rho}{|D\rho|})|D^c\rho| + \int_{J_\rho} (\int_{\rho_-(x)}^{\rho_+(x)} g^0(s,\nu_\rho(x))ds) d\mathcal{H}^{d-1}(x) = \int_\Omega |D^c\rho| + \int_{J_\rho} (\rho_+(x) - \rho_-(x)) d\mathcal{H}^{d-1}(x) \ge 0$$

so the singular terms are positive and we can write the fisher information for the absolute continuous part of $D(F'(\rho^h))$, i.e.

$$\int_0^T \int_\Omega \rho^h(t,y) \nabla c^*(D^{ac}(F'(\rho^h(t,y)))) \cdot D^{ac}(F'(\rho^h(t,y))) \le \int_\Omega [F(\rho_0(y)) - F(\rho^h(T,y))] dy.$$

### 14.4.3 Limiting equation

The following proposition represents the keystone which enables us to apply the Minty-Browder technique to complete the construction.

*Proposition* 14.4.5 (The continuous time limit). (i) The time derivative $\partial_t \overline{\rho}$ belongs to $L^1([0,T], BV(\Omega))'$ and is the time derivative of $\overline{\rho}$ in $L^1([0,T], BV(\Omega))'$, i.e. for any test function $\rho \in L^1([0,T], BV_2(\Omega))$ admitting a weak derivative $\Theta$ in $L^1_w([0,T], BV(\Omega)) \cup L^\infty([0,T]\times\Omega)$ as defined in [10], meaning

$$\rho(t,y) = \int_0^t \Theta(s,y) ds$$

we have

$$\int_0^T \int_\Omega \partial_t \overline{\rho}(t,y) \rho(t,y) dt dy = -\int_0^T \int_\Omega \overline{\rho}(t,y) \Theta(t,y) dt dy. \tag{14.48}$$

(ii) Let $\overline{A} \in L^\infty([0,T]\times\Omega)$ be the weak$^*$ limit of $A(\rho^h) = \rho^h \nabla c^*(D^{ac}(F'(\rho^h)))$, for any test function $\xi \ge 0$, we have then the corresponding localized inequality

$$\begin{aligned} -\int_\Omega \partial_t F(\overline{\rho}(t,y)) \xi(t,y) dt dy - \int_\Omega \nabla \xi(t,y) F'(\overline{\rho}(t,y)) \overline{A}(t,y) dt dy \\ \le \int_0^T \int_\Omega \xi(t,y) (\overline{A}(t,y); D^{ac}(F'(\overline{\rho}(t,y)))) dy dt + (T^s; \xi(t,y)) \end{aligned} \tag{14.49}$$

where $T^s$ is a purely singular measure. We shall use (14.49) in lieu of integration by parts.

*Remarque* 14.4.4. Note that we would originally expect instead of inequality (14.49) an equality obtained by integration by part from the equation

$$\partial_t \overline{\rho} = \text{div } (\overline{A})$$

multiplied by $\xi F'(\overline{\rho})$. The problem is that $F'(\overline{\rho})$ is not regular enough – since it belongs to $L^1_w([0,T], BV_2(\Omega))$ – to be used as a test function.

*Démonstration.* Let us prove (i) which is fundamental to obtain (ii). We want to prove that $\partial_t \overline{\rho}$ belongs to $L^1([0,T], BV(\Omega))'$. Remark that

$$\begin{aligned}\int_0^T \int_\Omega \partial_t \overline{\rho}(t,y)\xi(t,y)dtdy &= \lim_{h\to 0} \int_0^T \int_\Omega \partial_t \rho^h(t,y)\xi(t,y)dtdy \\ &= \lim_{h\to 0} \sum_1^{\frac{T}{h}} \int_{t_{i-1}}^{t_i} \int_\Omega \frac{(\rho_i^h(y) - \rho_{i-1}^h(y))}{h}\xi(t,y)dtdy.\end{aligned}$$

Indeed

$$\begin{aligned}|\sum_1^{\frac{T}{h}} \int_{t_{i-1}}^{t_i} \int_\Omega \frac{(\rho_i^h(y) - \rho_{i-1}^h(y))}{h}(\xi(t,y) - \xi(t_i,y))dtdy| &= |\sum_1^{\frac{T}{h}} \int_{t_{i-1}}^{t_i} \int_\Omega \frac{(\rho_i^h(y) - \rho_{i-1}^h(y))}{h}\partial_t\xi(t,y)(t_i - t)dtdy| + o(h) \\ &= ||\partial_t \xi||_{L^\infty([0,T]\times\Omega)} \lim_{\varepsilon\to 0} \delta(\varepsilon, h) + o(h).\end{aligned}$$

We therefore consider

$$\begin{aligned}\sum_1^{\frac{T}{h}} \int_{t_{i-1}}^{t_i} \int_\Omega \frac{(\rho_i^h(y) - \rho_{i-1}^h(y))}{h}\xi(t,y)dtdy &= \lim_{\varepsilon\to 0} \sum_{i=1}^{\frac{T}{h}} \int_{t_{i-1}}^{t_i} \int_\Omega \xi(t,y)\frac{(\rho_i^{\varepsilon h}(y) - \rho_{i-1}^{\varepsilon h}(y))}{h}dy \\ &= \lim_{\varepsilon\to 0} \sum_{i=1}^{\frac{T}{h}} \int_{t_{i-1}}^{t_i} \int_\Omega \xi(t,y)\frac{(\rho_i^{\varepsilon h}(y) - \rho_{i-1}^h(y))}{h}dy + \sum_{i=1}^{\frac{T}{h}} o_\varepsilon(1) \\ &= \lim_{\varepsilon\to 0} \sum_{i=1}^{\frac{T}{h}} \int_{t_{i-1}}^{t_i} \int_\Omega \frac{(\xi(t,y) - \xi(t, S_i^{\varepsilon h}(y))}{h}\rho_i^{\varepsilon h}(y)dy \\ &= \lim_{\varepsilon\to 0} \sum_{i=1}^{\frac{T}{h}} \int_{t_{i-1}}^{t_i} \int_\Omega \int_0^1 \nabla\xi(t, y + s(S_i^{\varepsilon h}(y) - y)) \cdot \frac{(S_i^{\varepsilon h}(y) - y)}{h}\rho_i^{\varepsilon h}(y)dyds.\end{aligned}$$

Then since $S_i^{\varepsilon h}(y) - y = h\nabla c^*(\nabla(F'(\rho_i^{\varepsilon h}(y)))) + \varepsilon\nabla(F'(\rho_i^{\varepsilon h}(y)))$ we can write

$$\begin{aligned}\sum_1^{\frac{T}{h}} \int_{t_{i-1}}^{t_i} \int_\Omega \frac{(\rho_i^h(y) - \rho_{i-1}^h(y))}{h}\xi(t,y)dtdy &\leq \lim_{\varepsilon\to 0} \int_0^1 \sum_{i=1}^{\frac{T}{h}} \int_{t_{i-1}}^{t_i} \int_\Omega |\nabla\xi(t, y + s(S_i^{\varepsilon h}(y) - y))|\rho_i^{\varepsilon h}(y)dyds \\ &\quad + \varepsilon||\nabla\xi||_{L^\infty([0,T]\times\Omega)}||\nabla\rho^{\varepsilon h}||_{L^1([0,T]\times\Omega)} \\ &\leq C||\nabla\xi||_{L^1([0,T]\times\Omega)}\end{aligned}$$

where $C$ does not depend on $h$. Moreover this integral may also be defined for any $\xi \in BV_2(\Omega)$ since for $h$ fixed, the piecewise constant function $\partial_h \rho^h = \dfrac{\rho_i^h - \rho_{i-1}^h}{h}$ on $[(i-1)h, ih]$ belongs to $L^\infty([0,T]\times\Omega)$. Then,

$$\sup_{t\in[0,T]} \int_\Omega \partial_h \rho^h \xi dy \leq C \quad \forall \xi \in BV_2(\Omega)$$

and the sequence $\partial_h \rho^h$ is bounded in $L^1([0,T], BV_2(\Omega))'$. This implies that $\partial_t \overline{\rho}$ is the w$^*$ limit of $\partial_h \rho^h$ and then belong to the dual of $L^1([0,T], BV_2(\Omega))$.

In fact, let us define the distribution $z^h$ by

$$< z^h; \zeta >= \int_0^T \int_\Omega \int_0^1 \zeta(t, y + sh\nabla c^*(\nabla(F'(\rho^h(t,y)))\nabla c^*(\nabla F'(\rho^h(t,y))\rho^h(t,y) ds dy dt.$$

The previous computation proves that $z^h$ is a bounded sequence in $L^\infty([0,T]\times\Omega)$ and that, for $h$ fixed, $\mathrm{div} z^h = \partial_h \rho^h$ in $L^1([0,T]\times\Omega)$.

In another hand, we have equality (14.48). Indeed,

$$\begin{aligned}
\int_0^T \int_\Omega \partial_t \overline{\rho}(t,y)\rho(t,y) dt dy &= \lim_{\delta\to 0}\lim_{h\to 0} \int_\delta^{T-\delta} \int_\Omega \partial_h \rho^h(t,y)\rho(t,y) dt dy \\
&= \lim_{\delta\to 0}\lim_{h\to 0} \left( \sum_2^{\frac{T}{h}-1} \int_{t_{i-1}}^{t_i} \int_\Omega \frac{(\rho_i^h(y) - \rho_{i-1}^h(y))}{h} \rho(t,y) dt dy \right. \\
&\quad \left. + \int_\delta^{t_1} \int_\Omega \frac{(\rho_1^h(y) - \rho_0^h(y))}{h} \rho(t,y) dt dy + \int_{\frac{T}{h}-1}^{t\frac{T}{h}-\delta} \int_\Omega \frac{(\rho_i^h(y) - \rho_{i-1}^h(y))}{h} \rho(t,y) dt dy \right) \\
&= -\lim_{\delta\to 0}\lim_{h\to 0} \int_\delta^{T-\delta} \int_\Omega \frac{(\rho(t-h,y) - \rho(t,y))}{h} \rho^h(t,y) dt dy \\
&= -\lim_{\delta\to 0}\lim_{h\to 0} \int_\delta^{T-\delta} \int_\Omega \frac{1}{h} \int_{t-h}^t \Theta(s,y) ds \rho^h(t,y) dt dy \\
&= -\int_0^T \int_\Omega \Theta(t,y)\overline{\rho}(t,y) dt dy.
\end{aligned}$$

Indeed, $\rho^h$ converges strongly in $L^1([0,T]\times\Omega)$ and $\dfrac{1}{h}\displaystyle\int_{t-h}^t \Theta(s,y) ds$ is bounded in $L^\infty([0,T]\times\Omega)$ — since $\Theta$ is bounded in $L^\infty([0,T]\times\Omega)$ — and converges toward $\Theta$.

(ii) First of all, $z^h$ is a bounded sequence in $L^\infty([0,T]\times\Omega)$ and then, up to a subsequence, converges in the sense of distributions toward a distribution $z$ which is equal to $\overline{A}$ in $L^\infty([0,T]\times\Omega)$.

Passing to the limit in the discrete equation (14.41), we obtain the limiting equation in the sense of distributions.

This point is an adaptation of the argument of [10].

We define as in the introduction for any $\rho \in L^1([0,T], BV_2(\Omega))$ and any test function $\zeta$

$$\begin{aligned}
&\int_0^T < (z, D\rho); \zeta > dt \\
&\qquad = - \int_0^T < \partial_t \overline{\rho}(t,y); \rho(t,y)\zeta(y) >_{L^1([0,T], BV_2(\Omega))', L^1([0,T], BV_2(\Omega))} dt - \int_0^T \int_\Omega \mathrm{div} z \rho(t,y)\zeta(y) dy dt.
\end{aligned}$$

As we noticed, $F'(\overline{\rho})$ does not belong to $L^1([0,T], BV_2(\Omega))$, so we need to introduce a regularization. In this step, we follow the proposition of [10].

Let the test function $\xi(t,y) = \eta(t)\zeta(y) \geq 0$ and introduce $F^\tau(t,y) = \dfrac{1}{\tau}\displaystyle\int_{t-\tau}^t \eta(s) F'(\overline{\rho}(s,y)) ds$. For $\tau$ fixed, small enough such that $\mathrm{spt}\eta \subset [\tau, T-\tau]$ , since $F'(\overline{\rho})$ is integrated in time, $F^\tau(t,\cdot)$ belongs to $C^0([0,T], BV_2(\Omega))$ (see [7]) and then belong to $L^1([0,T], BV_2(\Omega))$.

Let us now precise the argument of [10] by the following computation. First of all, equation (14.48) implies

$$\int_0^T \int_\Omega F(\overline{\rho}(t,y))\partial_t \xi(t,y)dtdy = \lim_{\tau\to 0}\int_0^T \int_\Omega \frac{\eta(t-\tau)-\eta(t)}{-\tau}F(\overline{\rho}(t,y)\zeta(y)dydt$$

$$= \lim_{\tau\to 0}\int_0^T \int_\Omega \frac{F(\overline{\rho}(t+\tau))-F(\overline{\rho}(t))}{-\tau}\eta(t)\zeta(y)dydt$$

and then since $F$ is a convex function

$$\int_0^T \int_\Omega \frac{F(\overline{\rho}(t+\tau))-F(\overline{\rho}(t))}{-\tau}\eta(t)\zeta(y)dydt \leq -\int_0^T \int_\Omega F'(\overline{\rho}(t,y))\frac{\overline{\rho}(t+\tau,y)-\overline{\rho}(t,y)}{\tau}\eta(t)\zeta(y)dtdy$$

$$= \int_0^T \int_\Omega \overline{\rho}(t,y)\zeta(y)\frac{F'(\overline{\rho}(t-\tau,y))\eta(t-\tau)-F'(\overline{\rho}(t,y))\eta(t)}{-\tau}dydt.$$

Since $\frac{F'(\overline{\rho}(t-\tau,y))\eta(t-\tau)-F'(\overline{\rho}(t,y))\eta(t)}{-\tau} \in L^1_w([0,T],BV_2(\Omega))\cap L^\infty([0,T]\times\Omega)$ is the time derivative of $F^\tau$ as defined in [10], i.e.

$$F^\tau(t,y) = \int_0^t \frac{F'(\overline{\rho}(s-\tau,y))\eta(s-\tau)-F'(\overline{\rho}(s,y))\eta(s)}{-\tau}ds$$

and since $F^\tau \in L^1([0,T],BV_2(\Omega))$ we obtain

$$\int_0^T \int_\Omega \overline{\rho}(t,y)\zeta(y)\partial_t F^\tau(t,y)dydt = -\int_0^T < \partial_t\overline{\rho},\zeta(y)F^\tau(t,y) > dt$$

$$= \int_0^T \int_\Omega F^\tau(t,y)z\cdot\nabla\zeta(y)dydt + \int_0^T \int_\Omega \zeta(y)(z;D(F^\tau(t,y)))dtdy$$

by definition of the distribution $(z,DF^\tau)$.

And then

$$\int_0^T \int_\Omega F(\overline{\rho}(t,y))\partial_t\xi(t,y)dtdy \leq \lim_{\tau\to 0}\int_0^T \int_\Omega F^\tau(t,y)z\cdot\nabla\zeta(y)dydt$$

$$+\lim_{\tau\to 0}\frac{1}{\tau}\int_0^T \int_\Omega \int_{t-\tau}^t \zeta(y)\eta(s)z\cdot D^{ac}(F'(\overline{\rho}(s,y))dtdyds$$

$$+\lim_{\tau\to 0}\frac{1}{\tau}\int_0^T \int_\Omega \zeta(y)(z;\int_{t-\tau}^t \eta(s)D^s(F'(\overline{\rho}(s,y)))dtdyds$$

First by (14.12) we write

$$\frac{1}{\tau}\int_0^T \int_\Omega \zeta(y)(z;\int_{t-\tau}^t \eta(s)D^s(F'(\overline{\rho}(s,y)))dtdyds \leq M\frac{1}{\tau}\int_{-\tau}^0 \int_0^T \int_\Omega \zeta(y)\eta(t+s)|D^s(F'(\overline{\rho}(t+s,y))|dtdyds$$

and then we use that for any $0\leq p\in L^1([0,T])$, $\lim_{\tau\to 0}\frac{1}{\tau}\int_{-\tau}^0 p(t+s)ds = p(t)$ in the sense of measure.

This yields the result since $z=\overline{A}$ which gives

$$\int_0^T \int_\Omega F(\overline{\rho}(t,y))\partial_t\xi(t,y)dtdy \leq \int_0^T \int_\Omega \eta(t)F'(\overline{\rho}(t,y))\overline{A}\cdot\nabla\zeta(y)dydt$$

$$+\int_0^T \int_\Omega \zeta(y)\eta(t)\overline{A}\cdot D^{ac}(F'(\overline{\rho}(t,y))dtdy$$

$$+||\overline{A}||_{L^\infty([0,T]\times\Omega)}\int_0^T \int_\Omega \eta(t)\zeta(y)|D^s(F'(\overline{\rho}(t,y))|dtdy$$

We denote $< T^s; \xi >$ the singular term $||\overline{A}||_{L^\infty([0,T]\times\Omega)} \int_0^T \int_\Omega \xi(t,y)|D^s(F'(\overline{\rho}(t,y))|dtdy$.

□

### 14.4.4 Identification of the limiting equation : a monotonicity argument

By a monotonicity argument, we achieve the Proof of Theorem (14.1.8) by proving the Proposition *Proposition* 14.4.6. The limiting measure $\overline{\rho}$ of the discrete in time measure $\rho^h$ satisfying equation (14.41) is solution in the sense of distributions to the following equation

$$\partial_t \overline{\rho} = \mathrm{div}(\overline{\rho}\nabla c^*(D(F'(\overline{\rho})))). \tag{14.50}$$

*Démonstration.* We want to prove that $\overline{A} = \overline{\rho}\nabla c^*(D(F'(\overline{\rho})))$. For that purpose, we want to use the Minty-Browder's argument which means that we have to prove that for any $z \in \mathbf{R}^d$,

$$(\overline{A} - A(\overline{\rho}, z))(D^{ac}(F'(\overline{\rho})) - z) \geq 0$$

and in fact, we want to prove for any test function $\xi(t,x) \geq 0$ and $\zeta(t,x) \in C^\infty(\mathbf{R}^d, \mathbf{R}^d)$

$$\int \xi(\overline{A} - A(\overline{\rho}, \zeta))(D^{ac}(F'(\overline{\rho})) - \zeta) \geq 0.$$

Indeed, this yields the expected equality since by taking $\zeta = D^{ac}(F'(\overline{\rho})) + \delta w$, we obtain

$$(\overline{A} - A(\overline{\rho}; D^{ac}(F'(\overline{\rho})) + \delta w))\delta w \geq 0$$

and then by passing successively to the limit $0 > \delta \to 0$ and $0 < \delta \to 0$, we obtain the equality $\overline{A} = A(\overline{\rho}, D^{ac}(F'(\overline{\rho})))$.

To prove this inequality, we want to pass to the limit when $\varepsilon$ and $h$ go to zero in the following inequality that comes from the convexity of $c^*$

$$\Lambda^{\varepsilon h} = \int_0^T \int_\Omega \rho^{\varepsilon h}(t,y)(\nabla c^*(\nabla(F'(\rho^{\varepsilon h}(t,y)))) - \nabla c^*(\zeta(t,y))) \cdot (\nabla(F'(\rho^{\varepsilon h}(t,y))) - \zeta(t,y))dydt \geq 0.$$

Let us develop this quantity as follows

$$\begin{aligned}
\Lambda^{\varepsilon h} &= \int_0^T\int_\Omega \xi(t,y)\rho^{\varepsilon h}(t,y)\nabla(F'(\rho^{\varepsilon h}(t,y)))\nabla c^*(\nabla(F'(\rho^{\varepsilon h}(t,y))))dydt \\
&+ \int_0^T \int_\Omega \xi(t,y)\rho^{\varepsilon h}(t,y)\zeta(t,y)) \cdot \nabla c^*(\zeta(t,y))dydt \\
&- \int_0^T\int_\Omega \xi(t,y)\rho^{\varepsilon h}(t,y)\nabla c^*(\zeta(t,y)) \cdot \nabla(F'(\rho^{\varepsilon h}(t,y)))dydt \\
&- \int_0^T \int_\Omega \xi(t,y)\rho^{\varepsilon h}(t,y)\nabla c^*(\nabla(F'(\rho^{\varepsilon h}(t,y)))) \cdot \zeta(t,y)dydt
\end{aligned}$$

**Term 1**

Proposition (14.4.3) gives us at the limit

$$\begin{aligned}
&\liminf_{\varepsilon h\to 0} \int_0^T\int_\Omega \xi(t,y)\rho^{\varepsilon h}(t,y)\nabla(F'(\rho^{\varepsilon h}(t,y)))\nabla c^*(\nabla(F'(\rho^{\varepsilon h}(t,y))))dydt \\
&\leq -\int_0^T \int_\Omega \xi(t,y)\partial_t(F(\overline{\rho}(t,y)))dtdy - \int_0^T \int_\Omega \nabla\xi(t,x)F'(\overline{\rho}(t,y))\overline{A} \\
&= \int_0^T\int_\Omega \xi(t,y)\overline{A}(t,y) \cdot \nabla(F'(\overline{\rho}(t,y))dydt + (T^s;\xi)
\end{aligned}$$

by (14.49).

**Term 2**

Since $\rho^{\varepsilon h}$ converges in $w^*L^\infty(\Omega)$ towards $\overline{\rho}$, we have

$$\int_0^T \int_\Omega \xi(t,x)\rho^{\varepsilon h}(t,y)\zeta(t,y)\cdot\nabla c^*(\zeta(t,y))dydt$$

$$\longrightarrow \int_0^T \int_\Omega \xi(t,x)\overline{\rho}(t,y)\zeta(t,y)\cdot\nabla c^*(\zeta(t,y))dydt$$

when $\varepsilon < h \to 0$.

**Term 3** Using Theorem 14.1.7 of De Cicco, Fusco and Verde [23] on the $L^1$ semicontinuity of functionals $J[\rho, D\rho]$ on $BV$ which are convex with respect to the gradient variable and continuous with respect to $\rho = F'(\rho^{\varepsilon h})$, we obtain

$$\liminf_{\varepsilon\to 0} \int_0^T\int_\Omega \xi(t,y)\nabla c^*(\zeta(t,y))\cdot\nabla(F'(\rho^{\varepsilon h}(t,y)))\ (F^*)'(F'(\rho^{\varepsilon h}(t,y)))dydt$$

$$\geq \liminf_{h\to 0} \int_0^T \int_\Omega \xi(t,y)\nabla c^*(\zeta(t,y))\cdot D(F'(\rho^h(t,y)))\rho^h(t,y)dydt$$

$$\geq \int_0^T \int_\Omega \xi(t,y)\nabla c^*(\zeta(t,y))\cdot D(F'(\overline{\rho}(t,y))\overline{\rho}(t,y)dydt.$$

In fact, since $J[\rho, D\rho]$ depends linearly on $D\rho$ in this case, we have that both $\liminf$s can be replace by limits which converge, and the inequality above becomes an equality.

**Term 4**

$$\int_0^T \int_\Omega \xi(t,x)\rho^{\varepsilon h}(t,y)\nabla c^*(\rho^{\varepsilon h}(t,y))\cdot\zeta(t,y))dydt \to \int_0^T \int_\Omega \xi(t,x)\overline{A}(t,y)\cdot\zeta(t,y))dydt$$

when $\varepsilon < h \to 0$.

Then the inequality $\Lambda^{\varepsilon h} \geq 0$ becomes

$$(\overline{A}(\nabla(F'(\overline{\rho})) - \zeta) - \overline{\rho}\nabla c^*(\zeta)(D(F'(\overline{\rho})) - \zeta) + T^s; \xi) \geq 0.$$

By Corollary 1 p.53 of [27], we know that $(\overline{A}(\nabla(F'(\overline{\rho}) - \zeta) - \overline{\rho}\nabla c^*(\zeta)(D^{ac}(F'(\overline{\rho})) - \zeta) + T^s$ is a Radon measure and then by the Lebesgue decomposition of Radon measure, we can conclude that it's absolute continuous part is positive which means that

$$(\overline{A} - A(\overline{\rho}, \zeta))(D^{ac}(F'(\overline{\rho})) - \zeta) \geq 0.$$

for all test function $\zeta$ which concludes the proof.

□

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