QUASINEUTRAL LIMIT FOR THE RELATIVISTIC VLASOV-MAXWELL SYSTEM

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1. Introduction

1.1. The Vlasov-Maxwell system. The modelling of plasmas, i.e. of rarefied ionized gases, leads naturally to the Vlasov-Maxwell system :

$$
\begin{aligned}
&\partial_t g + \frac{c\xi}{\sqrt{\xi^2+c^2}}\cdot\nabla_x g + \frac{Ze}{m_i}(E + \frac{c\xi}{\sqrt{\xi^2+c^2}}\wedge B)\cdot\nabla_\xi g = 0,\\
&\partial_t f + \frac{c\xi}{\sqrt{\xi^2+c^2}}\cdot\nabla_x f - \frac{e}{m_e}(E + \frac{c\xi}{\sqrt{\xi^2+c^2}}\wedge B)\cdot\nabla_\xi f = 0,\\
&\mathrm{div}_x B = 0, \quad \mathrm{div}_x E = \frac{e}{\epsilon_0}\int (Zg - f)d\xi,\\
&\partial_t B + \mathrm{curl}_x E = 0, \quad -\frac{1}{c^2}\partial_t E + \mathrm{curl}_x B = e\mu_0 \int \frac{c\xi}{\sqrt{\xi^2+c^2}}(Zg-f)d\xi,
\end{aligned}
\tag{1.1}
$$

where c denotes the light speed, $-e$ and Ze the respective charges of the electron and ion, μ_0 and $\epsilon_0 = 1/(c^2\mu_0)$ the permeability and the permittivity of vacuum. In those equations, f (resp. g) is a positive function which represents the probability for an electron (resp. for an ion) to be at time t at the point $x \in \Omega \subset \mathbf{R}^3$ with the momentum $\xi \in \mathbf{R}^3$.

The first two equations, the so-called Vlasov equations, express that charged particles evolve under the electromagnetic force :

$$
\frac{dX}{dt} = V,\qquad \frac{dP}{dt} = q(E + V\wedge B),
$$

where X and P are the position and the momentum of the particle at time t, and $V = cP/\sqrt{(cm)^2+P^2}$ its relativistic velocity.

The Maxwell equations take into account the wave propagation of the electromagnetic field. They can be possibly replaced by the Poisson equation in electrostatic regime, but not in relativistic regimes when the particles velocities are close to the light speed.

The understanding of this model is still incomplete : as the Vlasov-Maxwell system is both hyperbolic and nonlinear, we expect that singularities arise in finite time, and then propagate with finite speed. The mathematical study of such a system requires then a notion of solution with low regularity. The notion of weak solution we will use in this paper is not completely satisfactory since it does not ensure neither the uniqueness of the solution nor the fundamental physical properties.

Proposition 1.1. *Let (f, g) be a smooth solution of (1.1) on $[0, T]$. Then*

- *the global charge and the global energy are conserved on $[0, T]$*

(1.2)
$$\iint f(t,x,\xi)d\xi dx = Z\iint g(t,x,\xi)d\xi dx = \iint f_{in}(x,\xi)d\xi dx,$$
$$\iint c\sqrt{c^2+\xi^2}(m_e f + m_i g)(t,x,\xi)d\xi dx + \frac{1}{2}\int(\epsilon_0|E|^2 + \frac{1}{\mu_0}|B|^2)(t,x)dx$$
$$= \iint c\sqrt{c^2+\xi^2}(m_e f_{in} + m_i g_{in})(x,\xi)d\xi dx + \frac{1}{2}\int(\epsilon_0|E_{in}|^2 + \frac{1}{\mu_0}|B_{in}|^2)(x)dx$$

- *the following local conservation laws hold on $[0, T] \times \Omega$*

(1.3)
$$\begin{aligned}
&\partial_t \rho_e + div_x \hat{j}_e = \partial_t \rho_i + div_x \hat{j}_i = 0\\
&\partial_t j_e + div_x \int f\frac{c\xi\otimes\xi}{\sqrt{c^2+\xi^2}}d\xi + \frac{e}{m_e}\rho_e E + \frac{e}{m_e}\hat{j}_e \wedge B = 0\\
&\partial_t j_i + div_x \int g\frac{c\xi\otimes\xi}{\sqrt{c^2+\xi^2}}d\xi - \frac{Ze}{m_i}\rho_i E - \frac{Ze}{m_i}\hat{j}_i \wedge B = 0
\end{aligned}$$

with the notations $\rho_e = \int f d\xi$, $\rho_i = \int g d\xi$ for the densities of charges, $j_e = \int f\xi d\xi$, $j_i = \int f\xi d\xi$ for the momentum densities, and $\hat{j}_e = \int f\frac{c\xi}{\sqrt{c^2+\xi^2}}d\xi$, $\hat{j}_i = \int g\frac{c\xi}{\sqrt{c^2+\xi^2}}d\xi$ for the current densities.

1.2. **Scaling for the quasineutral regime.** In some regimes, we can actually obtain a correct description of the plasma behaviour through simplified models. Here we are interested in the derivation of such a model for a quasineutral plasma. This means that we consider the plasma on a space scale which is very large compared with the Debye length, which allows to neglect the plasma oscillations (created by the electric field). As a matter of fact, in order to isolate the phenomena which are directly linked with the quasineutrality constraint, we will assume in all the sequel that ions have infinite mass, and consider that the ion charge density $Z\rho_i = n(t,x)$ is fixed. To be completely rigorous, we have now to precise all the relations between the respective sizes of the various parameters.

The various scales arising in the equations are those listed below

- T the observation time scale;
- $T_o = L/c$ a characteristic time scale for the light propagation (where L is the observation space scale);
- $T_p = \sqrt{m\epsilon_0/ne^2}$ the reciprocal plasma frequency;

- $T_c = v^* \sqrt{ne^2\mu_0/m}$ the reciprocal gyrokinetic frequency (where $v^* = L/T$ is the characteristic bulk velocity).

The quasineutrality assumption reads

$$\frac{T_p}{T} = \epsilon << 1.$$

We assume moreover that the bulk motion is non-relativistic, i.e.

$$\frac{T_o}{T} = \gamma << 1.$$

Then we introduce a third nondimensional number

$$\alpha = \frac{T_c}{T} = \frac{\gamma}{\epsilon},$$

which measures whether or not the magnetic field is a relativistic effect.

Finally we come down to the study of the following nondimensional system

$$\begin{aligned}
&\partial_t f + \frac{\xi}{\sqrt{1+\gamma^2\xi^2}} \cdot \nabla_x f - (E + \alpha \frac{\xi}{\sqrt{1+\gamma^2\xi^2}} \wedge B) \cdot \nabla_\xi f = 0, \\
&\mathrm{div}_x B = 0, \quad \epsilon^2 \mathrm{div}_x E = n - \int f d\xi, \qquad (1.4) \\
&\alpha \partial_t B + \mathrm{curl}_x E = 0, \quad -\alpha\epsilon^2 \partial_t E + \mathrm{curl}_x B = -\alpha \int \frac{\xi}{\sqrt{1+\gamma^2\xi^2}} f d\xi,
\end{aligned}$$

for small values of the parameters ϵ and γ.

From a mathematical point of view, in order to understand the main features of the plasma behaviour for small values of ϵ, we are going to study the limit $\epsilon \to 0$: in this limiting process, some phenomena (such as oscillations) will be neglected but they should be only corrective terms.

In such a limit, we expect that the negative charge ρ_e equals everywhere the positive charge n (which justifies the terminolgy of quasineutral limit), with spatial oscillations on a characteristic length of order ϵ.

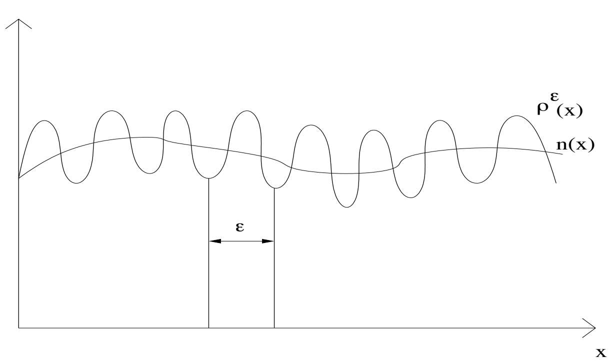

figure 1

The oscillations on the electron density generate of course oscillations on the electric and magnetic fields through the Maxwell equations.

The problem is to understand how these oscillations interfer, and in particular if the coupling by the nonlinear terms produces non-oscillating terms in the transport equation (which corresponds to an energy loss or an energy winning on the non-oscillating part due to turbulent effects).

In this paper, we will actually consider only a small part of this problem, for technical reasons and especially because the asymptotic behaviour of the system depends strongly on many conditions. We start by giving our two main assumptions, and then try to explain their physical meaning as well as the phenomena we expect to be controlled or to disappear under such assumptions.

(H1) the ion density of charge n is homogeneous and constant, take for instance $n = 1$.
This is of course a very particular case which could be extended a bit, assuming only that n depends on t and varies slowly with respect to x (typically on lengthes of order ϵ^{-1}).
If this assumption is not verified, the plasma will have a much more turbulent behaviour. Indeed, the singular perturbation

$$\mathrm{L}_n : (u, b, e) \mapsto \left(e, \frac{1}{\alpha} \,\mathrm{curl}_x e, -nu - \frac{1}{\alpha} \,\mathrm{curl}_x b \right)$$

does not preserve the Sobolev norms if $\nabla_x n \neq 0$. Then we expect that the regularity and compactness are lost instantaneously under this linear perturbation. Moreover, the kernel of such an operator L_n is much smaller if n is not homogeneous, which means that the averaged fields (i.e. the projections of the fields on $\mathrm{Ker}(\mathrm{L}_n)$) contain few informations (see [6] for an analogous study on rotating fluids).

(H2) the velocity profile is uniformly stable. More precisely, we assume that the initial data is almost monokinetic

$$\iint |\xi - u^0(x)|^2 f^0(x, \xi) d\xi dx \to 0.$$

We could also weaken this assumption, and suppose that the initial density is close to any global thermodynamic equilibrium (for instance a Maxwellian with density n and temperature 1). See [7] for a precise description of these equilibria.
If this assumption is not satisfied, instabilities will occur at the kinetic level (typically double-humped instabilities). These instabilities, which are still bad understood, cancel immediately the averaging process that we want to describe in this paper (see [8]).

1.3. **Description of the asymptotics.** Under assumptions $(H1)$ and $(H2)$, we are able to give a precise description of all oscillations, and of the possible coupling through the quadratic term of the transport equation (provided that there is no boundary).

The study of the asymptotics follows the following scheme. As the data are almost monokinetic, the density will remain close to a Dirac mass in v. We can prove that the moments behave as the solutions of

$$
\begin{aligned}
&\rho = 1 - \epsilon \mathrm{div}_x(\epsilon E), \quad \mathrm{div}_x B = 0 \\
&\partial_t j + \mathrm{div}_x \frac{1}{\rho}(j \otimes j) + \frac{1}{\epsilon}(1 - \epsilon \mathrm{div}_x(\epsilon E))\epsilon E + \alpha j \wedge B = 0 \\
&\partial_t B + \frac{1}{\alpha\epsilon} \, \mathrm{curl}_x(\epsilon E) = 0 \\
&\partial_t(\epsilon E) - \frac{1}{\alpha\epsilon} \, \mathrm{curl}_x B = \frac{1}{\epsilon} j,
\end{aligned}
\tag{1.5}
$$

since the relativistic correction vanish in the limit.

The second part of the job is based on the study of the singular perturbation L

$$
\mathrm{L} : (u, b, e) \mapsto \left(e, \frac{1}{\alpha} \, \mathrm{curl}_x e, -u - \frac{1}{\alpha} \, \mathrm{curl}_x b \right).
$$

As all coefficients are constant, we can work with the Fourier formulation, what allows to determine easily the eigenvalues and eigenvectors of L. The limiting triplet (u, e, b) belongs necessarily to Ker(L), so that

$$
u = -\frac{1}{\alpha} \, \mathrm{curl}_x b, \quad e = 0.
$$

Then we establish the limiting evolution equation by projecting the local conservation laws on the kernel of L. The difficulty consists in proving that there is no constructive resonance. The algebraic results we need here give conditions on the domains under which the convergence result holds.

The relative entropy method used in this paper, and whose principle is described in the first paragraph of Section 4, has already allowed to solve many problems concerning the gyrokinetic and quasineutral limits in plasma physics (see [1, 3, 7]). Some questions remained nevertheless still open, essentially because of a lack of understanding of the Vlasov-Maxwell system. In particular,

- the global conservation of energy and the local conservation of momentum are not ensured for weak solutions of (1.4), i.e. for the only solutions of (1.4) that are known to exist globally in time;
- the relativistic correction has to be taken into account in the initial model (1.4) except in electrostatic regimes, but it does not appear in the limiting system;
- the oscillations generated by the Maxwell equations have a much more complicated structure than the ones created by the coupling with the Poisson equation.

The various restrictions imposed in [1, 3, 7] concerning these three points are relaxed here, by analyzing precisely the analogy between the initial system (1.4) and the limiting system.

2. Main results

2.1. Weak solutions for the Vlasov-Maxwell system. We start by giving the mathematical framework of our study. We consider the spatial domain

$$\Omega = \frac{\mathbf{R}}{a_1\mathbf{Z}} \times \frac{\mathbf{R}}{a_2\mathbf{Z}} \times \frac{\mathbf{R}}{a_3\mathbf{Z}}$$

in order to avoid additional difficulties linked with the modelling of the boundary, in particular the appearance of boundary layers. We recall that the Cauchy problem (1.4) with initial data

$$f_{|t=0} = f_{in}, \quad E_{|t=0} = E_{in}, \quad B_{|t=0} = B_{in}, \tag{2.1}$$

satisfying the standard compatibility conditions

$$\epsilon^2 \mathrm{div}_x E_{in} = 1 - \int f_{in}(x,\xi)d\xi, \quad \mathrm{div}_x B_{in} = 0, \tag{2.2}$$

and the energy bound

$$\mathcal{E}_{in} = \frac{1}{\gamma^2}\iint \sqrt{1+\gamma^2\xi^2} f_{in} d\xi dx + \frac{1}{2}\int (\epsilon^2 |E_{in}|^2 + |B_{in}|^2)dx < +\infty, \tag{2.3}$$

is not known (nor expected) to have global strong solutions. Then we will work with a very weak notion of solution.

A *weak solution* of (1.4–2.1) is a triplet

$$\begin{aligned}(f,E,B) \in\ & L^\infty\big(\mathbf{R}^+, L^2(\Omega\times\mathbf{R}^3,\mathbf{R}^+) \times (L^2(\Omega))^2\big)\\ & \cap C\big(\mathbf{R}^+; w\text{-}L^2(\Omega\times\mathbf{R}^3)\times(w\text{-}L^2(\Omega))^2\big)\end{aligned}$$

which satisfies

$$\frac{1}{\gamma^2}\iint f(t)\sqrt{1+\gamma^2\xi^2}d\xi dx + \frac{1}{2}\int(\epsilon^2|E(t)|^2 + |B(t)|^2)dx \le \mathcal{E}_{in}\,, \quad \forall t>0\,, \tag{2.4}$$

and finally satisfies (1.4,2.1) in the sense of distributions.

The global existence of such weak solutions, as well as the local conservation of mass and the energy decay are established in [4]. Whether the local conservation of momentum holds in the sense of distributions on $\mathbf{R}_+^* \times \Omega$ is still unknown; this is one of the difficulties in rigorously deriving hydrodynamic models from the Vlasov-Maxwell system. The construction of [4] yields actually a solution which satisfies in addition a conservation law for the momentum and a global energy equality with defect measures coming from a possible lack of compactness of the sequence of approximating solutions.

Theorem 2.1. *For fixed $\epsilon,\alpha>0$, let $f_{in}^{\epsilon,\alpha}\in L^1\cap L^2(\Omega\times\mathbf{R}^3)$ be an ae. nonnegative function, and $E_{in}^{\epsilon,\alpha}, B_{in}^{\epsilon,\alpha}\in L^2(\Omega)$ be two vector fields satisfying the compatibility conditions (2.2) and the energy bound (2.3). Then there exists a weak solution $(f^{\epsilon,\alpha}, E^{\epsilon,\alpha}, B^{\epsilon,\alpha})$ to (1.4,2.1) which satisfies*

- *the local conservation of mass in the sense of distributions on* $\mathbf{R}_+^* \times \Omega$

$$\partial_t \int f^{\epsilon,\alpha} d\xi + div_x \int \frac{\xi}{\sqrt{1+\gamma^2\xi^2}} f^{\epsilon,\alpha} d\xi = 0\,, \tag{2.5}$$

- *the local conservation of momentum with symmetric nonnegative matrix-valued defect measures* $\mu_E, \mu_B \in L^\infty(\mathbf{R}^+, \mathcal{M}(\Omega))$*, a vector-valued defect measure* $\mu_{EB} \in L^\infty(\mathbf{R}^+, \mathcal{M}(\Omega))$*, and a defect measure* $m^{\epsilon,\alpha} \in L^\infty(\mathbf{R}^+, \mathcal{M}(\Omega \times S^2))$ *(coming from the approximation scheme of the Vlasov-Maxwell equations)*

$$\begin{aligned}
&\partial_t \left(\int \xi f^{\epsilon,\alpha}(t,x,\xi) d\xi + \gamma \int_{\sigma \in S^2} \sigma dm^{\epsilon,\alpha}(t,x,\sigma) \right) \\
&+ div_x \left(\int \frac{\xi \otimes \xi}{\sqrt{1+\gamma^2\xi^2}} f^{\epsilon,\alpha}(t,x,\xi) d\xi + \int_{\sigma \in S^2} \sigma \otimes \sigma dm^{\epsilon,\alpha}(t,x,\sigma) \right) \\
&+ nE^{\epsilon,\alpha} - div_x(\epsilon E^{\epsilon,\alpha} \otimes \epsilon E^{\epsilon,\alpha} - \frac{1}{2}|\epsilon E^{\epsilon,\alpha}|^2 Id) - div_x(\mu_E^{\epsilon,\alpha} - \frac{1}{2} tr(\mu_E^{\epsilon,\alpha}) Id) \\
&- div_x(B^{\epsilon,\alpha} \otimes B^{\epsilon,\alpha} - \frac{1}{2}|B^{\epsilon,\alpha}|^2 Id) - div_x(\mu_B^{\epsilon,\alpha} - tr(\mu_B^{\epsilon,\alpha}) Id) \\
&+ \epsilon\alpha \partial_t(\epsilon E^{\epsilon,\alpha} \wedge B^{\epsilon,\alpha}) + \epsilon\alpha \partial_t \mu_{EB}^{\epsilon,\alpha} = 0,
\end{aligned} \tag{2.6}$$

in the sense of distributions on $\mathbf{R}^+ \times \Omega$*,*

- *the global conservation of energy with the defect measures* $\mu_E^{\epsilon,\alpha}$*,* $\mu_B^{\epsilon,\alpha}$ *and* $m^{\epsilon,\alpha}$ *for all* $t > 0$

$$\begin{aligned}
&\frac{1}{\gamma^2} \iint \sqrt{1+\gamma^2\xi^2} f^{\epsilon,\alpha}(t,x,\xi) d\xi dx + \iint_{\sigma \in S^2} dm^{\epsilon,\alpha}(t,x,\sigma) \\
&+ \frac{1}{2} \int (\epsilon^2 |E^{\epsilon,\alpha}(t,x)|^2 + |B^{\epsilon,\alpha}(t,x)|^2) dx + \frac{1}{2} \int (tr(d\mu_E^{\epsilon,\alpha}) + tr(d\mu_B^{\epsilon,\alpha})) = \mathcal{E}_{in}^{\epsilon,\alpha}
\end{aligned} \tag{2.7}$$

as well as the inequality

$$\int d\mu_{EB}^{\epsilon,\alpha} g \le \frac{1}{2} \int g(tr(d\mu_E^{\epsilon,\alpha}) + tr(d\mu_B^{\epsilon,\alpha})), \tag{2.8}$$

for all positive test function $g(t,x)$*.*

2.2. **Strong convergence for well-prepared initial data.** In the case where the initial data are well prepared, which means that the initial velocity is essentially monokinetic, the initial fields converge strongly and the limiting fields belong to the kernel of the singular perturbation, we obtain the following strong convergence result.

Theorem 2.2.

- *Let* v_0, b_0 *be two divergence-free vector fields of* $H^s(\Omega)$ $(s > 2 + \frac{3}{2})$ *such that*

$$nv_0 + \frac{1}{\alpha}\ curl_x b_0 = 0,$$

for some homogeneous n. Then, there exist $T^* \in]0,+\infty]$ and a unique $(v,b) \in L^\infty_{loc}([0,T^*[, H^s(\Omega))$ solution of

$$\left\{ \begin{array}{r} \partial_t v + v\cdot\nabla v + \underline{e} + \alpha v\wedge b = 0 \\ \partial_t b + \frac{1}{\alpha}\ curl_x \underline{e} = 0 \\ nv + \frac{1}{\alpha}\ curl_x b = 0. \end{array} \right. \tag{2.9}$$

- *Let $(f_{in}^{\epsilon,\alpha})_{\epsilon,\alpha}$ be a family of nonnegative functions of $L^1\cap L^2(\Omega\times\mathbf{R}^3)$, and $(E_{in}^{\epsilon,\alpha})_{\epsilon,\alpha}$, $(B_{in}^{\epsilon,\alpha})_{\epsilon,\alpha}$ be two families of vector fields of $L^2(\Omega)$ satisfying the compatibility conditions (2.2) for the same constant n, as well as the convergence*

$$\iint \frac{(\xi-v_0)^2}{1+\sqrt{1+\gamma^2\xi^2}} f_{in}^{\epsilon,\alpha}(x,\xi)d\xi dx + \frac{1}{2}\int \epsilon^2 |E_{in}^{\epsilon,\alpha}(x)|^2 dx + \frac{1}{2}\int |B_{in}^{\epsilon,\alpha}(x)-b_0|^2 dx \to 0 \ as\ \epsilon\to 0.$$

Let $(f^{\epsilon,\alpha},E^{\epsilon,\alpha},B^{\epsilon,\alpha})$ be, for every $\epsilon>0,\alpha>0$, a solution of the scaled Vlasov-Maxwell system (1.4) with initial condition (2.1). Then, for all $T<T^*$, the current density converges weakly

$$\hat{j}^{\epsilon,\alpha} = \int f^{\epsilon,\alpha}\xi/\sqrt{1+\gamma^2\xi^2}d\xi \rightharpoonup nv \ in\ L^\infty([0,T],L^1(\Omega)),$$

the scaled electric field and the magnetic field converge strongly

$$\epsilon E^{\epsilon,\alpha} \to 0 \ and\ B^{\epsilon,\alpha}\to b \ in\ L^\infty([0,T],(L^2(\Omega))^2)$$

as $\epsilon\to 0, \alpha\to\bar\alpha < +\infty$, where $(v,b)\in L^\infty([0,T],H^s(\Omega))$ is the local strong solution of (2.9) with initial condition (v_0,b_0).

2.3. **Weak convergence for general initial data.** In the case where we do not assume that the limit of the initial fields belong to the kernel of the singular perturbation, we obtain a weak convergence result.

Theorem 2.3. *Let $(f^{\epsilon,\alpha},E^{\epsilon,\alpha},B^{\epsilon,\alpha})$ be, for every $\epsilon>0,\alpha>0$, a solution of the scaled Vlasov-Maxwell system (1.4) with initial condition (2.1) satisfying*

$$\iint \frac{(\xi-v_0)^2}{1+\sqrt{1+\gamma^2\xi^2}} f_{in}^{\epsilon,\alpha}(x,\xi)d\xi dx + \frac{1}{2}\int \epsilon^2 |E_{in}^{\epsilon,\alpha}(x)|^2 dx + \frac{1}{2}\int |B_{in}^{\epsilon,\alpha}(x)-b_0|^2 dx \to 0 \ as\ \epsilon\to 0$$

for some sufficiently smooth vector fields (v_0,e_0,b_0) such that $div_x b_o = 0$.

Then, for all $T<T^*$, the current density

$$\hat{j}^{\epsilon,\alpha} = \int f^{\epsilon,\alpha}\xi/\sqrt{1+\gamma^2\xi^2}d\xi \rightharpoonup nv \ in\ the\ sense\ of\ measures,$$

the scaled electric field and the magnetic field converge weakly

$$\epsilon E^{\epsilon,\alpha} \rightharpoonup 0 \ and\ B^{\epsilon,\alpha}\rightharpoonup b \ in\ L^\infty_{Loc}([0,T[,(L^2(\Omega))^2)$$

as $\epsilon \to 0, \alpha \to \bar{\alpha} < +\infty$, *where* $(v, b) \in L^\infty([0,T], H^s(\Omega))$ *is the local strong solution of (2.9) with initial condition*

$$\begin{cases} v_{in} = -\Delta_x(\alpha^2 Id - \Delta_x)^{-1} v_0 + \alpha \; curl_x (\alpha^2 Id - \Delta_x)^{-1} b_0 \\ b_{in} = -\alpha(\alpha^2 Id - \Delta_x)^{-1} \; curl_x v_0 + \alpha^2 (\alpha^2 Id - \Delta_x)^{-1} b_0 \end{cases}$$

which is the projection of (v_0, e_0, b_0) *on the kernel of the operator* L.

The proof of this theorem involves an oscillating quantity which describes the defect of strong convergence of the fields due to the more general initial data.

3. Refined theory for the Vlasov-Maxwell system

The aim of this section is to recall the main arguments used by DiPerna and Lions to prove the existence of renormalized solutions for the Vlasov-Maxwell system (1.4), and to show that their construction actually yields a solution which satisfies in addition a conservation law for the momentum (2.6) and a global energy equality (2.7) with defect measures. For the sake of simplicity, in this section, we will fix $\epsilon, \alpha > 0$ and drop the indices.

3.1. **A priori estimates.** Consider a sequence (f^n, E^n, B^n) of approximate solutions of (1.4,2.1), for instance

$$\partial_t f^n + \frac{\xi}{\sqrt{1+\gamma^2\xi^2}} \cdot \nabla_x f^n - (E^{n-1} + \alpha \frac{\xi}{\sqrt{1+\gamma^2\xi^2}} \wedge B^{n-1}) \cdot \nabla_\xi f^n = 0,$$

$$\mathrm{div}_x B^n = 0, \quad \epsilon^2 \mathrm{div}_x E^n = n - \int f^n d\xi,$$

$$\alpha \partial_t B^n + \mathrm{curl}_x E^n = 0, \quad -\alpha\epsilon^2 \partial_t E^n + \mathrm{curl}_x B^n = -\alpha \int \frac{\xi}{\sqrt{1+\gamma^2\xi^2}} f^n d\xi,$$

$$f^n_{|t=0} = f_{in}, \quad E^n_{|t=0} = E_{in} \text{ and } B^n_{|t=0} = B_{in}.$$

We have the following energy estimate

$$\sup_{t\in[0,T], n\in\mathbf{N}} \frac{1}{\gamma^2} \iint \sqrt{1+\gamma^2\xi^2} f^n(t,x,\xi) d\xi + \iint (\epsilon^2 |E^n|^2(t,x) + |B^n|^2(t,x)) dx < +\infty,$$

as well as the conservation of L^p norms for $p \in [1, \infty]$

$$\sup_{t\in\mathbf{R}^+, n\in\mathbf{N}} \iint |f^n|^p(t,x,\xi) dx d\xi < +\infty.$$

Then, up to extraction of a subsequence (still denoted by (f^n, E^n, B^n) for simplicity),

$$\begin{aligned} f^n &\rightharpoonup f \text{ in } w^* - L^\infty(\mathbf{R}^+, L^2(\Omega \times \mathbf{R}^3)), \\ \epsilon E^n &\rightharpoonup \epsilon E \text{ in } w^* - L^\infty(\mathbf{R}^+, L^2(\Omega)), \\ B^n &\rightharpoonup B \text{ in } w^* - L^\infty(\mathbf{R}^+, L^2(\Omega)). \end{aligned} \tag{3.1}$$

Of course, as the previous convergences hold only in weak spaces, we do not have

$$\|\epsilon E^n\|^2_{L^2(\Omega)} \to \|\epsilon E\|^2_{L^2(\Omega)}, \quad \|B^n\|^2_{L^2(\Omega)} \to \|B\|^2_{L^2(\Omega)},$$

but there exist matrix-valued defect measures μ_E, μ_B, a vector-valued defect measure μ_{EB} and a subsequence (E^n, B^n) such that, for all $\varphi \in C^0(\Omega)$ and for all $i, j \in \{1, 2, 3\}$

$$\begin{aligned}
\int \epsilon^2 E_i^n E_j^n \varphi(x) dx &\to \int \epsilon^2 E_i E_j \varphi(x) dx + \int \varphi d(\mu_E)_{ij},\\
\int B_i^n B_j^n \varphi(x) dx &\to \int B_i B_j \varphi(x) dx + \int \varphi d(\mu_B)_{ij},\\
\int \epsilon E^n \wedge B^n \varphi(x) dx &\to \int \epsilon E \wedge B \varphi(x) dx + \int \varphi d(\mu_{EB}).
\end{aligned} \tag{3.2}$$

It is easy to check that μ_E and μ_B are nonnegative symmetric matrices. Moreover, from the Cauchy-Schwarz inequality, we deduce that

$$\int d\mu_{EB} \leq \frac{1}{2} \int tr(d\mu_E) + tr(d\mu_B).$$

On the other hand, the sequence $(f^n|\xi|)$ is bounded in $L^\infty(\mathbf{R}^+, \mathcal{M}(\Omega \times \mathbf{R}^3))$. Thus the sequence

$$\nu_n = \int_0^\infty r f^n(t, x, r\sigma) r^2 dr$$

of push-forwards of f^n under the map $(t, x, \xi) \mapsto (t, x, \xi/|\xi|)$ is bounded in $L^\infty(\mathbf{R}^+, \mathcal{M}(\Omega \times S^2))$. Hence there exists a subsequence of (f^n) such that ν^n converges to ν in $L^\infty(\mathbf{R}^+, \mathcal{M}(\Omega \times S^2))$ weak-*. We next define the defect measure associated to the subsequence (f^n) by :

$$\iiint \psi(t, x, \sigma) dm(t, x, \sigma) = \iiint \psi(t, x, \sigma) d\nu(t, x, \sigma) - \iiint \psi(t, x, \frac{\xi}{|\xi|}) |\xi| f(t, x, \xi) dt dx d\xi$$

for every $\psi \in C_c^0(\mathbf{R}^+ \times \Omega \times S^2)$. It is easy to check that m is a positive measure.

Finally we are able to characterize the convergence defects of the approximation sequence through the various measures introduced above (for a more detailed review on these questions, we refer to [10]). This is the key argument which allows to obtain a conservation law for the momentum and a global energy equality, provided that we get a convenient formulation of these conservations for approximate solutions.

3.2. **The local conservation of momentum.** We have ssen that the following conservation law for the momentum holds formally :

$$\partial_t \int \xi f d\xi + \mathrm{div}_x \left(\int \frac{\xi \otimes \xi}{\sqrt{1 + \gamma^2 \xi^2}} f d\xi \right) + \rho E + \alpha \hat{j} \wedge B = 0.$$

In particular approximate solutions can be assumed sufficiently smooth to satisfy such a conservation. Nevertheless the weak compactness inferred from the a priori estimates does not allow to take limits in the quadratic terms ; the a priori bounds do not even allow to give sense to such terms since we have

$$\begin{aligned} &\rho, \hat{j} \in L^\infty([0,T], L^1 \cap L^{4/3}(\Omega)), \\ &E, B \in L^\infty([0,T], L^2(\Omega)). \end{aligned}$$

The following formal computations (performed on approximate solutions) provide an equivalent form for the momentum conservation, which seems to be more adapted for our job. From Maxwell's equations we deduce that

$$\begin{aligned} \rho E = &\ (n - \epsilon^2 \mathrm{div}_x E) E \\ \alpha \hat{j} \wedge B = &\ (\alpha \epsilon^2 \partial_t E - \ \mathrm{curl}_x B) \wedge B \end{aligned}$$

At this point, we use the identity

$$u \mathrm{div}_x u = \mathrm{div}_x (u \otimes u) + u \wedge \ \mathrm{curl}_x u - \nabla_x \frac{|u|^2}{2},$$

which implies in particular

$$\begin{aligned} (\epsilon^2 \mathrm{div}_x E) E = &\ \ \mathrm{div}_x \left((\epsilon E)^{\otimes 2} - \frac{1}{2} |\epsilon E|^2 Id \right) + \epsilon E \wedge \ \mathrm{curl}_x (\epsilon E) \\ = &\ \ \mathrm{div}_x \left((\epsilon E)^{\otimes 2} - \frac{1}{2} |\epsilon E|^2 Id \right) + \epsilon E \wedge (-\alpha \epsilon \partial_t B) \end{aligned}$$

and

$$\mathrm{curl}_x B \wedge B = \mathrm{div}_x \left(B^{\otimes 2} - \frac{1}{2} |B|^2 Id \right)$$

Replacing in the previous identities, we obtain

$$\begin{aligned} \rho E + \alpha \hat{j} \wedge B = &\ \ nE - \mathrm{div}_x \left((\epsilon E)^{\otimes 2} - \frac{1}{2} |\epsilon E|^2 Id \right) \\ &- \mathrm{div}_x \left(B^{\otimes 2} - \frac{1}{2} |B|^2 Id \right) + \epsilon \alpha \partial_t (\epsilon E \wedge B). \end{aligned}$$

Equipped with this identity, we can rewrite formally the local conservation of momentum

$$\begin{aligned} \partial_t \int \xi f d\xi + \mathrm{div}_x \left(\int \frac{\xi \otimes \xi}{\sqrt{1 + \gamma^2 \xi^2}} f d\xi \right) + nE - \mathrm{div}_x \left((\epsilon E)^{\otimes 2} - \frac{1}{2} |\epsilon E|^2 Id \right) \\ - \mathrm{div}_x \left(B^{\otimes 2} - \frac{1}{2} |B|^2 Id \right) + \epsilon \alpha \partial_t (\epsilon E \wedge B) = 0. \end{aligned} \tag{3.3}$$

3.3. **Passage to the limit.** Formally we have the local conservation of momentum (3.3) and the energy equality

$$\frac{1}{\gamma^2}\iint \frac{\xi^2}{\sqrt{1+\gamma^2\xi^2}} f d\xi dx + \frac{1}{2}\int (\epsilon^2|E|^2 + |B|^2)dx = \mathcal{E}_{in}$$

for all $t \in \mathbf{R}^+$, so that analogous statements hold rigorously for smooth approximate solutions. Taking limits in both equations and using the defect measures introduced in the previous paragraph, we get that the weak solutions constructed by DiPerna and Lions [4] satisfy (2.6) and (2.7).

4. The modulated energy method

4.1. **Description of the method.** The main results of this paper are obtained by a classical energy method. The principle is to modulate the energy of the system by test functions, and to obtain a stability inequality when these modulation functions are solutions of the limiting system.

The first step (paragraph 4.2) consists in computing the variation in time of the modulated energy defined as follows,

(4.1)

$$H_U^{\epsilon,\alpha}(t) = \iint \frac{|\xi - v|^2}{1+\sqrt{1+\gamma^2\xi^2}} f^{\epsilon,\alpha}(t,x,\xi) d\xi dx + \frac{1}{2}\int |\epsilon E^{\epsilon,\alpha} - e|^2 dx + \frac{1}{2}\int |B^{\epsilon,\alpha} - b|^2 dx.$$

for every test function $U = (v,e,b) \in C_c^\infty(\mathbf{R}^+ \times \Omega, (\mathbf{R}^3)^3)$ such that $\mathrm{div}_x b = 0$. Note that, in order to get a quantity which is uniformly bounded in the nonrelativistic limit, we do not work with the energy but with a Lyapunov functional which is computed from the energy and the global mass

$$\begin{aligned} H^{\epsilon,\alpha}(t) = &\ \mathcal{E}^{\epsilon,\alpha}(t) - \frac{1}{\gamma^2}\iint f^{\epsilon,\alpha}(t,x,\xi)d\xi \\ = &\ \iint \frac{\xi^2}{1+\sqrt{1+\gamma^2\xi^2}} f^{\epsilon,\alpha}(t,x,\xi)d\xi dx + \frac{1}{2}\int |\epsilon E^{\epsilon,\alpha} - e|^2 dx + \frac{1}{2}\int |B^{\epsilon,\alpha} - b|^2 dx. \end{aligned}$$

Moreover, as we expect the limiting flow to be monokinetic and nonrelativistic, we just introduce a slight perturbation in the kinetic energy.

The second step (paragraph 4.3) consists in establishing a bound on this quantity involving an acceleration operator that is small when (v,e,b) is a solution of

$$\begin{aligned} \partial_t v + v\cdot\nabla v + \frac{e}{\epsilon} + \alpha v \wedge b = 0 \\ \partial_t b + \frac{1}{\alpha\epsilon}\mathrm{curl}\ e = 0 \\ \partial_t e - \frac{1}{\alpha\epsilon}\mathrm{curl}\ b - n\frac{v}{\epsilon} + v \mathrm{div}\ e = 0 \\ \mathrm{div}\ b = 0. \end{aligned} \tag{4.2}$$

The last step in the convergence proof is then to study the asymptotic behaviour of the solutions of (4.2) as $\epsilon \to 0$ (depending on the size of α). For some particular data, the so-called well-prepared data, the singular perturbation has little effect on the system, no oscillatory behaviour occurs and we will easily prove a strong convergence result (section 5). For general initial data, we have to describe precisely the oscillations and to study their coupling (section 6) in order to characterize the asymptotic behaviour (section 7).

4.2. **Derivation of the modulated energy.**

Proposition 4.1. *The modulated energy $H_U^{\epsilon,\alpha}(t)$ satisfies*

$$
\begin{aligned}
H_U^{\epsilon,\alpha}(t) - H_U^{\epsilon,\alpha}(0) \quad & + \iint dm^{\epsilon,\alpha}(t,x,\sigma) + \frac{1}{2}\int tr(d\mu_E^{\epsilon,\alpha}) + tr(d\mu_B^{\epsilon,\alpha}) \\
& = \int_0^t \int D_x v : (h_U^{\epsilon,\alpha}(s,x)dx + d\nu^{\epsilon,\alpha}(s,x))ds \\
& + \int A^{\epsilon,\alpha}(U) \cdot \begin{pmatrix} \rho^{\epsilon,\alpha} v - \hat{j}^{\epsilon,\alpha} \\ b - B^{\epsilon,\alpha} \\ e - \epsilon E^{\epsilon,\alpha} \end{pmatrix} dxds + R^{\epsilon,\alpha}(t)
\end{aligned}
\tag{4.3}
$$

where the flux terms $h_U^{\epsilon,\alpha}$ and $\nu^{\epsilon,\alpha}$ are defined by

$$
\begin{aligned}
h_U^{\epsilon,\alpha} \quad = \quad & -\int f^{\epsilon,\alpha} \frac{(\xi - v)\otimes(\xi - v)}{\sqrt{1+\gamma^2\xi^2}} d\xi \\
& + [(\epsilon E^{\epsilon,\alpha} - e)\otimes(\epsilon E^{\epsilon,\alpha} - e) - \frac{1}{2}|\epsilon E^{\epsilon,\alpha} - e|^2 Id] \\
& + [(B^{\epsilon,\alpha} - b)\otimes(B^{\epsilon,\alpha} - b) - \frac{1}{2}|B^{\epsilon,\alpha} - b|^2 Id],
\end{aligned}
$$

$$
\begin{aligned}
d\nu^{\epsilon,\alpha} \quad = \quad & -\int_{\sigma\in S^2} \sigma\otimes\sigma dm^{\epsilon,\alpha} \\
& + [d\mu_E^{\epsilon,\alpha} - \frac{1}{2} tr(d\mu_E^{\epsilon,\alpha}) Id] \\
& + [d\mu_B^{\epsilon,\alpha} - \frac{1}{2} tr(d\mu_B^{\epsilon,\alpha}) Id]
\end{aligned}
$$

the acceleration term $A^{\epsilon,\alpha}(U)$ is given by

$$
A^{\epsilon,\alpha}(U) = \begin{pmatrix}
\partial_t v + v\cdot\nabla v + \frac{e}{\epsilon} + \alpha v\wedge b \\
\partial_t b + \frac{1}{\alpha\epsilon} curl\ e + (\alpha\epsilon\partial_t e - curl\ b)\wedge v \\
\partial_t e - \frac{1}{\alpha\epsilon} curl\ b - n\frac{v}{\epsilon} + v\, div\ e - (\alpha\epsilon\partial_t b + curl\ e)\wedge v
\end{pmatrix}
$$

and all the correction terms are grouped in a remainder $R^{\epsilon,\alpha}$ as follows

$$
\begin{aligned}
R^{\epsilon,\alpha}(t) =& -\left[\iint v\xi(\frac{2}{1+\sqrt{1+\gamma^2\xi^2}}-1)f^{\epsilon,\alpha}d\xi dx\right]_0^t + \left[\iint v^2(\frac{1}{1+\sqrt{1+\gamma^2\xi^2}}-\frac{1}{2})f^{\epsilon,\alpha}d\xi dx\right]_0^t \\
&+\int_0^t\iint D_xvf^{\epsilon,\alpha}v\otimes v(\frac{1}{\sqrt{1+\gamma^2\xi^2}}-1)d\xi dxds + \int_0^t\iint \xi(-1+\frac{1}{\sqrt{1+\gamma^2\xi^2}})f^{\epsilon,\alpha}\partial_tvd\xi dxds \\
&+\left[\int \alpha\epsilon v\cdot[(\epsilon E^{\epsilon,\alpha}-e)\wedge(B^{\epsilon,\alpha}-b)dx + d\mu_{EB}^{\epsilon,\alpha}]\right]_0^t \\
&-\int_0^t\int \alpha\epsilon\partial_t v\cdot[(\epsilon E^{\epsilon,\alpha}-e)\wedge(B^{\epsilon,\alpha}-b)dx + d\mu_{EB}^{\epsilon,\alpha}]\,dxds \\
&+\gamma\left[\iint v.\sigma dm^{\epsilon,\alpha}\right]_0^t - \gamma\int_0^t\iint \partial_t v.\sigma dm^{\epsilon,\alpha} - \int_0^t\iint D_xv:\sigma\otimes\sigma dm^{\epsilon,\alpha}.
\end{aligned}
$$

Proof. The previous identity is obtained by a direct computation of the time derivative of the modulated energy. We first rewrite

$$
\begin{aligned}
H_U^{\epsilon,\alpha}(t) &= \iint \frac{\xi^2}{1+\sqrt{1+\gamma^2\xi^2}}f^{\epsilon,\alpha}d\xi dx - \iint \frac{2v\xi}{1+\sqrt{1+\gamma^2\xi^2}}f^{\epsilon,\alpha}d\xi dx \\
&\quad + \iint \frac{v^2}{1+\sqrt{1+\gamma^2\xi^2}}f^{\epsilon,\alpha}d\xi dx + \frac{1}{2}\int(|\epsilon E^{\epsilon,\alpha}-e|^2+|B^{\epsilon,\alpha}-b|^2)dx \\
&= H^{\epsilon,\alpha}(t) - \iint \frac{2v\xi}{1+\sqrt{1+\gamma^2\xi^2}}f^{\epsilon,\alpha}d\xi dx \\
&\quad + \iint \frac{v^2}{1+\sqrt{1+\gamma^2\xi^2}}f^{\epsilon,\alpha}d\xi dx + \frac{1}{2}\int((e^2-2\epsilon E^{\epsilon,\alpha}.e)+(b^2-2B^{\epsilon,\alpha}.b))dx
\end{aligned}
$$

By (2.7), it follows that

$$
\begin{aligned}
H_U^{\epsilon,\alpha}(t) + & \iint dm^{\epsilon,\alpha}(t,x,\sigma) + \frac{1}{2}\int(tr(d\mu_E^{\epsilon,\alpha}+tr(d\mu_B^{\epsilon,\alpha})) - H_U^{\epsilon,\alpha}(0) \\
=& -\int_0^t\frac{d}{dt}\iint v\xi f^{\epsilon,\alpha}d\xi dxds + \int_0^t\frac{d}{dt}\iint \frac{v^2}{2}f^{\epsilon,\alpha}d\xi dxds \\
&-\int_0^t\frac{d}{dt}\iint v\xi(\frac{2}{1+\sqrt{1+\gamma^2\xi^2}}-1)f^{\epsilon,\alpha}d\xi dxds \\
&+\int_0^t\frac{d}{dt}\iint v^2(\frac{1}{1+\sqrt{1+\gamma^2\xi^2}}-\frac{1}{2})f^{\epsilon,\alpha}d\xi dxds \\
&+\int_0^t\frac{d}{dt}\int(-\epsilon E^{\epsilon,\alpha}e+\frac{e^2}{2}-B^{\epsilon,\alpha}b+\frac{b^2}{2})dxds.
\end{aligned}
$$

Denote by C_1 the first part of the remainder

$$
C_1 = -\left[\iint v\xi(\frac{2}{1+\sqrt{1+\gamma^2\xi^2}}-1)f^{\epsilon,\alpha}d\xi dx + \iint v^2(\frac{1}{1+\sqrt{1+\gamma^2\xi^2}}-\frac{1}{2})f^{\epsilon,\alpha}d\xi dx\right]_0^t
$$

Recall that $\rho^{\epsilon,\alpha} = \int f^{\epsilon,\alpha} d\xi$, $j^{\epsilon,\alpha} = \int f^{\epsilon,\alpha}\xi d\xi$ and $\hat{j}^{\epsilon,\alpha} = \int f^{\epsilon,\alpha}\xi/\sqrt{1+\gamma^2\xi^2}d\xi$. Then,

$$\begin{aligned}
H_U^{\epsilon,\alpha}(t) + & \iint dm^{\epsilon,\alpha}(t,x,\sigma) + \frac{1}{2}\int (tr(d\mu_E^{\epsilon,\alpha} + tr(d\mu_B^{\epsilon,\alpha})) - H_U^{\epsilon,\alpha}(0) \\
= & -\int_0^t \int (j^{\epsilon,\alpha}\partial_t v + v\partial_t j^{\epsilon,\alpha})dxds + \int_0^t \int (\rho^{\epsilon,\alpha} v\partial_t v + \frac{v^2}{2}\partial_t \rho^{\epsilon,\alpha})dxds + C_1 \\
& + \int_0^t \int (\partial_t e(e - \epsilon E^{\epsilon,\alpha}) + \partial_t b(b - B^{\epsilon,\alpha}))dxds - \int_0^t (e\partial_t(\epsilon E^{\epsilon,\alpha}) + b\partial_t B^{\epsilon,\alpha})dxds.
\end{aligned}$$

From the local conservations of mass and momentum (2.5,2.6), we deduce that

$$\begin{aligned}
H_U^{\epsilon,\alpha}(t) + & \iint dm^{\epsilon,\alpha}(t,x,\sigma) + \frac{1}{2}\int (tr(d\mu_E^{\epsilon,\alpha} + tr(d\mu_B^{\epsilon,\alpha})) - H_U^{\epsilon,\alpha}(0) \\
= & -\int_0^t \int \hat{j}^{\epsilon,\alpha}\partial_t v dxds + \int_0^t \int (\rho^{\epsilon,\alpha} v\partial_t v - \frac{v^2}{2}\mathrm{div}_x \hat{j}^{\epsilon,\alpha})dxds + C_2 + C_1 \\
& + \int_0^t \int v \left(\mathrm{div}_x \int \frac{\xi\otimes\xi}{\sqrt{1+\gamma^2\xi^2}} f^{\epsilon,\alpha} d\xi + F^{\epsilon,\alpha}\right) dxds \\
& + \int_0^t \int v \left(\gamma\partial_t \int \sigma dm^{\epsilon,\alpha} + \mathrm{div}_x \int \sigma\otimes\sigma dm^{\epsilon,\alpha}\right) \\
& + \int_0^t \int v \left(-\mathrm{div}_x (d\mu_E^{\epsilon,\alpha} - \frac{1}{2}tr(d\mu_E^{\epsilon,\alpha})Id) - \mathrm{div}_x (d\mu_B^{\epsilon,\alpha} - tr(d\mu_B^{\epsilon,\alpha})Id) + \epsilon\alpha\partial_t d\mu_{EB}^{\epsilon,\alpha}\right) \\
& + \int_0^t \int (\partial_t e(e - \epsilon E^{\epsilon,\alpha})e + \partial_t b(b - B^{\epsilon,\alpha}))dxds - \int_0^t (e\partial_t(\epsilon E^{\epsilon,\alpha}) + b\partial_t B^{\epsilon,\alpha})dxds \\
= & -\int_0^t \int \hat{j}^{\epsilon,\alpha}\partial_t v dxds + \int_0^t \int (\rho^{\epsilon,\alpha} v\partial_t v + D_x v : v\otimes \hat{j}^{\epsilon,\alpha})dxds + C_2 + C_1 \\
& - \int_0^t \iint D_x v : \int \frac{\xi\otimes\xi}{\sqrt{1+\gamma^2\xi^2}} f^{\epsilon,\alpha} d\xi dxds + \int_0^t \int v\cdot F^{\epsilon,\alpha} dxds \\
& + \gamma\left[\iint v.\sigma dm^{\epsilon,\alpha}\right]_0^t - \gamma\int_0^t \iint \partial_t v.\sigma dm^{\epsilon,\alpha} - \int_0^t \iint D_x v : \sigma\otimes\sigma dm^{\epsilon,\alpha} \\
& + \int_0^t \int D_x v : \left(d\mu_E^{\epsilon,\alpha} + d\mu_B^{\epsilon,\alpha} - \frac{1}{2}tr(d\mu_E^{\epsilon,\alpha} + d\mu_B^{\epsilon,\alpha})Id\right) \\
& + \epsilon\alpha\left[\int v.d\mu_{EB}^{\epsilon,\alpha}\right]_0^t - \epsilon\alpha\int_0^t \int \partial_t v d\mu_{EB}^{\epsilon,\alpha} \\
& + \int_0^t \int (\partial_t e(e - \epsilon E^{\epsilon,\alpha})e + \partial_t b(b - B^{\epsilon,\alpha}))dxds - \int_0^t (e\partial_t(\epsilon E^{\epsilon,\alpha}) + b\partial_t B^{\epsilon,\alpha})dxds,
\end{aligned}$$

where the force field $F^{\epsilon,\alpha}$ equals

$$F^{\epsilon,\alpha} = nE^{\epsilon,\alpha} - \mathrm{div}_x(\epsilon E^{\epsilon,\alpha}\otimes\epsilon E^{\epsilon,\alpha} - \frac{1}{2}|\epsilon E^{\epsilon,\alpha}|^2 Id) - \mathrm{div}_x(B^{\epsilon,\alpha}\otimes B^{\epsilon,\alpha} - \frac{1}{2}|B^{\epsilon,\alpha}|^2 Id) + \epsilon\alpha\partial_t(\epsilon E^{\epsilon,\alpha}\wedge B^{\epsilon,\alpha})$$

and the second part of the remainder C_2 is defined by

$$C_2 = -\int_0^t \int (j^{\epsilon,\alpha} - \hat{j}^{\epsilon,\alpha})\partial_t v dx ds = -\int_0^t \iint \xi(1 - \frac{1}{\sqrt{1+\gamma^2\xi^2}}) f^{\epsilon,\alpha}\partial_t v d\xi dx ds.$$

Then, by Maxwell's equations, we obtain

$$\begin{aligned}
H_U^{\epsilon,\alpha}(t) + & \iint dm^{\epsilon,\alpha}(t,x,\sigma) + \frac{1}{2}\int (tr(d\mu_E^{\epsilon,\alpha} + tr(d\mu_B^{\epsilon,\alpha})) - H_U^{\epsilon,\alpha}(0) \\
= & -\int_0^t \int \hat{j}^{\epsilon,\alpha}\partial_t v dx ds + \int_0^t \int (\rho^{\epsilon,\alpha} v\partial_t v + D_x v : v \otimes \hat{j}^{\epsilon,\alpha}) dx ds + C_2 + C_1 \\
& -\int_0^t \iint D_x v : \int \frac{\xi \otimes \xi}{\sqrt{1+\gamma^2\xi^2}} f^{\epsilon,\alpha} d\xi dx ds + \int_0^t \int v \cdot F^{\epsilon,\alpha} dx ds \\
& +\gamma \left[\iint v.\sigma dm^{\epsilon,\alpha}\right]_0^t - \gamma \int_0^t \iint \partial_t v.\sigma dm^{\epsilon,\alpha} - \int_0^t \iint D_x v : \sigma \otimes \sigma dm^{\epsilon,\alpha} \\
& + \int_0^t \int D_x v : \left(d\mu_E^{\epsilon,\alpha} + d\mu_B^{\epsilon,\alpha} - \frac{1}{2} tr(d\mu_E^{\epsilon,\alpha} + d\mu_B^{\epsilon,\alpha}) Id \right) \\
& + \epsilon\alpha \left[\int v.d\mu_{EB}^{\epsilon,\alpha}\right]_0^t - \epsilon\alpha \int_0^t \int \partial_t v d\mu_{EB}^{\epsilon,\alpha} \\
& + \int_0^t \int (\partial_t e(e - \epsilon E^{\epsilon,\alpha})e + \partial_t b(b - B^{\epsilon,\alpha})) dx ds \\
& - \int_0^t \left(e\left(\frac{1}{\alpha\epsilon}\text{ curl}_x B^{\epsilon,\alpha} + \frac{1}{\epsilon}\hat{j}^{\epsilon,\alpha}\right) + b\left(-\frac{1}{\alpha\epsilon}\text{ curl}_x \epsilon E^{\epsilon,\alpha}\right)\right) dx ds,
\end{aligned}$$

which can be rewritten

(4.4)

$$\begin{aligned}
H_U^{\epsilon,\alpha}(t) + & \iint dm^{\epsilon,\alpha}(t,x,\sigma) + \frac{1}{2}\int (tr(d\mu_E^{\epsilon,\alpha} + tr(d\mu_B^{\epsilon,\alpha})) - H_U^{\epsilon,\alpha}(0) \\
= & -\int_0^t \int \hat{j}^{\epsilon,\alpha}\partial_t v dx ds + \int_0^t \int (\rho^{\epsilon,\alpha} v\partial_t v + D_x v : v \otimes \hat{j}^{\epsilon,\alpha}) dx ds + C_3 + C_2 + C_1 \\
& -\int_0^t \int D_x v : \left(\int \frac{\xi \otimes \xi}{\sqrt{1+\gamma^2\xi^2}} f^{\epsilon,\alpha} d\xi dx + d\nu^{\epsilon,\alpha} \right) ds + \int_0^t \int v \cdot F^{\epsilon,\alpha} dx ds \\
& + \int_0^t \int (\partial_t e(e - \epsilon E^{\epsilon,\alpha})e + \partial_t b(b - B^{\epsilon,\alpha})) dx ds \\
& - \int_0^t \left(e\left(\frac{1}{\alpha\epsilon}\text{ curl}_x B^{\epsilon,\alpha} + \frac{1}{\epsilon}\hat{j}^{\epsilon,\alpha}\right) + b\left(-\frac{1}{\alpha\epsilon}\text{ curl}_x \epsilon E^{\epsilon,\alpha}\right)\right) dx ds,
\end{aligned}$$

introducing the following groups of defect terms

$$d\nu^{\epsilon,\alpha} = \int_{\sigma \in S^2} \sigma \otimes \sigma dm^{\epsilon,\alpha} - \left(d\mu_E^{\epsilon,\alpha} + d\mu_B^{\epsilon,\alpha} - \frac{1}{2} tr(d\mu_E^{\epsilon,\alpha} + d\mu_B^{\epsilon,\alpha}) Id \right)$$

and

$$C_3 = \gamma \left[\iint v.\sigma dm^{\epsilon,\alpha} \right]_0^t - \gamma \int_0^t \iint \partial_t v.\sigma dm^{\epsilon,\alpha}$$
$$+ \epsilon\alpha \left[\int v.d\mu_{EB}^{\epsilon,\alpha} \right]_0^t - \epsilon\alpha \int_0^t \int \partial_t v d\mu_{EB}^{\epsilon,\alpha}$$

The second step of the proof consists in introducing systematically the modulations $\xi - v$, $\hat{j}^{\epsilon,\alpha} - \rho^{\epsilon,\alpha} v$, $\epsilon E^{\epsilon,\alpha} - e$ and $B^{\epsilon,\alpha} - b$ in order to get the structure of the right hand side in (4.4).

$$\begin{aligned}
H_U^{\epsilon,\alpha}(t) + & \iint dm^{\epsilon,\alpha}(t,x,\sigma) + \frac{1}{2}\int (tr(d\mu_E^{\epsilon,\alpha} + tr(d\mu_B^{\epsilon,\alpha})) - H_U^{\epsilon,\alpha}(0) \\
= & -\int_0^t \int (\hat{j}^{\epsilon,\alpha} - \rho^{\epsilon,\alpha} v)\partial_t v dx ds + \int_0^t \int D_x v : v \otimes \hat{j}^{\epsilon,\alpha} dx ds + C_3 + C_2 + C_1 \\
& - \int_0^t \int D_x v : \left(\int \frac{(\xi - v + v)^{\otimes 2}}{\sqrt{1+\gamma^2\xi^2}} f^{\epsilon,\alpha} d\xi dx + d\nu^{\epsilon,\alpha} \right) ds + \int_0^t \int v \cdot F^{\epsilon,\alpha} dx ds \\
& + \int_0^t \int (\partial_t e(e - \epsilon E^{\epsilon,\alpha}) + \partial_t b(b - B^{\epsilon,\alpha})) dx ds \\
& - \int_0^t \int e \left(\frac{1}{\alpha\epsilon} \operatorname{curl}_x (B^{\epsilon,\alpha} - b + b) + \frac{1}{\epsilon}(\hat{j}^{\epsilon,\alpha} - \rho^{\epsilon,\alpha} v + \rho^{\epsilon,\alpha} v) \right) dx ds \\
& + \int_0^t \int b \left(\frac{1}{\alpha\epsilon} \operatorname{curl}_x (\epsilon E^{\epsilon,\alpha} - e + e) \right) dx ds,
\end{aligned}$$

which can be rewritten

(4.5)
$$\begin{aligned}
H_U^{\epsilon,\alpha}(t) + & \iint dm^{\epsilon,\alpha}(t,x,\sigma) + \frac{1}{2}\int (tr(d\mu_E^{\epsilon,\alpha} + tr(d\mu_B^{\epsilon,\alpha})) - H_U^{\epsilon,\alpha}(0) \\
= & -\int_0^t \int (\hat{j}^{\epsilon,\alpha} - \rho^{\epsilon,\alpha} v) \left(\partial_t v + (v.\nabla_x) v + \frac{1}{\epsilon} e + \alpha v \wedge b \right) dx ds + C_3 + C_2 + C_1 \\
& - \int_0^t \int D_x v : \left(\int \frac{(\xi - v)^{\otimes 2}}{\sqrt{1+\gamma^2\xi^2}} f^{\epsilon,\alpha} d\xi dx + d\nu^{\epsilon,\alpha} \right) ds + C_4 \\
& + \int_0^t \int v \cdot \left(F^{\epsilon,\alpha} - \frac{1}{\epsilon}\rho^{\epsilon,\alpha} e - \alpha \hat{j}^{\epsilon,\alpha} \wedge b \right) dx ds \\
& + \int_0^t \int (e - \epsilon E^{\epsilon,\alpha}) \left(\partial_t e - \frac{1}{\alpha\epsilon} \operatorname{curl}_x b \right) dx ds \\
& + \int_0^t \int (b - B^{\epsilon,\alpha}) \left(\partial_t b + \frac{1}{\alpha\epsilon} \operatorname{curl}_x e \right) dx ds,
\end{aligned}$$

with

$$C_4 = \int_0^t \iint D_x v : f^{\epsilon,\alpha} v \otimes v \left(\frac{1}{\sqrt{1+\gamma^2\xi^2}} - 1 \right) d\xi dx ds.$$

It remains then to deal with the nonlinear term

$$
\begin{aligned}
&F^{\epsilon,\alpha} - \frac{1}{\epsilon}\rho^{\epsilon,\alpha} e - \alpha\hat{j}^{\epsilon,\alpha}\wedge b \\
&\quad = -\mathrm{div}_x(\epsilon E^{\epsilon,\alpha}\otimes \epsilon E^{\epsilon,\alpha} - \frac{1}{2}|\epsilon E^{\epsilon,\alpha}|^2 Id) - \mathrm{div}_x(B^{\epsilon,\alpha}\otimes B^{\epsilon,\alpha} - \frac{1}{2}|B^{\epsilon,\alpha}|^2 Id) \\
&\quad + \epsilon\alpha\partial_t(\epsilon E^{\epsilon,\alpha}\wedge B^{\epsilon,\alpha}) + nE^{\epsilon,\alpha} - \frac{1}{\epsilon}\rho^{\epsilon,\alpha} e - \alpha\hat{j}^{\epsilon,\alpha}\wedge b.
\end{aligned}
$$

It becomes when we introduce the modulation

$$
\begin{aligned}
&F^{\epsilon,\alpha} - \frac{1}{\epsilon}\rho^{\epsilon,\alpha} e - \alpha\hat{j}^{\epsilon,\alpha}\wedge b \\
&= -\mathrm{div}_x((\epsilon E^{\epsilon,\alpha} - e)^{\otimes 2} - \frac{1}{2}|\epsilon E^{\epsilon,\alpha} - e|^2 Id) - \mathrm{div}_x((B^{\epsilon,\alpha} - b)^{\otimes 2} - \frac{1}{2}|B^{\epsilon,\alpha} - b|^2 Id) \\
&+ \epsilon\alpha\partial_t((\epsilon E^{\epsilon,\alpha} - e)\wedge(B^{\epsilon,\alpha} - b)) + \frac{1}{\epsilon}n(\epsilon E^{\epsilon,\alpha} - e) - \frac{1}{\epsilon}(\rho^{\epsilon,\alpha} - n)e - \alpha\hat{j}^{\epsilon,\alpha}\wedge b \\
&- \mathrm{div}_x(\epsilon E^{\epsilon,\alpha}\otimes e + e\otimes \epsilon E^{\epsilon,\alpha} - e\cdot\epsilon E^{\epsilon,\alpha} Id) \\
&- \mathrm{div}_x(B^{\epsilon,\alpha}\otimes b + b\otimes B^{\epsilon,\alpha} - b\cdot B^{\epsilon,\alpha} Id) \\
&+ \mathrm{div}_x(e^{\otimes 2} - \frac{1}{2}|e|^2 Id) + \mathrm{div}_x(b^{\otimes 2} - \frac{1}{2}|b|^2 Id) \\
&+\epsilon\alpha\Big(\partial_t e\wedge(B^{\epsilon,\alpha} - b) + (\epsilon E^{\epsilon,\alpha} - e)\wedge\partial_t b + e\wedge\partial_t B^{\epsilon,\alpha} + \partial_t(\epsilon E^{\epsilon,\alpha})\wedge b\Big)
\end{aligned}
$$

Using the relation

$$
\mathrm{div}_x(a\otimes b + b\otimes a - a\cdot b Id) = \mathrm{div}_x(a)b + \mathrm{div}_x(b)a + \ \mathrm{curl}_x(a)\wedge b + \ \mathrm{curl}_x(b)\wedge a
$$

leads to

$$
\begin{aligned}
&F^{\epsilon,\alpha} - \frac{1}{\epsilon}\rho^{\epsilon,\alpha} e - \alpha\hat{j}^{\epsilon,\alpha}\wedge b \\
&= -\mathrm{div}_x((\epsilon E^{\epsilon,\alpha} - e)^{\otimes 2} - \frac{1}{2}|\epsilon E^{\epsilon,\alpha} - e|^2 Id) - \mathrm{div}_x((B^{\epsilon,\alpha} - b)^{\otimes 2} - \frac{1}{2}|B^{\epsilon,\alpha} - b|^2 Id) \\
&+ \epsilon\alpha\partial_t((\epsilon E^{\epsilon,\alpha} - e)\wedge(B^{\epsilon,\alpha} - b)) + \frac{1}{\epsilon}n(\epsilon E^{\epsilon,\alpha} - e) - \frac{1}{\epsilon}(\rho^{\epsilon,\alpha} - n)e - \alpha\hat{j}^{\epsilon,\alpha}\wedge b \\
&- \mathrm{div}_x(\epsilon E^{\epsilon,\alpha})e - \mathrm{div}_x(e)\epsilon E^{\epsilon,\alpha} - \ \mathrm{curl}_x(\epsilon E^{\epsilon,\alpha})\wedge e - \ \mathrm{curl}_x e\wedge\epsilon E^{\epsilon,\alpha} \\
&- \ \mathrm{curl}_x(B^{\epsilon,\alpha})\wedge b - \ \mathrm{curl}_x b\wedge B^{\epsilon,\alpha} \\
&+ e\,\mathrm{div}_x e + \ \mathrm{curl}_x e\wedge e + \ \mathrm{curl}_x b\wedge b \\
&+ \epsilon\alpha\Big(\partial_t e\wedge(B^{\epsilon,\alpha} - b) + (\epsilon E^{\epsilon,\alpha} - e)\wedge\partial_t b) + e\wedge\partial_t B^{\epsilon,\alpha} + \partial_t(\epsilon E^{\epsilon,\alpha})\wedge b\Big)
\end{aligned}
$$

or equivalently

$$
\begin{aligned}
&F^{\epsilon,\alpha} - \frac{1}{\epsilon}\rho^{\epsilon,\alpha} e - \alpha \hat{j}^{\epsilon,\alpha} \wedge b \\
&\quad = -\mathrm{div}_x((\epsilon E^{\epsilon,\alpha} - e)^{\otimes 2} - \frac{1}{2}|\epsilon E^{\epsilon,\alpha} - e|^2 Id) - \mathrm{div}_x((B^{\epsilon,\alpha} - b)^{\otimes 2} - \frac{1}{2}|B^{\epsilon,\alpha} - b|^2 Id) \\
&\quad + \epsilon\alpha\partial_t((\epsilon E^{\epsilon,\alpha} - e) \wedge (B^{\epsilon,\alpha} - b)) + \frac{1}{\epsilon} n(\epsilon E^{\epsilon,\alpha} - e) - \frac{1}{\epsilon}\Big(\rho^{\epsilon,\alpha} - n + \epsilon^2 \mathrm{div}_x E^{\epsilon,\alpha}\Big) e \\
&\quad - \mathrm{div}_x(e)(\epsilon E^{\epsilon,\alpha} - e) - \Big(\ \mathrm{curl}_x(\epsilon E^{\epsilon,\alpha}) + \epsilon\alpha\partial_t B^{\epsilon,\alpha}\Big) \wedge e - \ \mathrm{curl}_x e \wedge (\epsilon E^{\epsilon,\alpha} - e) \\
&\quad + \Big(-\ \mathrm{curl}_x(B^{\epsilon,\alpha}) + \partial_t(\epsilon^2 \alpha E^{\epsilon,\alpha}) - \alpha \hat{j}^{\epsilon,\alpha}\Big) \wedge b - \ \mathrm{curl}_x b \wedge (B^{\epsilon,\alpha} - b) \\
&\quad + \epsilon\alpha\Big(\partial_t e \wedge (B^{\epsilon,\alpha} - b) + (\epsilon E^{\epsilon,\alpha} - e) \wedge \partial_t b\Big)
\end{aligned}
$$

By Maxwell's equations, we finally obtain

(4.6)

$$
\begin{aligned}
&F^{\epsilon,\alpha} - \frac{1}{\epsilon}\rho^{\epsilon,\alpha} e - \alpha \hat{j}^{\epsilon,\alpha} \wedge b \\
&\quad = -\mathrm{div}_x((\epsilon E^{\epsilon,\alpha} - e)^{\otimes 2} - \frac{1}{2}|\epsilon E^{\epsilon,\alpha} - e|^2 Id) - \mathrm{div}_x((B^{\epsilon,\alpha} - b)^{\otimes 2} - \frac{1}{2}|B^{\epsilon,\alpha} - b|^2 Id) \\
&\quad + \epsilon\alpha\partial_t((\epsilon E^{\epsilon,\alpha} - e) \wedge (B^{\epsilon,\alpha} - b)) + \frac{1}{\epsilon} n(\epsilon E^{\epsilon,\alpha} - e) \\
&\quad - \mathrm{div}_x(e)(\epsilon E^{\epsilon,\alpha} - e) - \ \mathrm{curl}_x e \wedge (\epsilon E^{\epsilon,\alpha} - e) - \ \mathrm{curl}_x b \wedge (B^{\epsilon,\alpha} - b) \\
&\quad + \epsilon\alpha\left(\partial_t e \wedge (B^{\epsilon,\alpha} - b) + (\epsilon E^{\epsilon,\alpha} - e) \wedge \partial_t b\right)
\end{aligned}
$$

Plugging (4.6) into (4.5) provides then

(4.7)

$$
\begin{aligned}
&H_U^{\epsilon,\alpha}(t) + \iint dm^{\epsilon,\alpha}(t,x,\sigma) + \frac{1}{2}\int (tr(d\mu_E^{\epsilon,\alpha} + tr(d\mu_B^{\epsilon,\alpha}) - H_U^{\epsilon,\alpha}(0) \\
= \ & -\int_0^t \int (\hat{j}^{\epsilon,\alpha} - \rho^{\epsilon,\alpha} v)\left(\partial_t v + (v.\nabla_x)v + \frac{1}{\epsilon} e + \alpha v \wedge b\right) dxds + C_3 + C_2 + C_1 \\
& -\int_0^t \int D_x v : \left(\int \frac{(\xi - v)^{\otimes 2}}{\sqrt{1 + \gamma^2 \xi^2}} f^{\epsilon,\alpha} d\xi dx + d\nu^{\epsilon,\alpha}\right) ds + C_4 + C_5 \\
& + \int_0^t \int D_x v : \left((\epsilon E^{\epsilon,\alpha} - e)^{\otimes 2} - \frac{1}{2}|\epsilon E^{\epsilon,\alpha} - e|^2 Id) + (B^{\epsilon,\alpha} - b)^{\otimes 2} - \frac{1}{2}|B^{\epsilon,\alpha} - b|^2 Id\right) dxds \\
& + \int_0^t \int (e - \epsilon E^{\epsilon,\alpha})\left(\partial_t e - \frac{1}{\alpha\epsilon}\ \mathrm{curl}_x b - \frac{1}{\epsilon} nv + v\mathrm{div}_x e + v \wedge \ \mathrm{curl}_x e + \alpha\epsilon v \wedge \partial_t b\right) dxds \\
& + \int_0^t \int (b - B^{\epsilon,\alpha})\left(\partial_t b + \frac{1}{\alpha\epsilon}\ \mathrm{curl}_x e + v \wedge \ \mathrm{curl}_x b - \epsilon\alpha v \wedge \partial_t e\right) dxds,
\end{aligned}
$$

where

$$\begin{aligned} C_5 &= \epsilon\alpha \int_0^t \int v.\partial_t((\epsilon E^{\epsilon,\alpha} - e) \wedge (B^{\epsilon,\alpha} - b))dxds \\ &= \epsilon\alpha \left[\int v((\epsilon E^{\epsilon,\alpha} - e) \wedge (B^{\epsilon,\alpha} - b))dx \right]_0^t - \epsilon\alpha \int_0^t \int \partial_t v.((\epsilon E^{\epsilon,\alpha} - e) \wedge (B^{\epsilon,\alpha} - b))dxds. \end{aligned}$$

Checking that $R^{\epsilon,\alpha} = C_1 + C_2 + C_3 + C_4 + C_5$ concludes the proof. □

4.3. **The stability inequality.**

The particular structure of the identity established in the previous Proposition (4.1) allows to prove the following stability inequality.

Corollary 4.1. *The modulated energy $H_U^{\epsilon,\alpha}(t)$ satisfies the Gronwall inequality*

$$\begin{aligned} H_U^{\epsilon,\alpha}(t) \quad &+ \iint dm^{\epsilon,\alpha}(t,x,\sigma) + \frac{1}{2}\int (tr(d\mu_E^{\epsilon,\alpha} + tr(d\mu_B^{\epsilon,\alpha}) \\ &\leq H_U^{\epsilon,\alpha}(0) \exp\left(6\int_0^t ||D_x v||_{L^\infty} d\tau\right) + O(\gamma) + O(\epsilon) \\ &+ \int_0^t \exp\left(6\int_0^s ||D_x v||_{L^\infty} d\tau\right) \int A^{\epsilon,\alpha}(U) \cdot \begin{pmatrix} \rho^{\epsilon,\alpha} v - \hat{j}^{\epsilon,\alpha} \\ b - B^{\epsilon,\alpha} \\ e - \epsilon E^{\epsilon,\alpha} \end{pmatrix} dxds. \end{aligned} \tag{4.8}$$

Proof. This corollary is based essentially on the control of the relativistic corrections which are expected to converge to 0 as $\gamma \to 0$. With the notations of the previous paragraph, we have

$$\begin{aligned} |C_1| \quad \leq \quad & ||v||_{L^\infty} \int\int \frac{\gamma|\xi|^2}{1 + \sqrt{1 + \gamma^2|\xi|^2}} f^{\epsilon,\alpha} dxd\xi \\ & + ||v||_{L^\infty}^2 \int\int \frac{\gamma|\xi|}{2(1 + \sqrt{1 + \gamma^2|\xi|^2})} f^{\epsilon,\alpha} dxd\xi \end{aligned}$$

because

$$\left| \frac{2}{1 + \sqrt{1 + \gamma^2\xi^2}} - 1 \right| = \left| \frac{1 - \sqrt{1 + \gamma^2\xi^2}}{1 + \sqrt{1 + \gamma^2\xi^2}} \right| \leq \frac{\gamma|\xi|}{1 + \sqrt{1 + \gamma^2\xi^2}}.$$

Since the global mass and the energy $H^{\epsilon,\alpha}$ are conserved, we get by Cauchy Schwarz inequality

$$|C_1| \leq C_{in}\gamma(||v||_{L^\infty} + ||v||_{L^\infty}^2).$$

In the same way, we obtain for all $t \in [0, T]$

$$|C_2| \quad \leq \quad ||\partial_t v||_{L^\infty} \int_0^t \iint \frac{2\gamma|\xi|^2}{1 + \sqrt{1 + \gamma^2|\xi|^2}} f^{\epsilon,\alpha} dxd\xi ds \leq 2TC_{in}\gamma||\partial_t v||_{L^\infty},$$

and finally

$$|C_4| \leq ||D_x v||_{L^\infty} ||v||_{L^\infty}^2 \int_0^t \iint \frac{2\gamma|\xi|}{1 + \sqrt{1 + \gamma^2|\xi|^2}} f^{\epsilon,\alpha} dxd\xi ds \leq 2TC_{in}\gamma||\nabla v||_{L^\infty} ||v||_{L^\infty}^2.$$

Then this part of the remainder $R^{\epsilon,\alpha}$ satisfies

$$|C_1| + |C_2| + |C_4| = O(\gamma)$$

where the constants depend only on the initial mass and energy, and on the $W^{1,\infty}(\mathbf{R}^+ \times \Omega)$ norm of v.

The second step of the proof consists in estimating the other terms in the remainder $R^{\epsilon,\alpha}$. By the energy inequality,

$$\begin{aligned}
|C_3| &\leq 2\gamma m^{\epsilon,\alpha}(\Omega \times S^2)||v||_{L^\infty} + \gamma T m^{\epsilon,\alpha}(\Omega \times S^2)||\partial_t v||_{L^\infty} \\
&\quad + 2\epsilon\alpha|\mu_{EB}|(\Omega)||v||_{L^\infty} + \epsilon\alpha T|\mu_{EB}|(\Omega)||\partial_t v||_{L^\infty} \\
&\leq 4C_{in}(\epsilon\alpha + \gamma)\left(||v||_{L^\infty} + T||\partial_t v||_{L^\infty}\right),
\end{aligned}$$

$$\begin{aligned}
|C_5| &\leq 2\alpha\epsilon||\epsilon E^{\epsilon,\alpha} - e||_{L^2}||B^{\epsilon,\alpha} - b||_{L^2}||v||_{L^\infty} + \alpha\epsilon T||\epsilon E^{\epsilon,\alpha} - e||_{L^2}||B^{\epsilon,\alpha} - b||_{L^2}||\partial_t v||_{L^\infty} \\
&\leq 4C_{in}\epsilon\left(||v||_{L^\infty} + T||\partial_t v||_{L^\infty}\right),
\end{aligned}$$

from which we deduce that

$$|C_3| + |C_5| = O(\gamma) + O(\epsilon)$$

where the constants depend only on the initial mass and energy, and on the $W^{1,\infty}(\mathbf{R}^+ \times \Omega)$ norm of v.

Then, in order to apply Gronwall's lemma, it remains to prove that the flux terms $h_U^{\epsilon,\alpha}$ and $\nu^{\epsilon,\alpha}$ can be controlled in terms of the modulated energy

$$H_U^{\epsilon,\alpha} + \iint dm^{\epsilon,\alpha} + \frac{1}{2}\int tr(d\mu_E^{\epsilon,\alpha} + d\mu_B^{\epsilon,\alpha}).$$

We have clearly

$$\left\|\int f^{\epsilon,\alpha}\frac{(\xi - v)^{\otimes 2}}{\sqrt{1+\gamma^2\xi^2}}d\xi\right\|_{L^1} \leq 6\iint f^{\epsilon,\alpha}\frac{|\xi - v|^2}{1+\sqrt{1+\gamma^2\xi^2}}d\xi dx,$$

$$||(\epsilon E^{\epsilon,\alpha} - e)^{\otimes 2} - \frac{1}{2}|\epsilon E^{\epsilon,\alpha} - e|^2 Id||_{L^1} \leq 3\int |\epsilon E^{\epsilon,\alpha} - e|^2 dx,$$

and

$$||(B^{\epsilon,\alpha} - b)^{\otimes 2} - \frac{1}{2}|B^{\epsilon,\alpha} - b|^2 Id||_{L^1} \leq 3\int |B^{\epsilon,\alpha} - b|^2 dx,$$

from which we deduce that

$$||h_U^{\epsilon,\alpha}||_{L^1} \leq 6H_U^{\epsilon,\alpha}.$$

In the same way, as μ_E and μ_B are non negative symetric matrix

$$\iint |\sigma \otimes \sigma dm^{\epsilon,\alpha}| \leq 6m^{\epsilon,\alpha}(\Omega \times S^2),$$

$$\int |\mu_E^{\epsilon,\alpha} - \frac{1}{2}tr(d\mu_E^{\epsilon,\alpha} Id) \leq 3tr(\mu_E)(\Omega),$$

$$\int |d\mu_B^{\epsilon,\alpha} - \frac{1}{2}tr(d\mu_B^{\epsilon,\alpha})Id| \leq 3tr(\mu_B)(\Omega),$$

so that

$$|\nu^{\epsilon,\alpha}|(\Omega) \leq 6\left(\iint dm^{\epsilon,\alpha} + \frac{1}{2}\int tr(d\mu_E^{\epsilon,\alpha} + d\mu_B^{\epsilon,\alpha})\right).$$

Thus, applying Gronwall's lemma concludes the proof. □

5. Strong convergence for well prepared initial data

In this section we restrict our attention to initial data for which solutions of (4.2) do not present any fast oscillation (nonnegligible in norm).

Indeed, we suppose that the initial data (v_0, e_0, b_0) belongs to the kernel of the singular perturbation, i.e.

$$e_0 = 0, \quad nv_0 + \frac{1}{\alpha}\,\mathrm{curl}_x b_0 = 0.$$

Since $e_0 = 0$, we modulate $H^{\epsilon,\alpha}(t)$ with $U_\epsilon = (v, \epsilon \underline{e}, b)$ instead of $U = (v, 0, b)$. We obtain a similar result but in a more precise form wich will be useful in the sequel.

Then we expect that any solution (v, e, b) of (4.2) with initial data $(v_0, 0, b_0)$ behaves asymptotically as $\epsilon \to 0$ as the solution $(v, 0, b)$ of

$$\left\{\begin{aligned} \partial_t v + v\cdot\nabla v + \underline{e} + \alpha v \wedge b = 0\\ \partial_t b + \frac{1}{\alpha}\mathrm{curl}\ \underline{e} = 0\\ nv + \frac{1}{\alpha}\ \mathrm{curl} b = 0 \end{aligned}\right. \tag{5.1}$$

where $\underline{e} \sim e/\epsilon$ is the Lagrange multiplier associated to the constraint $nv + \frac{1}{\alpha}\ \mathrm{curl} b = 0$.

In particular, if the following convergence holds

(5.2)

$$\iint \frac{(\xi - v_0)^2}{1+\sqrt{1+\gamma^2\xi^2}} f_{in}^{\epsilon,\alpha}(x,\xi)d\xi dx + \frac{1}{2}\int \epsilon^2 |E_{in}^{\epsilon,\alpha}(x)|^2 dx + \frac{1}{2}\int |B_{in}^{\epsilon,\alpha}(x) - b_0|^2 dx$$
$$= H_U^{\epsilon,\alpha}(0) \xrightarrow{\epsilon\to 0} 0,$$

which means that

- the initial velocity profile is essentially monokinetic;
- the initial fields converge strongly;
- and the limiting fields belong to the kernel of the singular perturbation;

then we expect that

$$\hat{j}^{\epsilon,\alpha} \sim \rho^{\epsilon,\alpha} v_{\epsilon,\alpha} \to nv;$$

$$\epsilon E^{\epsilon,\alpha} \sim e_{\epsilon,\alpha} \to 0;$$

$$B^{\epsilon,\alpha} \sim b_{\epsilon,\alpha} \to b;$$

where $\rho^{\epsilon,\alpha}$, $\hat{j}^{\epsilon,\alpha}$ are the density and momentum associated with any solution of (1.4), $(v_{\epsilon,\alpha}, e_{\epsilon,\alpha}, b_{\epsilon,\alpha})$ is the solution of the asymptotic system (4.2) and (v, b) satisfy the mean field equation (5.1). The mathematical formulation of this result is given in the following

Proposition 5.1. *Let* $(f_{in}^{\epsilon,\alpha})_{\epsilon,\alpha}$ *be a family of nonnegative functions of* $L^1 \cap L^2(\Omega\times\mathbf{R}^3)$, *and* $(E_{in}^{\epsilon,\alpha})_{\epsilon,\alpha}$, $(B_{in}^{\epsilon,\alpha})_{\epsilon,\alpha}$ *be two families of vector fields of* $L^2(\Omega)$ *satisfying the compatibility conditions (2.2) for some constant* n *and the uniform energy bound*

$$\sup_{\epsilon,\alpha} H^{\epsilon,\alpha}(0) < +\infty.$$

Assume that there exist two divergence-free vector fields $v_0, b_0 \in H^s(\Omega)$ *(for* $s > 2+3/2$*) such that*

$$nv_0 + \frac{1}{\alpha}\ curl_x b_0 = 0,$$

$$\iint \frac{(\xi - v_0)^2}{1+\sqrt{1+\gamma^2\xi^2}} f_{in}^{\epsilon,\alpha}(x,\xi)d\xi dx + \frac{1}{2}\int \epsilon^2 |E_{in}^{\epsilon,\alpha}(x)|^2 dx + \frac{1}{2}\int |B_{in}^{\epsilon,\alpha}(x) - b_0|^2 dx \to 0 \ as\ \epsilon \to 0.$$

Let $(f^{\epsilon,\alpha}, E^{\epsilon,\alpha}, B^{\epsilon,\alpha})$ *be, for every* $\epsilon > 0, \alpha > 0$, *a solution of the scaled Vlasov-Maxwell system (1.4) with initial condition (2.1). Then, there exists* $T > 0$ *such that the current density* $\hat{j}^{\epsilon,\alpha} = \int f^{\epsilon,\alpha}\xi/\sqrt{1+\gamma^2\xi^2}d\xi$ *converges weakly in* $L^\infty([0,T], L^1(\Omega))$, *the scaled electric field* $\epsilon E^{\epsilon,\alpha}$ *and the magnetic field* $B^{\epsilon,\alpha}$ *converge strongly in* $L^\infty_{Loc}([0,T[, (L^2(\Omega))^2)$ *to* $(nv, 0, b)$ *as* $\epsilon \to 0, \alpha \to \bar{\alpha} < +\infty$, *where* $(v,b) \in L^\infty([0,T], H^s(\Omega))$ *is the local strong solution of (5.1) with initial condition* (v_0, b_0).

In other words, this result shows that

- the space $\mathrm{Ker}(\mathrm{L}_n)$ is invariant under the flow of (4.2) (up to corrections of higher order in ϵ) : if the initial data belongs to $\mathrm{Ker}(\mathrm{L}_n)$, then the solution of (4.2) is close to $\mathrm{Ker}(\mathrm{L}_n)$;
- the mean field equation corresponding to (4.2) (i.e. its projection on $\mathrm{Ker}(\mathrm{L}_n)$) is (5.1);
- for well-prepared initial data, the strong convergence of the fields and the monokinetic profile are preserved : more precisely a convergence result such as (5.2) holds for all $t > 0$.

The proof of this result is based on the Gronwall's inequality obtained in the previous section, and on the study of the limiting system. Indeed, if we were able to exhibit a smooth solution $U_{\epsilon,\alpha} = (v_{\epsilon,\alpha}, e_{\epsilon,\alpha}, b_{\epsilon,\alpha})$ of (4.2) for all $\epsilon, \alpha > 0$, then we would have

$$A^{\epsilon,\alpha}(U_{\epsilon,\alpha}) = 0$$

and by corollary 4.1, $U_{\epsilon,\alpha}$ would give a good approximation of $(\hat{j}^{\epsilon,\alpha}/\rho^{\epsilon,\alpha}, \epsilon E^{\epsilon,\alpha}, B^{\epsilon,\alpha})$.

5.1. **Theory for the limiting system.** The first step consists then in characterizing the asymptotic system (4.2), and more precisely the associated mean field system (5.1) (since initial data are well-prepared).

Lemma 5.1. *Let v_0, b_0 be two divergence-free vector fields of $H^s(\Omega)$ $(s > 1 + \frac{3}{2})$ such that*

$$nv_0 + \frac{1}{\alpha}\ curl_x b_0 = 0,$$

for some homogeneous n. Then, there exist $T^* \in]0, +\infty]$ and a unique $(v, b) \in L^\infty_{loc}([0, T^*[, H^s)$ solution of (5.1).

Proof. If n is homogeneous, because of the constraint,

$$\mathrm{div}_x v = -\frac{1}{\alpha n}\mathrm{div}_x(\ \mathrm{curl}_x b).$$

Then,

$$\partial_t\ \mathrm{curl}_x v + (v \cdot \nabla_x)\ \mathrm{curl}_x v - (\ \mathrm{curl}_x v \cdot \nabla_x)v +\ \mathrm{curl}_x \underline{e} + \alpha\ \mathrm{curl}_x(v \wedge b) = 0,$$

and

$$\mathrm{curl}_x(v \wedge b) = (b \cdot \nabla_x)v - (v \cdot \nabla_x)b.$$

Let us introduce

$$w =\ \mathrm{curl}_x v - \alpha b.$$

The system (5.1) can be rewritten

$$\partial_t w + v \cdot \nabla w - w \cdot \nabla v = 0$$
$$-\Delta v + \alpha^2 n v =\ \mathrm{curl}_x w$$
$$nv + \frac{1}{\alpha}\ \mathrm{curl} b = 0,\ \mathrm{div}\ b = 0.$$

Consider the mapping

$$M : v \in H^s \mapsto \tilde{v} = M(v) \in H^s$$

where $\tilde{v}$ is defined as the solution of the elliptic equation

$$-\Delta_x \tilde{v} + \alpha^2 n \tilde{v} = \tilde{w}$$

and $\tilde{w}$ satisfies the following transport equation

$$\partial_t \tilde{w} + v \cdot \nabla_x \tilde{w} - \tilde{w} \cdot \nabla_x v = 0.$$

For $s > 1 + 3/2$, and for T sufficiently small, we can prove that M is a contraction. Therefore, the existence results are the same as for the incompressible Euler equations. □

If n is not homogeneous, the limiting system is much more complicated and we do not have any existence result for strong solutions. However, we should have a weak convergence result if we introduce a notion of dissipative solutions for the limiting system. From now on, we assume that n is homogeneous.

5.2. **Weak compactness results.** In order to get the asymptotic behaviour of the macroscopic quantities $(\hat{j}^{\epsilon,\alpha}, \epsilon E^{\epsilon,\alpha}, B^{\epsilon,\alpha})$, we have first to establish the relative compactness of the families.

From the energy inequality we deduce that

$$\sup_{\epsilon,\alpha} \|\epsilon E^{\epsilon,\alpha}\|^2_{L^\infty(\mathbf{R}^+,L^2(\Omega))} < +\infty,$$

$$\sup_{\epsilon,\alpha} \|B^{\epsilon,\alpha}\|^2_{L^\infty(\mathbf{R}^+,L^2(\Omega))} < +\infty,$$

$$\sup_{\epsilon,\alpha} \sup_{t\in\mathbf{R}^+} \iint \frac{\xi^2}{1+\sqrt{1+\gamma^2\xi^2}} f^{\epsilon,\alpha} dx d\xi < +\infty,$$

while the conservation of mass provides

$$\iint f^{\epsilon,\alpha} dx d\xi = n|\Omega|.$$

Then we obtain a uniform bound on $\hat{j}^{\epsilon,\alpha}$ by interpolation

$$\begin{aligned} |\hat{j}^{\epsilon,\alpha}| &\le \left(\int f^{\epsilon,\alpha} dx d\xi\right)^{1/2} \left(\int \frac{\xi^2}{1+\gamma^2\xi^2} f^{\epsilon,\alpha} dx d\xi\right)^{1/2} \\ &\le \left(\int f^{\epsilon,\alpha} dx d\xi\right)^{1/2} \left(\int \frac{2\xi^2}{1+\sqrt{1+\gamma^2\xi^2}} f^{\epsilon,\alpha} dx d\xi\right)^{1/2} \end{aligned}$$

which implies in particular that

$$\sup_{\epsilon,\alpha} \|\hat{j}^{\epsilon,\alpha}\|_{L^\infty(\mathbf{R}^+,L^1(\Omega))} < +\infty.$$

Up to extraction of a subsequence, we then have

$$\epsilon E^{\epsilon,\alpha} \rightharpoonup E, \quad B^{\epsilon,\alpha} \rightharpoonup B \text{ in } w^* - L^\infty(\mathbf{R}^+, L^2(\Omega))$$

$$\rho^{\epsilon,\alpha} \rightharpoonup \rho, \quad \hat{j}^{\epsilon,\alpha} \rightharpoonup j \text{ in the sense of measures.}$$

Moreover, using the Poisson equation, we obtain that

$$\rho = n.$$

5.3. **Convergence results.** Let (v,b) be the solution of (5.1) with initial condition (v_0,b_0). Comparing both systems (5.1) and (4.2) shows that $U_\epsilon=(v,\epsilon\underline{e},b)$ is an ϵ-approximate solution of (4.2), and more precisely that it satisfies

$$A^{\epsilon,\alpha}(U_\epsilon)=\begin{pmatrix}\partial_t v+v\cdot\nabla v+\underline{e}+\alpha v\wedge b\\ \partial_t b+\frac{1}{\alpha}\text{curl }\underline{e}+(\alpha\epsilon^2\partial_t\underline{e}-\text{curl }b)\wedge v\\ \epsilon\partial_t\underline{e}-\frac{1}{\alpha\epsilon}\text{curl }b-n\frac{v}{\epsilon}+\epsilon v\text{div }\underline{e}-(\alpha\epsilon\partial_t b+\epsilon\text{curl }\underline{e})\wedge v\end{pmatrix}$$
$$=\begin{pmatrix}0\\ (\alpha\epsilon^2\partial_t\underline{e})\wedge v\\ \epsilon\partial_t\underline{e}+\epsilon v\text{div }\underline{e}\end{pmatrix}.$$

As $\underline{e}=-(\partial_t v+v\cdot\nabla_x v+\alpha v\wedge b)$, it is uniformly bounded in $W^{1,\infty}([0,T]\times\Omega)$. Then, since $\iint dm^{\epsilon,\alpha}(t,x,\sigma)+\frac{1}{2}\int tr(d\mu_E^{\epsilon,\alpha})+tr(d\mu_B^{\epsilon,\alpha})\geq 0$, plugging this last identity in the stability inequality (4.8) leads to

$$H_{U_\epsilon}^{\epsilon,\alpha}(t)\leq H_{U_\epsilon}^{\epsilon,\alpha}(0)\exp(6\int_0^t\|D_x v(s)\|_{L^\infty}ds)+O(\gamma)+O(\epsilon).$$

The assumption on the initial data gives exactly

$$H_{U_\epsilon}^{\epsilon,\alpha}(0)=H_U^{\epsilon,\alpha}(0)+O(\epsilon)\to 0,$$

with $U=(v,0,b)$, from which we deduce

$$H_{U_\epsilon}^{\epsilon,\alpha}(t)\to 0 \text{ as } \epsilon\to 0,$$

or equivalently

$$H_U^{\epsilon,\alpha}(t)\to 0 \text{ as } \epsilon\to 0$$

for all $t<T^*$ where T^* is the lifespan of the strong solution of (5.1). In particular,

$$\epsilon E^{\epsilon,\alpha}\to 0,\quad B^{\epsilon,\alpha}\to b \text{ strongly in } L^\infty([0,T],L^2(\Omega)).$$

To get the convergence result on the current, we use a - by now standard - argument of convexity. Define

$$h_v^{\epsilon,\alpha}(t)=K(\hat{j}^{\epsilon,\alpha}-v\hat{\rho}^{\epsilon,\alpha};\hat{\rho}^{\epsilon,\alpha})=\int\frac{|\hat{j}^{\epsilon,\alpha}-v\hat{\rho}^{\epsilon,\alpha}|^2}{\hat{\rho}^{\epsilon,\alpha}}dx$$

with

$$K(m,\sigma)=\int\sup_{b\in\mathcal{D}([0,T]\times\Omega)}\frac{1}{2}<\sigma(t,x);|b(t,x)|^2>+<m(t,x);b(t,x)>dxdt,$$

and

$$\hat{\rho}^{\epsilon,\alpha}=\int\frac{1}{\sqrt{1+\gamma^2\xi^2}}f^{\epsilon,\alpha}d\xi.$$

From the Cauchy-Schwarz inequality we deduce that

$$|\hat{j}^{\epsilon,\alpha} - v\hat{\rho}^{\epsilon,\alpha}|^2 \leq \left(\int \frac{(\xi - v)^2}{1+\sqrt{1+\gamma^2\xi^2}} f^{\epsilon,\alpha} d\xi\right)\left(\int \frac{1+\sqrt{1+\gamma^2\xi^2}}{1+\gamma^2\xi^2} f^{\epsilon,\alpha} d\xi\right)$$

and consequently

$$h_v^{\epsilon,\alpha}(t) \leq 2H_{U_\epsilon}^{\epsilon,\alpha}(t).$$

Note in particular that $\hat{j}^{\epsilon,\alpha} - \hat{\rho}^{\epsilon,\alpha} v \to 0$ strongly in $L^\infty([0,T], L^1(\Omega))$.

On the other hand, it is easy to check that

$$\hat{\rho}^{\epsilon,\alpha} \to n$$

in the sense of measures. Indeed, the relativistic correction $\rho^{\epsilon,\alpha} - \hat{\rho}^{\epsilon,\alpha}$ can be estimated as the remainder C_4 in section 4.3. Then since K is lower semi-continuous for the convergence in the vague sense of measures

$$\int \frac{|j - nv|^2}{n} dx \leq \lim h_v^{\epsilon,\alpha}(t) \leq \lim H_U^{\epsilon,\alpha}(t) = 0$$

which proves that $j = nv$.

The uniqueness of the limit point ensures that the whole family $(\hat{j}^{\epsilon,\alpha}, \epsilon E^{\epsilon,\alpha}, B^{\epsilon,\alpha})$ converges strongly to $(nv, 0, b)$.

6. Study of the singular perturbation

In view of the previous results, for general initial data, we expect $(\hat{j}^{\epsilon,\alpha}, \epsilon E^{\epsilon,\alpha}, B^{\epsilon,\alpha})$ to behave as the solution of the following singular system

$$\begin{aligned} \partial_t v + v \cdot \nabla v + \frac{e}{\epsilon} + \alpha v \wedge b = 0 \\ \partial_t b + \frac{1}{\alpha\epsilon} \text{curl } e = 0 \\ \partial_t e - \frac{1}{\alpha\epsilon} \text{curl } b - \frac{v}{\epsilon} + v \text{div } e = 0 \\ \text{div } b = 0, \end{aligned} \tag{6.1}$$

that is to oscillate under the linear penalization L around a mean state satisfying (5.1). (Recall that we restrict our attention to the case where n is constant, and for simplicity $n \equiv 1$.)

In order to obtain a complete description of the asymptotic, we will first describe the oscillations generated by the singular perturbation and then, we will check that generically the oscillations do not bring any contribution in the evolution of the mean field.

6.1. **Description of the oscillations.** Define the linear operator S

$$S : \begin{pmatrix} \mathrm{div}_x v \\ \mathrm{div}_x e \\ \mathrm{curl}_x v \\ \mathrm{curl}_x e \\ (-\Delta_x)^{1/2} b \end{pmatrix} \mapsto \begin{pmatrix} \mathrm{div}_x e \\ -\mathrm{div}_x v \\ \mathrm{curl}_x e \\ \frac{1}{\alpha}\Delta_x b - \mathrm{curl}_x v \\ \frac{1}{\alpha}(-\Delta_x)^{1/2}\, \mathrm{curl}_x e \end{pmatrix} \tag{6.2}$$

that is a variant of L which has a simpler spectral theory. Describing the oscillations generated by the linear penalization requires a good understanding of the structure of S and in particular a precise description of its kernel.

Lemma 6.1. *Consider the bounded antiselfadjoint operator S defined on $L^2(\Omega)$ by (6.2). Then,*

- *for all $k \in \mathbf{Z}^3$, the symbol S_k of S admits the following purely imaginar eigenvalues*

$$\begin{aligned} &\lambda_1(k) = i, \quad \lambda_2(k) = -i, \\ &\lambda_3(k) = 0, \quad \lambda_4(k) = i\sqrt{\frac{|k|^2}{\alpha^2} + 1}, \quad \lambda_5(k) = -i\sqrt{\frac{|k|^2}{\alpha^2} + 1}, \end{aligned} \tag{6.3}$$

  with the notation $|k|^2 = \frac{k_1^2}{a_1^2} + \frac{k_2^2}{a_2^2} + \frac{k_3^2}{a_3^2}$.
- *the projections $\Pi_{k,j}$ ($j \in \{1, .., 5\}$) on the eigenspaces of S_k are bounded uniformly in $k \in \mathbf{Z}^3$. In particular, the projection on the kernel of S is the pseudo-differential operator $\mathcal{P}$ of order 0 defined by*

$$\mathcal{P} = \begin{pmatrix} 0 & 0 & & & \\ 0 & 0 & & & \\ & & (-\Delta_x)(\alpha^2 Id - \Delta_x)^{-1} & 0\,Id & -\alpha(-\Delta_x)^{1/2}(\alpha^2 Id - \Delta_x)^{-1} \\ & & 0\,Id & 0\,Id & 0\,Id \\ & & -\alpha(-\Delta_x)^{1/2}(\alpha^2 Id - \Delta_x)^{-1} & 0\,Id & \alpha^2(\alpha^2 Id - \Delta_x)^{-1} \end{pmatrix};$$

- *the group of isometries $\mathcal{S}$ generated by S preserves all Sobolev norms.*

Proof. As S is a linear pseudo-differential operator with constant coefficients, a natural idea to study its spectral properties is to consider its symbol, i.e. the following block diagonal matrix

$$S_k = \begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & 0\,Id & Id & 0\,Id \\ & & -Id & 0\,Id & -\frac{|k|}{\alpha} Id \\ & & 0\,Id & \frac{|k|}{\alpha} Id & 0\,Id \end{pmatrix}$$

where Id denotes the identity of $\mathbf{C}^3$.

• In order to describe the oscillating modes, it is enough to consider separately each block. The eigenvalues of

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

are i and $-i$ which we denote by $\lambda_1(k)$ and $\lambda_2(k)$. The eigenvalues of

$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & -\frac{|k|}{\alpha} \\ 0 & \frac{|k|}{\alpha} & 0 \end{pmatrix}$$

are $\lambda_3(k) = 0$, $\lambda_4(k) = i\sqrt{\frac{|k|^2}{\alpha^2}+1}$ and $\lambda_5(k) = -i\sqrt{\frac{|k|^2}{\alpha^2}+1}$. As all eigenvalues are purely imaginar, it is easy to check that S_k generates a group of isometries $(\exp(tS_k)_{t\in\mathbf{R}})$.

• To obtain the projections $\Pi_{k,j}$ $(j \in \{1,..,5\})$ on the eigenspaces of S_k, we have to compute the eigenvectors and the transfer matrices, which is a technical step performed in Appendix A. Then the explicit formula for $\Pi_{k,j}$ shows that its norm is uniformly bounded with respect to j and k : in particular, the projections are pseudo-differential operators of order 0.

• The projection on the kernel of S satisfies for all $k \in \mathbf{Z}^3$

$$\mathcal{P}_k = \Pi_{k,3} = \begin{pmatrix} 0 \\ 0 \\ \frac{-|k|}{\alpha\lambda} Id \\ 0\, Id \\ \frac{1}{\lambda} Id \end{pmatrix} \left(0, 0, \frac{-|k|}{\alpha\lambda} Id, 0\, Id, \frac{1}{\lambda} Id\right),$$

Then,

$$\mathcal{P} = \begin{pmatrix} 0 \\ 0 \\ -(-\Delta_x)^{1/2}(\alpha^2 Id - \Delta_x)^{-1/2} \\ 0\, Id \\ \alpha(\alpha^2 Id - \Delta_x)^{-1/2} \end{pmatrix} \left(0, 0, -(-\Delta_x)^{1/2}(\alpha^2 Id - \Delta_x)^{-1/2}, 0\, Id, \alpha(\alpha^2 Id - \Delta_x)^{-1/2}\right)$$

which concludes the proof. □

Lemma 6.1 shows that the linear part of the system (6.1) generates a group of isometries $\mathcal{S}$. Conjugating the equations (6.1) by $\mathcal{S}$ allows then to remove the fast temporal oscillations. Indeed we introduce the new variable

$$W_\epsilon = \mathcal{S}(\frac{t}{\epsilon})U_\epsilon \tag{6.4}$$

which is expected to have uniformly bounded time derivatives.

Lemma 6.2. *Let $U^\epsilon = (div_x v^\epsilon, div_x e^\epsilon,\ curl_x v^\epsilon,\ curl_x e^\epsilon, (-\Delta_x)^{1/2} b^\epsilon)$ where $(v^\epsilon, e^\epsilon, b^\epsilon)$ is a solution of (6.1). Then the vector field W^ϵ defined by (6.4) satisfies the non-autonomous system of nonlinear equations*

$$\partial_t W^\epsilon + Q(\frac{t}{\epsilon}, W^\epsilon, W^\epsilon) = 0 \tag{6.5}$$

where the operator Q is defined by its Fourier coefficients

$$\forall k \in \mathbf{Z}^3 \quad Q_k(t, V, W) = \sum_{l+m=k} \sum_{\eta \in [[1;5]]^3} exp(t\omega_\eta(k; l; m)) s_\eta(k; l; m)) V_l W_m$$

with

$$\omega_\eta(k; l; m) = \lambda_{\eta_1}(k) - \lambda_{\eta_2}(k) - \lambda_{\eta_3}(k). \tag{6.6}$$

Moreover, the tensor $s_\eta(l+m; l; m)$ satisfies

$$|s_\eta(l+m; l; m)| \leq C(|l| + |m|)$$

for some non negative constant C independant of l and m.

Proof. We can rewrite the equation satisfied by U^ϵ in the following way

$$\partial_t U^\epsilon + \frac{1}{\epsilon} S U^\epsilon + R(U^\epsilon; U^\epsilon) = 0 \tag{6.7}$$

where R is the symmetric bilinear operator obtained by polarization of the quadratic forms

$$R(U, U) = \begin{pmatrix} \mathrm{div}_x(v \cdot \nabla v + \alpha v \wedge b) \\ \mathrm{div}_x(v \mathrm{div}_x e) \\ \mathrm{curl}_x(v \cdot \nabla v + \alpha v \wedge b) \\ \mathrm{curl}_x(v \mathrm{div}_x e) \\ 0 \end{pmatrix} \tag{6.8}$$

where U is the vector field of coordinates $(\mathrm{div}_x v, \mathrm{div}_x e,\ \mathrm{curl}_x v,\ \mathrm{curl}_x e, (-\Delta_x)^{1/2} b)$, and S is the linear penalization defined in Lemma 6.1.

Conjugating (6.7) by the group $\mathcal{S}$ provides

$$\mathcal{S}(\frac{t}{\epsilon}) \partial_t U^\epsilon + \frac{1}{\epsilon} \mathcal{S}(\frac{t}{\epsilon}) S U^\epsilon + \mathcal{S}(\frac{t}{\epsilon}) R(U^\epsilon; U^\epsilon) = 0,$$

or equivalently

$$\partial_t (\mathcal{S}(\frac{t}{\epsilon}) U^\epsilon) + \mathcal{S}(\frac{t}{\epsilon}) R(U^\epsilon; U^\epsilon) = 0.$$

Then

$$\partial_t W^\epsilon + \mathcal{S}(\frac{t}{\epsilon}) R(\mathcal{S}(\frac{-t}{\epsilon}) W^\epsilon; \mathcal{S}(\frac{-t}{\epsilon}) W^\epsilon) = 0,$$

which can be rewritten on the Fourier side

$$\partial_t W_k^\epsilon + \mathcal{S}_k(\frac{t}{\epsilon}) \sum_{l+m=k} \tilde{R}_{lm}[\mathcal{S}_l(\frac{-t}{\epsilon}) W_l^\epsilon; \mathcal{S}_m(\frac{-t}{\epsilon}) W_m^\epsilon] = 0$$

where R_{lm} are the Fourier coefficients of R. By definition (6.8) of R

$$|R_{lm}| \leq C(|l| + |m|).$$

In order to describe explicitly the dependence with respect to time, we decompose the penalization according to the different modes. From Lemma 6.1 we deduce that

$$\mathcal{S}_k(t) = P_k^{-1} \begin{pmatrix} \lambda_1(k) & 0 & 0 & 0 & 0 \\ 0 & \lambda_2(k) & 0 & 0 & 0 \\ 0 & 0 & \lambda_3(k)Id & 0 & 0 \\ 0 & 0 & 0 & \lambda_4(k)Id & 0 \\ 0 & 0 & 0 & 0 & \lambda_5(k)Id \end{pmatrix} P_k$$

where Id denotes the identity of $\mathbf{C}^3$. Then

$$\partial_t W_k^\epsilon + \sum_{l+m=k} \sum_{\eta \in [[1;5]]^3} \exp(\frac{t}{\epsilon}\omega_\eta(k,l,m)) s_\eta(k,l,m)[W_l^\epsilon; W_m^\epsilon] = 0$$

where the phase ω is defined by

$$\omega_\eta(k,l,m) = \lambda_{\eta_1}(k) - \lambda_{\eta_2}(l) - \lambda_{\eta_3}(m)$$

and the tensor s satisfies

$$s_\eta(k,l,m)[W_l; W_m] = \Pi_{k,\eta_1} R_{lm}[\Pi_{l,\eta_2} W_l; \Pi_{m,\eta_3} W_m].$$

Since $||\Pi_{k,j}|| \leq C$ and $|R_{lm}| \leq C(|l|+|m|)$, we check that $|s_\eta(k,l,m)| \leq C(|l|+|m|)$. □

Lemma 6.2 seems to indicate that W^ϵ as well as its time derivatives should be uniformly bounded with respect to ϵ. Such a result would imply in particular that (up to extraction of a subsequence) the family W^ϵ converges strongly to some W, which is supposed to solve the following limit system

$$\partial_t W + \bar{Q}(W,W) = 0 \tag{6.9}$$

where the autonomous bilinear operator $(V,W) \mapsto \bar{Q}(V,W)$ is defined by its Fourier coefficients

$$\forall k \in \mathbf{Z}^3, \quad \bar{Q}_k(V,W) = \sum_{\substack{l+m=k \\ \omega_\eta(k,l,m)=0}} s_\eta(k,l,m)[V_l; W_m], \tag{6.10}$$

the other terms converging weakly to 0 because of fast oscillations (formal time averaging).

Of course, we are not able to justify directly such an asymptotic. The idea consists then in describing precisely the formal limiting system (6.9) and then in constructing approximate solutions to the original system (6.5). A stability result will then allow to conclude.

6.2. **Description of the resonances.** In order to characterize and to obtain precise estimates on the solution W of the system (6.9), the first step is to understand its structure, in particular the type of coupling produced by the quadratic term. We start by a precise description of the resonances.

Lemma 6.3. *For all $\eta \in \{1,..,5\}^3$ and all $k,l,m \in \mathbf{Z}^3$, define $\omega_\eta(k,l,m)$ by (6.6). Then there exists a set $\mathcal{A} \subset (\mathbf{R}^+_*)^3$ of Lebesgue measure zero such that for all $(a_1,a_2,a_3) \in (\mathbf{R}^+_*)^3 \setminus \mathcal{A}$,*

- $\omega_\eta(l+m,l,m) = 0$ *implies that* $\eta_1 = 3$ *or* $\eta_2 = 3$ *or* $\eta_3 = 3$
- $\omega_\eta(l+m,l,m) = 0$ *with* $\eta_1 = 3 \Rightarrow \{\eta_2,\eta_3\} = \{1,2\}$ *or* $\{\eta_2,\eta_3\} = \{4,5\}$ *or* $\eta_2 = \eta_3 = 3$*;*
- $\omega_\eta(l+m,l,m) = 0$ *with* $\eta_2 = 3 \Rightarrow \eta_1 = \eta_3$*;*
- $\omega_\eta(l+m,l,m) = 0$ *with* $\eta_3 = 3 \Rightarrow \eta_1 = \eta_2$*.*

Moreover, if $\omega_\eta(l+m,l,m) \neq 0$,

$$|\omega_\eta(l+m,l,m)|^{-1} \leq C(1+|l|)^s(1+|m|)^s, \tag{6.11}$$

for some nonnegative constants C and s depending only on (a_1,a_2,a_3).

Proof. The results concerning the resonances, i.e. the solutions of the dispersion equation

$$\omega_\eta(l+m,l,m) = 0$$

come from algebraic properties of the functions $k \mapsto \lambda_j(k)$ defined by (6.3) : the main argument is the small divisor estimate stated in Appendix B.

We expect all resonances to involve generically at least one zero eigenvalue. To establish such a claim, we consider

$$q(l,m) = \Pi_{\eta \in \{1,2,4,5\}^3}\, \omega_\eta(l+m,l,m)$$

By (6.6), $q(l,m)$ is a polynomial with respect to the variables $\lambda_j(k)$ (for $j \in \{1,2,4,5\}$ and $k \in \{l,m,l+m\}$). Considerations of symmetry ensure that it is actually a polynomial with respect to $\sigma_j(k)$ ($j \in [\![1,4]\!], k \in \{l,m,l+m\}$), where $(\sigma_j)_{j\in[\![1,4]\!]}$ are the symmetrical functions associated to $(\lambda_j)_{j\in\{1,2,4,5\}}$. Computing these elementary symmetrical functions shows that $q(l,m)$ is a polynomial with respect to $|l|^2$, $|m|^2$ and $|l+m|^2$. Then there exists a polynomial P such that

$$q(l,m) = P\left(\frac{l_1}{a_1}, \frac{l_2}{a_2}, \frac{l_3}{a_3}, \frac{m_1}{a_1}, \frac{m_2}{a_2}, \frac{m_3}{a_3}\right)$$

By Proposition 9.1, there exist a set $\mathcal{A} \subset (\mathbf{R}^+_*)^3$ of Lebesgue measure zero and $\Omega \subset \mathbf{Z}^6$ such that for all $(a_1,a_2,a_3) \in (\mathbf{R}^*_+)^3 \setminus \mathcal{A}$, $\exists (C,s)$,

$$\begin{aligned} &\forall (l,m) \in \Omega,\ q(l,m) \equiv 0 \\ &\forall (l,m) \in \mathbf{Z}^6 \setminus \Omega,\ |q(l,m)|^{-1} \leq C(1+|l|)^s(1+|m|)^s \end{aligned} \tag{6.12}$$

In particular, as $|\omega_\eta(l+m,l,m)| \le 3\left(1+\frac{2}{\alpha}\right)(1+|l|)(1+|m|)$ for all $\eta \in \{1,2,4,5\}^3$ and all $l,m \in \mathbf{Z}^3$, this implies

$$\forall (l,m) \in \mathbf{Z}^6 \setminus \Omega,\ \forall \eta \in \{1,2,4,5\}^3,\ |\omega_\eta(l+m,l,m)|^{-1} \le C'(1+|l|)^{s'}(1+|m|)^{s'}$$

for some nonnegative constants C' and s'.

It remains then to prove that $\Omega = \emptyset$. Consider $(l,m) \in \mathbf{Z}^6$. If η_1, η_2 and η_3 are not equal to 3, $\lambda_{\eta_1}(l+m)$, $\lambda_{\eta_2}(l)$ and $\lambda_{\eta_3}(m)$ tend to 1 or -1 as $a_1,a_2,a_3 \to +\infty$. Then, for all η, $\omega_\eta(l+m,l,m)$ converges to an odd number as $a_1,a_2,a_3 \to +\infty$. Thus $q(l,m) \not\equiv 0$ and $(l,m) \notin \Omega$. Then (6.12) provides the first assertion in Lemma 6.3, i.e. a necessary condition for having a resonance.

In order to complete our characterization of the resonances, we have then to consider the cases where (at least) one of the eigenvalues is zero. By a symmetry argument, it is enough to study one case, for instance $\eta_1 = 3$.

Then, $\omega_\eta(l+m,l,m) = 0$ if and only if

$$\lambda_{\eta_2}(l) + \lambda_{\eta_3}(m) = 0,$$

which is obviously equivalent to the following

- $\eta_2 = \eta_3 = 3$;
- $\{\eta_2,\eta_3\} = \{1,2\}$;
- $\{\eta_2,\eta_3\} = \{4,5\}$ and $|l| = |m|$.

In order to obtain an estimate for $|\omega_\eta(l+m,l,m)|^{-1}$ when $\lambda_{\eta_2}(l) + \lambda_{\eta_3}(m) \neq 0$, we compute it. If $\lambda_{\eta_2}(l)$ and $\lambda_{\eta_3}(m)$ have the same sign, $|\omega_\eta(l+m,l,m)| \ge 2$. If they have opposite signs,

- either $|\omega_\eta(l+m,l,m)| = \left|\sqrt{1+\frac{|l|^2}{\alpha^2}} - 1\right| \ge C$;
- or $|\omega_\eta(l+m,l,m)| = \left|\sqrt{1+\frac{|m|^2}{\alpha^2}} - 1\right| \ge C$;
- or $|\omega_\eta(l+m,l,m)| = \left|\sqrt{1+\frac{|l|^2}{\alpha^2}} - \sqrt{1+\frac{|m|^2}{\alpha^2}}\right| \ge C(1+|l|)^{-1}(1+|m|)^{-1}$;

where C depends only on a_1,a_2,a_3. □

Note that the mean-field, that is the projection of W on the kernel of S, seems to play a particular role. In the sequel, we will denote $\bar{W}$ this projection and $W_{osc} = W - \bar{W}$. Of course, W_{osc} does not depend on ϵ, but we call it the oscillating part of W since it gives the oscillating part of U :

$$\bar{U} = \mathcal{S}(-\frac{t}{\epsilon})\bar{W} = \bar{W},$$

$$U_{osc} = \mathcal{S}(-\frac{t}{\epsilon})W_{osc},$$

should be respectively the weak limit of U^ϵ, and the oscillating part of U^ϵ which describes exactly the defect of strong convergence of U^ϵ to $\bar{U}$.

6.3. **Structure of the limiting system.** Equipped with the previous result on the resonances, we are now able to analyze the structure of the limiting system. More precisely, we will prove the following

Proposition 6.1. *There exists a set $\mathcal{A} \subset (\mathbf{R}_*^+)^3$ of Lebesgue measure zero such that for all $(a_1, a_2, a_3) \in (\mathbf{R}_*^+)^3 \setminus \mathcal{A}$, any solution W of (6.9) can be decomposed in*

$$W = \bar{W} + W_{osc}$$

where $\bar{W}$ solves a locally well-posed nonlinear autonomous equation (which is more or less equivalent to the limiting equation (5.1) under a convenient change of variables), and W_{osc} is governed by a linear system of equations whose coefficients depend on $\bar{W}$.

6.3.1. *The mean-field equation.* The mean-field $\bar{W}$ is the projection of W on the kernel $\mathrm{Ker}(S)$, which means that

$$\bar{W}_k = \Pi_{k,3} W_k.$$

Then from (6.9) we deduce that

$$\partial_t \overline{W}_k + \Pi_{k,3} \sum_{\substack{l+m=k, \eta \in [[1;5]]^3, \\ \omega_\eta(k,l,m)=0,}} s_\eta(k;l;m)[W_l; W_m] = 0$$

or equivalently

$$\partial_t \overline{W}_k + \sum_{\substack{l+m=k, \eta \in [[1;5]]^3, \\ \omega_\eta(k,l,m)=0, \eta_1=3}} s_\eta(k;l;m)[W_l; W_m] = 0 \tag{6.13}$$

By Lemma 6.3, we know that

$$\omega_\eta(k,l,m) = 0 \text{ with } \lambda_{\eta_1}(k) = 0$$

if and only if $\{\eta_2, \eta_3\} = \{1, 2\}$, or $\eta_2 = \eta_3 = 3$, or $\{\eta_2, \eta_3\} = \{4, 5\}$ with $|l| = |m|$.

Then (6.13) can be rewritten

$$\begin{aligned} &\partial_t \overline{W}_k + \sum_{l+m=k} s_{3,3,3}(k;l;m)[\bar{W}_l; \bar{W}_m] \\ &= - \sum_{\substack{l+m=k, \\ \{\eta_2,\eta_3\}=\{1,2\}, \eta_1=3}} s_\eta(k;l;m)[W_l; W_m] - \sum_{\substack{l+m=k, |l|=|m| \\ \{\eta_2,\eta_3\}=\{4,5\}, \eta_1=3}} s_\eta(k;l;m)[W_l; W_m] \end{aligned}$$

Let us then introduce the coordinates of W_k in the following basis of eigenvectors

$$W_k = \begin{pmatrix} 1 \\ i \\ 0 \\ 0 \\ 0 \end{pmatrix} \mu_k + \begin{pmatrix} 1 \\ -i \\ 0 \\ 0 \\ 0 \end{pmatrix} \nu_k + \begin{pmatrix} 0 \\ 0 \\ \frac{-|k|}{\alpha\lambda_k} Id \\ 0\, Id \\ \frac{1}{\lambda_k} Id \end{pmatrix} \phi_k + \begin{pmatrix} 0 \\ 0 \\ \frac{1}{\lambda_k} Id \\ -i\, Id \\ \frac{-|k|}{\alpha\lambda_k} Id \end{pmatrix} \chi_k + \begin{pmatrix} 0 \\ 0 \\ \frac{1}{\lambda_k} Id \\ i\, Id \\ \frac{-|k|}{\alpha\lambda_k} Id \end{pmatrix} \psi_k,$$

where $\mu_k, \nu_k \in \mathbf{C}$ and $\phi_k, \chi_k, \psi_k \in \mathbf{C}^3$. Note that, since W is real-valued, $W_k^* = W_{-k}$ and

$$\mu_k^* = \nu_{-k}, \quad \psi_k^* = \chi_{-k} \quad \text{and} \quad \phi_k^* = \phi_{-k}.$$

Moreover, with these notations, we have

$$\bar{W}_k = \begin{pmatrix} 0 \\ 0 \\ \frac{-|k|}{\alpha\lambda_k} Id \\ 0\, Id \\ \frac{1}{\lambda_k} Id \end{pmatrix} \phi_k.$$

The problem is now to determine the equation for the vector-valued coordinate ϕ_k :

$$\phi_k = \left(0, 0, \frac{-|k|}{\alpha\lambda_k} Id, 0\, Id, \frac{1}{\lambda_k} Id\right) W_k = \left(0, 0, \frac{-|k|}{\alpha\lambda_k} Id, 0\, Id, \frac{1}{\lambda_k} Id\right) \bar{W}_k.$$

By definition of $s_\eta(k, l, m)$, we have

$$\begin{aligned}
&\partial_t \phi_k + \left(0, 0, \frac{-|k|}{\alpha\lambda_k} Id, 0\, Id, \frac{1}{\lambda_k} Id\right) \sum_{l+m=k} R_{lm}[\bar{W}_l, \bar{W}_m] \\
&= -\left(0, 0, \frac{-|k|}{\alpha\lambda_k} Id, 0\, Id, \frac{1}{\lambda_k} Id\right) \sum_{\substack{l+m=k, \\ \{\eta_2,\eta_3\}=\{1,2\}}} R_{lm}[\Pi_{l,\eta_2} W_l, \Pi_{m,\eta_3} W_m] \\
&\quad - \left(0, 0, \frac{-|k|}{\alpha\lambda_k} Id, 0\, Id, \frac{1}{\lambda_k} Id\right) \sum_{\substack{l+m=k, \\ \{\eta_2,\eta_3\}=\{4,5\}, \quad |l|=|m|}} R_{lm}[\Pi_{l,\eta_2} W_l, \Pi_{m,\eta_3} W_m]
\end{aligned}$$

which can be rewritten

$$\partial_t \phi_k + \left(0, 0, \frac{-|k|}{\alpha\lambda_k} Id, 0\, Id, \frac{1}{\lambda_k} Id\right) \sum_{k=l+m} R_{lm} \left[\begin{pmatrix} 0 \\ 0 \\ \frac{-|l|}{\alpha\lambda_l} Id \\ 0\, Id \\ \frac{1}{\lambda_l} Id \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \frac{-|m|}{\alpha\lambda_m} Id \\ 0\, Id \\ \frac{1}{\lambda_m} Id \end{pmatrix} \right] \phi_l \otimes \phi_m = -T_k^{\mu\nu} - T_k^{\chi\psi}$$

where

$$T_k^{\mu\nu} = 2\left(0, 0, \frac{-|k|}{\alpha\lambda_k} Id, 0\, Id, \frac{1}{\lambda_k} Id\right) \sum_{k=l+m} R_{lm} \left[\begin{pmatrix} 1 \\ i \\ 0\, Id \\ 0\, Id \\ 0\, Id \end{pmatrix}, \begin{pmatrix} 1 \\ -i \\ 0\, Id \\ 0\, Id \\ 0\, Id \end{pmatrix} \right] \mu_l \nu_m$$

$$T_k^{\chi\psi} = 2\left(0, 0, \frac{-|k|}{\alpha\lambda_k} Id, 0\, Id, \frac{1}{\lambda_k} Id\right) \sum_{k=l+m} R_{lm} \left[\begin{pmatrix} 0 \\ 0 \\ \frac{1}{\lambda_l} Id \\ -i\, Id \\ \frac{|k|}{\alpha\lambda_l} Id \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \frac{1}{\lambda_m} Id \\ i\, Id \\ \frac{|k|}{\alpha\lambda_m} Id \end{pmatrix}\right] \chi_l \otimes \psi_m.$$

Note that we only need to compute two (vector-valued) components of R_{lm}.

A symmetry argument (developped in Appendix C) allows actually to prove that both terms $T_k^{\mu\nu}$ and $T_k^{\chi\psi}$ vanish, so that ϕ satisfies an autonomous equation.

Keeping only the terms involving ϕ, we get

$$\partial_t \phi_k + \frac{-|k|}{\alpha\lambda_k} ik \wedge \sum_{k=l+m} \left[\left(\frac{-il \wedge \phi_l}{\alpha|l|\lambda_l}\right).im\left(\frac{-im \wedge \phi_m}{\alpha|m|\lambda_m}\right) + \alpha\left(\frac{-il \wedge \phi_l}{\alpha|l|\lambda_l}\right) \wedge \frac{\phi_m}{|m|\lambda_m}\right] = 0$$

From this equation, we can deduce the equation satisfied by the mean field

$$\bar{U} = \bar{W} = \begin{pmatrix} 0 \\ 0 \\ -(-\Delta_x)^{1/2}(\alpha^2 Id - \Delta_x)^{-1/2}\phi \\ 0 \\ \alpha(\alpha^2 Id - \Delta_x)^{-1/2}\phi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \operatorname{curl}_x \bar{v} \\ 0 \\ (-\Delta_x)^{1/2}\bar{b} \end{pmatrix}$$

which is exactly the limiting system (5.1) obtained for well-prepared initial data :

$$\partial_t(\operatorname{curl}_x \bar{v} - \alpha\bar{b}) + \operatorname{curl}_x(\bar{v}.\nabla_x \bar{v} + \alpha\bar{v} \wedge \bar{b}) = 0 \tag{6.14}$$

since $(\operatorname{curl}_x \bar{v} - \alpha\bar{b})_k = -\frac{\lambda\alpha}{|k|}\phi_k$.

6.3.2. *The equation for the oscillating part.* By definition, the oscillating part W_{osc} of W satisfies $W_{osc} = W - \bar{W}$ where $\bar{W} = \mathcal{P}W$ is the projection of W on $\operatorname{Ker}(S)$: in other words, W_{osc} is the sum of the projections of W on the other eigenspaces of S. The goal of this paragraph is then to determine the equations governing the coordinates μ_k, ν_k, ψ_k and χ_k.

From (6.9) we deduce that

$$\partial_t \Pi_{k,1} W_k + \Pi_{k,1} \sum_{\substack{l+m=k, \eta\in[[1;5]]^3,\\ \omega_\eta(k,l,m)=0,}} s_\eta(k;l;m)[W_l; W_m] = 0$$

or equivalently

$$\partial_t \mu_k + \left(\frac{1}{2}, -\frac{i}{2}, 0\, Id, 0\, Id, 0\, Id\right) \sum_{\substack{l+m=k, \eta\in[[1;5]]^3,\\ \omega_\eta(k,l,m)=0, \eta_1=1}} s_\eta(k;l;m)[W_l; W_m] = 0 \tag{6.15}$$

By Lemma 6.3, we know that

$$\omega_\eta(k,l,m) = 0 \text{ with } \eta_1 = 1$$

if and only if $\eta_2 = 3$ and $\eta_3 = 1$, or $\eta_2 = 1$ and $\eta_3 = 3$. Then (6.15) can be rewritten

$$\partial_t \mu_k + 2\left(\frac{1}{2}, -\frac{i}{2}, 0\,Id, 0\,Id, 0\,Id\right) \sum_{l+m=k} R_{lm}\left[\begin{pmatrix} 0 \\ 0 \\ \frac{-|l|}{\alpha\lambda_l}\,Id \\ 0\,Id \\ \frac{1}{\lambda_l}\,Id \end{pmatrix}, \begin{pmatrix} 1 \\ i \\ 0\,Id \\ 0\,Id \\ 0\,Id \end{pmatrix}\right] \phi_l \mu_m = 0$$

or in explicit form

(6.16)

$$\partial_t \mu_k + ik\cdot \sum_{k=l+m} \Bigg[\left(\frac{il}{\alpha\lambda_l|l|}\wedge(-|l|)\phi_l\right)\cdot im\left(\frac{-im}{|m|^2}\mu_m\right) + \left(\frac{-im}{|m|^2}\mu_m\right)\cdot il\left(\frac{il}{\alpha\lambda_l|l|}\wedge(-|l|)\phi_l\right)$$
$$+\alpha\left(\frac{-im}{|m|^2}\mu_m\wedge\frac{\phi_l}{|l|\lambda_l}\right) - i\left(\frac{il}{\alpha\lambda_l|l|}\wedge(-|l|)\phi_l\right) i\mu_m\Bigg] = 0$$

In the same way,

(6.17)

$$\partial_t \nu_k + ik\cdot \sum_{k=l+m} \Bigg[\left(\frac{il}{\alpha\lambda_l|l|}\wedge(-|l|)\phi_l\right)\cdot im\left(\frac{-im}{|m|^2}\nu_m\right) + \left(\frac{-im}{|m|^2}\nu_m\right)\cdot il\left(\frac{il}{\alpha\lambda_l|l|}\wedge(-|l|)\phi_l\right)$$
$$+\alpha\left(\frac{-im}{|m|^2}\nu_m\wedge\frac{\phi_l}{|l|\lambda_l}\right) + i\left(\frac{il}{\alpha\lambda_l|l|}\wedge(-|l|)\phi_l\right)(-i\nu_m)\Bigg] = 0$$

From (6.9) we also deduce that

$$\partial_t \chi_k + \left(0, 0, \frac{1}{2\lambda_k}\,Id, \frac{i}{2}\,Id, \frac{|k|}{2\alpha\lambda_k}\,Id\right) \sum_{\substack{l+m=k,\eta\in[[1;5]]^3,\\ \omega_\eta(k,l,m)=0,}} s_\eta(k;l;m))[W_l;W_m] = 0.$$

By Lemma 6.3, we know that

$$\omega_\eta(k,l,m) = 0 \text{ with } \eta_1 = 4$$

if and only if $\eta_2 = 3$, $\eta_3 = 4$ and $|k| = |m|$, or $\eta_2 = 4$, $\eta_3 = 3$ and $|k| = |l|$. Then

$$\partial_t \chi_k + 2\left(0, 0, \frac{1}{2\lambda_k}\,Id, \frac{i}{2}\,Id, \frac{|k|}{2\alpha\lambda_k}\,Id\right) \sum_{\substack{l+m=k,\\ |k|=|m|}} R_{lm}\left[\begin{pmatrix} 0 \\ 0 \\ \frac{-|l|}{\alpha\lambda_l}\,Id \\ 0\,Id \\ \frac{1}{\lambda_l}\,Id \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \frac{1}{\lambda_m}\,Id \\ -i\,Id \\ \frac{|m|}{\alpha\lambda_m}\,Id \end{pmatrix}\right] \phi_l \chi_m = 0,$$

which can be rewritten
(6.18)
$$\partial_t\chi_k + \frac{1}{\lambda_k} ik \wedge \sum_{\substack{l+m=k,\\ |k|=|m|}} \left[\left(\frac{-il\wedge\phi_l}{\alpha|l|\lambda_l}\right)\cdot im\left(\frac{im\wedge\chi_m}{|m|^2\lambda_m}\right) + \alpha\left(\frac{-il\wedge\phi_l}{\alpha|l|\lambda_l}\right)\wedge\frac{\chi_m}{\alpha\lambda_m} \right.$$
$$\left. + \left(\frac{im\wedge\chi_m}{|m|^2\lambda_m}\right)\cdot il\left(\frac{-il\wedge\phi_l}{\alpha|l|\lambda_l}\right) + \alpha\left(\frac{im\wedge\chi_m}{|m|^2\lambda_m}\right)\wedge\frac{\phi_l}{|l|\lambda_l}\right] = 0.$$
And in the same way,
(6.19)
$$\partial_t\psi_k + \frac{1}{\lambda_k} ik \wedge \sum_{\substack{l+m=k,\\ |k|=|m|}} \left[\left(\frac{-il\wedge\phi_l}{\alpha|l|\lambda_l}\right)\cdot im\left(\frac{im\wedge\psi_m}{|m|^2\lambda_m}\right) + \alpha\left(\frac{-il\wedge\phi_l}{\alpha|l|\lambda_l}\right)\wedge\frac{\psi_m}{\alpha\lambda_m} \right.$$
$$\left. + \left(\frac{im\wedge\psi_m}{|m|^2\lambda_m}\right)\cdot il\left(\frac{-il\wedge\phi_l}{\alpha|l|\lambda_l}\right) + \alpha\left(\frac{im\wedge\psi_m}{|m|^2\lambda_m}\right)\wedge\frac{\phi_l}{|l|\lambda_l}\right] = 0.$$

In particular, the equations governing μ, ν, χ and ψ are linear pseudo-differential equations whose coefficients depend (linearly) on ϕ and its first derivatives, which concludes the proof of Proposition 6.1.

6.3.3. *Theory for the asymptotic system.* The previous analysis of the structure of the limiting system (6.9) allows to prove that the Cauchy problem is well-posed for smooth initial data.

Proposition 6.2. *Let $(a_1, a_2, a_3) \in (\mathbf{R}^+_*)^3 \setminus \mathcal{A}$ where $\mathcal{A}$ is the set of Lebesgue measure zero defined in Proposition 6.1, and $(W^{in}) \in C^{r+1}(\Omega)$ with $r > 2$. Then there exists $T^* > 0$ such that the system (6.9) admits a unique strong solution $W \in L^\infty_{loc}([0, T^*[, H^r(\Omega))$ with initial data W^{in}.*

Proof. By Proposition 6.1, for all $(a_1, a_2, a_3) \in (\mathbf{R}^+_*)^3 \setminus \mathcal{A}$, any solution W of (6.9) can be decomposed in
$$W = \bar{W} + W_{osc}$$
where $\bar{W}$ solves a well-posed nonlinear autonomous equation (which is more or less equivalent to the limiting equation (5.1) under a convenient change of variables), and W_{osc} is governed by a linear system of equations whose coefficients depend on $\bar{W}$.

If $W^{in} \in C^{r+1}(\Omega)$, then $\bar{W}^{in} = \mathcal{P}W^{in} \in C^{r+1}(\Omega)$ because $\mathcal{P}$ is a pseudo-differential operator of order 0. Moreover, we have identified the autonomous equation for $\bar{W} = (0, 0, \operatorname{curl}_x\bar{v}, 0, (-\Delta_x)^{1/2}\bar{b})^T$ to be exactly the system (5.1) for $(\bar{v}, \bar{b})$. We have seen in Lemma 5.1 that (5.1) can be studied exactly as the 3D incompressible Euler equation. Then there exists $T^* > 0$ such that there exists $\bar{W}$ solution of (6.13) on $[0, T^*[$ and for all $T < T^*$,
$$\|\bar{W}\|_{L^\infty([0,T],C^{r+1})} + \left\|\partial_t\bar{W}\right\|_{L^\infty([0,T],C^r)} \leq \|(\bar{v}, \bar{b})\|_{L^\infty([0,T],C^{r+2})} \leq C,$$

where C depends on T and on $||(\bar{v}^{in}, \bar{b}^{in})||_{C^{r+2}} \leq ||W^{in}||_{C^{r+1}}$.

The equations governing μ, ν, χ and ψ are linear equations of the form

$$\partial_t W_{osc} + \alpha(\phi).\nabla_x W_{osc} = \beta(W_{osc}, \phi)$$

where α is a linear pseudodifferential operator of order 0, β is a bilinear pseudodifferential operator of order 0 with respect to W_{osc} and of order 1 with respect to ϕ, and such that

$$\mathrm{div}_x \alpha(\phi) = 0.$$

Then it is easy to check that

$$\frac{d}{dt}||W_{osc}||^2_{H^s} \leq C||W_{osc}||^2_{H^s}||\phi||_{C^{s+1}},$$

and by Gronwall's inequality

$$||W_{osc}||_{L^\infty([0,T],H^s)} \leq ||W^{in}||^2_{H^s} \exp(CT||\bar{W}||_{L^\infty([0,T],C^{s+1})}).$$

Then, for all $T < T^*$, there exists $C_T > 0$ such that

$$||W||_{L^\infty([0,T],H^r)} + ||\partial_t W||_{L^\infty([0,T],H^{r-1})} \leq C_T. \tag{6.20}$$

□

7. Weak convergence for general initial data

7.1. **Error estimates.** In this section, we want to prove that the solution W of (6.9) provides a good estimate of any solution W^ϵ of (6.5). A natural idea is to use a stability argument of Gronwall type to prove that

$$||W^\epsilon - W||_{L^\infty(\Omega)} \to 0.$$

For instance, if

$$\partial_t W + Q(\frac{t}{\epsilon}, W, W) \to 0 \tag{7.1}$$

the following would hold

$$\partial_t(W - W^\epsilon) + Q(\frac{t}{\epsilon}, W - W^\epsilon, W + W^\epsilon) \to 0$$

and then, by definition of Q,

$$\frac{d}{dt}||W - W^\epsilon||_{L^\infty} \leq C||W - W^\epsilon||_{L^\infty}||W||_{W^{1\infty}} + \mathrm{O}(\epsilon),$$

which would give the expected convergence result by the Gronwall Lemma, since W satisfies the strong estimate $||W||_{W^{1,\infty}} \leq C$.

Nevertheless the convergence (7.1) does not hold in norm but only in a weak sense (recall that oscillating terms have been neglected in order to obtain the asymptotic system). A standard method to study singular perturbation consists then in introducing a small quantity ϵy_ϵ with $\|y_\epsilon\| = O(1)$ in a convenient norm, and such that

$$\partial_t(W+\epsilon y^\epsilon) + Q(\frac{t}{\epsilon}, (W+\epsilon y^\epsilon), (W+\epsilon y^\epsilon)) \to 0. \tag{7.2}$$

The previous stability argument allows then to conclude since $W+\epsilon y^\epsilon$ obviously has the same behaviour as W.

Lemma 7.1. *Let $(W^{in}) \in C^{r+1}(\Omega)$ with $r > 3$. Denote by $W \in L^\infty([0,T], H^r)$ the solution of (6.9) with initial data W^{in}; define y^ϵ by its Fourier coefficients*

$$\forall k \in \mathbf{Z}^3,\ y^\epsilon_k = - \sum_{\substack{l+m=k,\ |l|+|m| \leq |\log \epsilon|,\\ \eta \in [\![1,5]\!]^3,\ \omega_\eta(k,l,m)\neq 0}} \frac{\exp\left(\frac{t}{\epsilon}\omega_\eta(k,l,m)\right)}{\omega_\eta(k,l,m)} s_\eta(k,l,m) W_l W_m \tag{7.3}$$

Then,

- *there exists a nonnegative constant C such that*
$$\|y^\epsilon\|_{L^\infty([0,T],H^r)} \leq C|\log \epsilon|^{2s+1}$$
where s depends only on $(a_1,a_2,a_3) \in (\mathbf{R}^+_*)^3 \setminus \mathcal{A}$.
- *there exists $\delta \in C(\mathbf{R}^+)$ with $\delta(0)=0$ such that*
$$\left\|\partial_t(W+\epsilon y^\epsilon) + Q\left(\frac{t}{\epsilon}, (W+\epsilon y^\epsilon), (W+\epsilon y^\epsilon)\right)\right\|_{L^\infty([0,T],H^2)} \leq \delta(\epsilon)\,.$$

Proof. By Proposition 6.2, the solution W of (6.9) with initial data $W^{in} \in C^{r+1}(\Omega)$ satisfies the following regularity estimate

$$\|W\|_{L^\infty([0,T],H^r)} + \|\partial_t W\|_{L^\infty([0,T],H^{r-1})} \leq C_T$$

for some nonnegative constant C_T, as soon as T is strictly less than the lifespan T^*.

On the other hand, by Lemma 6.3, there exists nonnegative constants (C,s) such that

$$\forall (l,m,\eta),\ \omega_\eta(l+m,l,m) \neq 0 \Rightarrow |\omega_\eta(l+m,l,m)|^{-1} \leq C(1+|l|)^s(1+|m|)^s\,.$$

By Lemma 6.2,

$$|s_\eta(l+m,l,m)| \leq C(|l|+|m|)$$

Combining these last two estimates leads to

$$\begin{aligned} |y^\epsilon_k| &\leq C \sum_{l+m=k,\, |l|+|m| \leq |\log\epsilon|} (1+|l|)^s(1+|m|)^s(|l|+|m|)|W_l||W_m| \\ &\leq C|\log\epsilon|^{2s+1} \sum_{l+m=k} |W_l||W_m|\,. \end{aligned}$$

Then

$$(1+|k|^2)^{r/2}|y^\epsilon_k| \leq C|\log\epsilon|^{2s+1} \sum_{l+m=k} (1+|l|^2)^{r/2}|W_l|(1+|m|^2)^{r/2}|W_m|$$

which, together with (6.20) gives the expected bound on $\|y^\epsilon\|_{L^\infty([0,T],H^r)}$.

In the next step we check that $W+\epsilon y^\epsilon$ approximately verifies (6.5). By (6.9) and (7.3),

$$\begin{aligned}
\partial_t(W+\epsilon y^\epsilon)_k = &-\sum_{\substack{l+m=k,\\ \omega_\eta(k,l,m)=0}} s_\eta(k,l,m)[W_l,W_m] \\
&-\sum_{\substack{l+m=k,\ |l|+|m|\le|\log\epsilon|,\\ \omega_\eta(k,l,m)\neq 0}} \exp\left(\frac{t}{\epsilon}\omega_\eta(k,l,m)\right) s_\eta(k,l,m)[W_l,W_m] \\
&-\epsilon \sum_{\substack{l+m=k,\ |l|+|m|\le|\log\epsilon|,\\ \omega_\eta(k,l,m)\neq 0}} \frac{\exp\left(\frac{t}{\epsilon}\omega_\eta(k,l,m)\right)}{\omega_\eta(k,l,m)} s_\eta(k,l,m)\partial_t[W_l,W_m]
\end{aligned}$$

from which we deduce that

$$\begin{aligned}
&\partial_t(W+\epsilon y^\epsilon)_k + Q_k(\frac{t}{\epsilon}, W+\epsilon y^\epsilon, W+\epsilon y^\epsilon) \\
&= \sum_{\substack{l+m=k,|l|+|m|>|\log\epsilon|\\ \omega_\eta(k,l,m)\neq 0}} \exp\left(\frac{t}{\epsilon}\omega_\eta(k,l,m)\right) s_\eta(k,l,m)[W_l,W_m] + \epsilon Q_k\left(\frac{t}{\epsilon}, y^\epsilon, 2W+\epsilon y^\epsilon\right) \\
&-\epsilon \sum_{\substack{l+m=k,\ |l|+|m|\le|\log\epsilon|,\\ \omega_\eta(k,l,m)\neq 0}} \frac{\exp\left(\frac{t}{\epsilon}\omega_\eta(k,l,m)\right)}{\omega_\eta(k,l,m)} s_\eta(k,l,m)\partial_t[W_l,W_m] \,.
\end{aligned}$$

The estimates on $s_\eta(l+m,l,m)$ and $\omega_\eta(l+m,l,m)$ give then

$$\begin{aligned}
(1+|k|^2)&\left|\partial_t(W+\epsilon y^\epsilon)_k + Q_k\left(\frac{t}{\epsilon}, W+\epsilon y^\epsilon, W+\epsilon y^\epsilon\right)\right| \\
&\le C \sum_{l+m=k,|l|+|m|>|\log\epsilon|} (|l|+|m|)^{-(r-3)}(1+|l|^2)^{r/2}|W_l|(1+|m|^2)^{r/2}|W_m| \\
&+ C\epsilon \sum_{l+m=k} (1+|l|+|m|)^3 |y_l^\epsilon||(2W+\epsilon y^\epsilon)_m| \\
&+ C\epsilon \sum_{\substack{l+m=k,\ |l|+|m|\le|\log\epsilon|,\\ \omega_\eta(k,l,m)\neq 0}} (1+|l|)^s(1+|m|)^s(1+|l|+|m|)^3|\partial_t W_l||W_m|
\end{aligned}$$

which can be rewritten

$$\begin{aligned}
\left\|\partial_t(W+\epsilon y^\epsilon)_k + Q_k\left(\frac{t}{\epsilon}, W+\epsilon y^\epsilon, W+\epsilon y^\epsilon\right)\right\|_{H^2} &\le C|\log\epsilon|^{-(r-3)}\|W\|_{H^r}^2 \\
&+ C\epsilon\|y^\epsilon\|_{H^3}\|2W+\epsilon y^\epsilon\|_{H^3} \\
&+ C\epsilon|\log\epsilon|^{2s+3}\|\partial_t W\|_{L^2}\|W\|_{L^2} \,.
\end{aligned}$$

Using the a priori estimates on W and y^ϵ leads to the expected result. □

7.2. **Convergence result.** The weak compactness results established in the well-prepared case are still valid : up to extraction of a subsequence, we then have

$$\epsilon E^{\epsilon,\alpha} \rightharpoonup e, \quad B^{\epsilon,\alpha} \rightharpoonup b \text{ in } w^* - L^\infty(\mathbf{R}^+, L^2(\Omega))$$

$$\rho^{\epsilon,\alpha} \rightharpoonup \rho, \quad \hat{j}^{\epsilon,\alpha} \rightharpoonup j = \rho v \text{ in the sense of measures.}$$

Moreover, using the Poisson equation, we obtain that

$$\rho = 1.$$

It remains then to identify the weak limits v, e and b, and more precisely to establish that $e = 0$, and (v, b) is a solution of (5.1), or in other words that

$$\begin{pmatrix} 0 \\ 0 \\ \operatorname{curl}_x v \\ 0 \\ (-\Delta_x)^{1/2} b \end{pmatrix} = \bar{W} = \mathcal{P} W$$

where W is the solution of (6.9).

As the stability method used here can only provide strong convergence results, we actually need a more precise description of the asymptotic behaviour. As $W + \epsilon y^\epsilon$ approximately verifies the macroscopic system (6.5), we expect $U_\epsilon = \mathcal{S}\left(-\frac{t}{\epsilon}\right)(W + \epsilon y^\epsilon)$ to provide a good approximation of the macroscopic quantities

$$(\operatorname{div}_x \hat{j}^{\epsilon,\alpha}, \operatorname{div}_x(\epsilon E^{\epsilon,\alpha}),\ \operatorname{curl}_x \hat{j}^{\epsilon,\alpha},\ \operatorname{curl}_x(\epsilon E^{\epsilon,\alpha}), (-\Delta_x)^{1/2} B^{\epsilon,\alpha})$$

defined from the Vlasov-Maxwell system. Indeed we will prove that

$$\begin{aligned} \hat{j}^{\epsilon,\alpha} &\sim u_{\epsilon,\alpha} \equiv -\nabla_x(-\Delta_x)^{-1} U_{\epsilon,1} + \operatorname{curl}_x(-\Delta_x)^{-1} U_{\epsilon,3}, \\ \epsilon E^{\epsilon,\alpha} &\sim e_{\epsilon,\alpha} \equiv -\nabla_x(-\Delta_x)^{-1} U_{\epsilon,2} + \operatorname{curl}_x(-\Delta_x)^{-1} U_{\epsilon,4}, \\ B^{\epsilon,\alpha} &\sim b_{\epsilon,\alpha} \equiv (-\Delta_x)^{-1/2} U_{\epsilon,5}. \end{aligned}$$

By Lemma 7.1, we know that there exists $\delta \in C(\mathbf{R}^+)$ with $\delta(0) = 0$ such that

$$\left\| \partial_t (W + \epsilon y^\epsilon) + Q\left(\frac{t}{\epsilon}, (W + \epsilon y^\epsilon), (W + \epsilon y^\epsilon)\right) \right\|_{L^\infty([0,T],H^2)} \leq \delta(\epsilon)\,,$$

from which we deduce that

$$\left\| \partial_t U_\epsilon + \frac{1}{\epsilon} S U_\epsilon + R(U_\epsilon, U_\epsilon) \right\|_{L^\infty([0,T],H^2)} \leq \delta(\epsilon)\,.$$

Comparing both systems (6.7) and (4.2) shows that

$$A^{\epsilon,\alpha}(u_{\epsilon,\alpha}, e_{\epsilon,\alpha}, b_{\epsilon,\alpha}) = O(\delta(\epsilon))$$

in $L^\infty([0,T]\times\Omega)$. Then plugging this last identity in the stability inequality (4.8) leads to

$$H^{\epsilon,\alpha}_{u_{\epsilon,\alpha},e_{\epsilon,\alpha},b_{\epsilon,\alpha}}(t) \leq H^{\epsilon,\alpha}_{u_{\epsilon,\alpha},e_{\epsilon,\alpha},b_{\epsilon,\alpha}}(0)\exp(6\int_0^t \|D_x v(s)\|_{L^\infty} ds) + O(\gamma) + O(\delta(\epsilon)).$$

The assumption on the initial data gives exactly

$$H^{\epsilon,\alpha}_{u_{\epsilon,\alpha},e_{\epsilon,\alpha},b_{\epsilon,\alpha}}(0) = H^{\epsilon,\alpha}_{u,e,b}(0) + O(\epsilon) \to 0$$

with

$$u \equiv -\nabla_x(-\Delta_x)^{-1}U_1^{in} + \text{ curl}_x(-\Delta_x)^{-1}U_3^{in},$$
$$e \equiv -\nabla_x(-\Delta_x)^{-1}U_2^{in} + \text{ curl}_x(-\Delta_x)^{-1}U_4^{in},$$
$$b \equiv (-\Delta_x)^{-1/2}U_5^{in} \text{ where } U^{in} = W^{in}.$$

We then deduce

$$H^{\epsilon,\alpha}_{u_{\epsilon,\alpha},e_{\epsilon,\alpha},b_{\epsilon,\alpha}}(t) \to 0 \text{ as } \epsilon,\gamma \to 0,$$

for all $t < T^*$ where T^* is the lifespan of the strong solution of (5.1). In particular,

$$\epsilon E^{\epsilon,\alpha} \rightharpoonup 0, \quad B^{\epsilon,\alpha} \rightharpoonup b \text{ in } w - L^\infty([0,T], L^2(\Omega)).$$

To get the convergence result on the current, a natural idea is to use the following estimate

$$\int \frac{|\hat{j}^{\epsilon,\alpha} - \hat{\rho}^{\epsilon,\alpha}u_{\epsilon,\alpha}|^2}{\hat{\rho}^{\epsilon,\alpha}} \leq 2H^{\epsilon,\alpha}_{u_{\epsilon,\alpha},e_{\epsilon,\alpha},b_{\epsilon,\alpha}}$$

as in the well-prepared case. Nevertheless, since $u_{\epsilon,\alpha}$ has an oscillating part, we should need some regularity with respect to time on $\hat{\rho}^{\epsilon,\alpha}$ and $\sqrt{\hat{\rho}^{\epsilon,\alpha}}$ to conclude directly, which seems to be a difficult issue. Here we will rather use an argument of weak convergence. Taking weak limits in the Ampere equation (the last equation in (1.4)) shows that the limiting macroscopic quantities (v, e, b) belongs to the kernel of the penalization L, in particular

$$\text{curl}_x b = -\alpha v.$$

We have like this identified the weak limits v, e and b, which concludes the proof of theorem 2.3.

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8. Appendix A
The singular perturbation

The study of the asymptotic behaviour of a singular system such as (6.1) requires a good understanding of the linear perturbation. This first appendix is devoted to technical computations leading to the precise description of the oscillating modes of the linear operator S with symbol

$$S_k = \begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & 0Id & Id & 0Id \\ & & -Id & 0Id & -\frac{|k|}{\alpha}Id \\ & & 0Id & \frac{|k|}{\alpha}Id & 0Id \end{pmatrix}.$$

- We are first interested in the diagonalization of the first block

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

which is actually independent of k.

We have seen that its eigenvalues are i and $-i$. Correponding eigenvectors are $(1,i)$ and $(1,-i)$, from which we deduce that the transfer matrices are

$$P = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \quad \text{and} \quad P^{-1} = \begin{pmatrix} 1/2 & -i/2 \\ 1/2 & i/2 \end{pmatrix}$$

and that the projections $\Pi_{k,1}$ and $\Pi_{k,2}$ can be written

$$\Pi_{k,1} = \begin{pmatrix} 1 \\ i \\ 0\,Id \\ 0\,Id \\ 0\,Id \end{pmatrix} \left(\frac{1}{2}, -\frac{i}{2}, 0\,Id, 0\,Id, 0\,Id\right),$$

$$\Pi_{k,2} = \begin{pmatrix} 1 \\ -i \\ 0\,Id \\ 0\,Id \\ 0\,Id \end{pmatrix} \left(\frac{1}{2}, \frac{i}{2}, 0\,Id, 0\,Id, 0\,Id\right).$$

- The second block can be tensorized as follows

$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & -\frac{|k|}{\alpha} \\ 0 & \frac{|k|}{\alpha} & 0 \end{pmatrix} \otimes Id$$

where Id denotes the identity of $\mathbf{C}^3$.

The eigenvalues of such a matrix are $\lambda_3(k) = 0$, $\lambda_4(k) = i\sqrt{\frac{|k|^2}{\alpha^2}+1}$ and $\lambda_5(k) = -i\sqrt{\frac{|k|^2}{\alpha^2}+1}$. The transfer matrices diagonalizing the first factor can be written

$$P_k = \begin{pmatrix} \frac{-|k|}{\alpha\lambda} & \frac{1}{\lambda} & \frac{1}{\lambda} \\ 0 & -i & i \\ \frac{1}{\lambda} & \frac{|k|}{\alpha\lambda} & \frac{|k|}{\alpha\lambda} \end{pmatrix} \quad \text{and} \quad P_k^{-1} = \begin{pmatrix} \frac{-|k|}{\alpha\lambda} & 0 & \frac{1}{\lambda} \\ \frac{1}{2\lambda} & \frac{i}{2} & \frac{|k|}{2\alpha\lambda} \\ \frac{1}{2\lambda} & -\frac{i}{2} & \frac{|k|}{2\alpha\lambda} \end{pmatrix},$$

with the notation $\lambda = \sqrt{1+|k|^2/\alpha^2}$ from which we deduce the explicit formula for the projections $\Pi_{k,3}$, $\Pi_{k,4}$ and $\Pi_{k,5}$:

$$\Pi_{k,3} = \begin{pmatrix} 0 \\ 0 \\ \frac{-|k|}{\alpha\lambda}\,Id \\ 0\,Id \\ \frac{1}{\lambda}\,Id \end{pmatrix} \left(0, 0, \frac{-|k|}{\alpha\lambda}\,Id, 0\,Id, \frac{1}{\lambda}\,Id\right),$$

$$\Pi_{k,4} = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{\lambda}\,Id \\ -i\,Id \\ \frac{|k|}{\alpha\lambda}\,Id \end{pmatrix} \left(0, 0, \frac{1}{2\lambda}\,Id, \frac{i}{2}\,Id, \frac{|k|}{2\alpha\lambda}\,Id\right),$$

$$\Pi_{k,5} = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{\lambda}\,Id \\ i\,Id \\ \frac{|k|}{\alpha\lambda}\,Id \end{pmatrix} \left(0, 0, \frac{1}{2\lambda}\,Id, -\frac{i}{2}\,Id, \frac{|k|}{2\alpha\lambda}\,Id\right).$$

9. Appendix B
Small divisor estimate

In order to describe the coupling between the various oscillating components through the nonlinear terms, a basic tool is the study of the resonances, i.e. of the solutions $(l,m,\eta) \in (\mathbf{Z}^3)^2 \times [\![1,5]\!]^3$ of the dispersion equation

$$\omega_\eta(l+m,l,m) = \lambda_{\eta_1}(l+m) - \lambda_{\eta_2}(l) - \lambda_{\eta_3}(m) = 0,$$

where $(\lambda_j(k))_{j\in[\![1,5]\!]}$ denote the eigenvalues of R_k for all $k \in \mathbf{Z}^3$. As these eigenvalues are the roots of a polynom with polynomial coefficients in k_1/a_1, k_2/a_2 and k_3/a_3, the following small divisor estimate [5] plays a crucial role to obtain the structure of the limiting equation.

Proposition 9.1. *Let $P(l,m)$ be a polynomial in a_3^{-1} with coefficients that are polynomials in l,m. Then, there exist $\mathcal{A} \subset \mathbf{R}_*^+$ of Lebesgue measure zero and $\Omega \subset \mathbf{Z}^6$ such that*

$$\forall (l,m) \in \Omega,\ P(l,m) \equiv 0$$

$$\forall a_3 \in \mathbf{R}_+^* \setminus \mathcal{A},\ \exists (C,s),\ \forall (l,m) \in \mathbf{Z}^6 \setminus \Omega,\ |P(l,m)|^{-1} \leq C(1+|l|)^s(1+|m|)^s$$

10. Appendix C
Symmetry properties of the bilinear operator R

In order to prove that the mean field satisfies an autonomous equation, we have to establish that there does not exist any constructive coupling between oscillating terms, which is due to the particular form of the nonlinear term.

The explicit formula for R_{lm} shows that

$$\begin{aligned}
&\left(0,0,\frac{-|k|}{\alpha\lambda_k}\,Id, 0\,Id, \frac{1}{\lambda}\,Id\right) R_{l,m}[X_l,Y_m] \\
&= \frac{-|k|}{2\alpha\lambda_k} ik \wedge \left(\left(\frac{-il}{|l|^2} X_{l,1} + \frac{il}{|l|^2} \wedge X_{l,3} \right) .im \left(\frac{-im}{|m|^2} Y_{m,1} + \frac{im}{|m|^2} \wedge Y_{m,3} \right) \right.\\
&\left. +\alpha \left(\frac{-il}{|l|^2} X_{l,1} + \frac{il}{|l|^2} \wedge X_{l,3} \right) \wedge \frac{1}{|m|} Y_{m,5} \right) \\
&+ \frac{-|k|}{2\alpha\lambda_k} ik \wedge \left(\left(\frac{-im}{|m|^2} Y_{m,1} + \frac{im}{|m|^2} \wedge Y_{m,3} \right) .il \left(\frac{-il}{|l|^2} X_{l,1} + \frac{il}{|l|^2} \wedge X_{l,3} \right) \right.\\
&\left. +\alpha \left(\frac{-im}{|m|^2} Y_{m,1} + \frac{im}{|m|^2} \wedge Y_{m,3} \right) \wedge \frac{1}{|l|} X_{l,5} \right)
\end{aligned}$$

Plugging $X_l = (\mu_l, i\mu_l, 0, 0, 0)^T$ and $Y_m = (\nu_m, -i\nu_m, 0, 0, 0)^T$ into the previous identity leads to

$$\begin{aligned} T_k^{\mu\nu} &= \frac{-|k|}{2\alpha\lambda_k} ik \wedge \sum_{k=l+m} \mu_l \nu_m \left(\frac{il}{|l|^2}.im\left(\frac{im}{|m|^2}\right) + \frac{im}{|m|^2}.il\left(\frac{il}{|l|^2}\right)\right) \\ &= +\frac{|k|}{2\alpha\lambda_k} ik \wedge \sum_{k=l+m} \mu_l \nu_m \frac{il.m}{|l|^2|m|^2} k = 0 \end{aligned}$$

In the same way, for $X_l = (0, 0, \frac{1}{\lambda_l}\chi_l, -i\chi_l, \frac{|l|}{\alpha\lambda_l}\chi_l)^T$ and $Y_m = (0, 0, \frac{1}{\lambda_m}\psi_m, i\psi_m, \frac{|m|}{\alpha\lambda_m}\psi_m)^T$ with $|l| = |m|$, we obtain

$$\begin{aligned} &\left(0, 0, \frac{-|k|}{\alpha\lambda_k} Id, 0\, Id, \frac{1}{\lambda_k} Id\right) R_{l,m}[X_l, Y_m] \\ &= \frac{-|k|}{2\alpha\lambda_k} ik \wedge \left(\left(\frac{il}{\lambda|l|^2} \wedge \chi_l\right).im\left(\frac{im}{\lambda|m|^2} \wedge \psi_m\right) + \alpha\left(\frac{il}{\lambda|l|^2} \wedge \chi_l\right) \wedge \frac{1}{\alpha\lambda}\psi_m\right) \\ &+ \frac{-|k|}{2\alpha\lambda_k} ik \wedge \left(\left(\frac{im}{\lambda|m|^2} \wedge \psi_m\right).il\left(\frac{il}{\lambda|l|^2} \wedge \chi_l\right) + \alpha\left(\frac{im}{\lambda|m|^2} \wedge \psi_m\right) \wedge \frac{1}{\alpha\lambda}\chi_l\right) \end{aligned}$$

where $\lambda = \lambda_l = \lambda_m$, from which we deduce

$$\begin{aligned} T_k^{\chi\psi} &= \frac{-|k|}{2\alpha\lambda_k} k \wedge \sum_{k=l+m,\,|l|=|m|} \frac{1}{\lambda|l|^4} \left((l \wedge \chi_l).m\,(m \wedge \psi_m) + (m \wedge \psi_m).l\,(l \wedge \chi_l)\right) \\ &+ \frac{|k|}{2\alpha\lambda_k} k \wedge \sum_{k=l+m,\,|l|=|m|} \frac{1}{\lambda|l|^2} \left((l \wedge \chi_l) \wedge \psi_m + (m \wedge \psi_m) \wedge \chi_l\right) \\ &= \frac{-|k|}{2\alpha\lambda_k} k \wedge \sum_{k=l+m,\,|l|=|m|} \frac{1}{\lambda|l|^4} \left((l \wedge \chi_l).m\,(m \wedge \psi_m) + (m \wedge \psi_m).l\,(l \wedge \chi_l)\right) \\ &+ \frac{|k|}{2\alpha\lambda_k} k \wedge \sum_{k=l+m,\,|l|=|m|} \frac{1}{\lambda|l|^4} \left((l \wedge \chi_l) \wedge ((m \wedge \psi_m) \wedge m) + (m \wedge \psi_m) \wedge ((l \wedge \chi_l) \wedge l)\right) \end{aligned}$$

Indeed, we have

$$(m \wedge \psi_m) \wedge m = |m|^2 \psi_m \text{ and } (l \wedge \chi_l) \wedge l = |l|^2 \chi_l$$

by the divergence-free conditions $m.\psi_m = l.\chi_l = 0$. Then, developping the vectorial product leads to

$$\begin{aligned}
T_k^{\chi\psi} &= \frac{-|k|}{2\alpha\lambda_k} k \wedge \sum_{k=l+m,\,|l|=|m|} \frac{1}{\lambda|l|^4} \left((l\wedge\chi_l).m\,(m\wedge\psi_m) + (m\wedge\psi_m).l\,(l\wedge\chi_l)\right)\\
&\quad + \frac{|k|}{2\alpha\lambda_k} k \wedge \sum_{k=l+m,\,|l|=|m|} \frac{1}{\lambda|l|^4} \left((l\wedge\chi_l).m(m\wedge\psi_m) + (m\wedge\psi_m).l(l\wedge\chi_l)\right)\\
&\quad - \frac{|k|}{2\alpha\lambda_k} k \wedge \sum_{k=l+m,\,|l|=|m|} \frac{1}{\lambda|l|^4} \left((l\wedge\chi_l).(m\wedge\psi_m)m + (m\wedge\psi_m).(l\wedge\chi_l)l\right)\\
&= -\frac{|k|}{2\alpha\lambda_k} k \wedge \sum_{k=l+m,\,|l|=|m|} \frac{1}{\lambda|l|^4} (l\wedge\chi_l).(m\wedge\psi_m)k = 0
\end{aligned}$$

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