Diffusion approximation for Fokker Planck with heavy tail equilibria : a spectral method in dimension 1

Gilles Lebeau, Marjolaine Puel

Laboratoire J.-A. Dieudonné
Université de Nice Sophia-Antipolis
Parc Valrose, 06108 Nice Cedex 02, France
lebeau@unice.fr;mpuel@unice.fr

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Abstract

This paper is devoted to the diffusion approximation for the 1-d Fokker Planck equation with a heavy tail equilibria of the form $(1+v^2)^{-\beta/2}$, in the range $\beta \in]1,5[$. We prove that the limit diffusion equation involves a fractional Laplacian $\kappa|\Delta|^{\frac{\beta+1}{6}}$, and we compute the value of the diffusion coefficient κ. This extends previous results of E. Nasreddine and M. Puel [17] in the case $\beta > 5$, and of P. Cattiaux, E. Nasreddine and M. Puel [7] in the case $\beta = 5$.
[1]

Contents

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1 Introduction

1.1 Setting of the problem

In this present paper, we deal with the equation

$$\partial_t f + v \cdot \nabla_x f = Q(f), \quad \text{provided } f_0 \geq 0, \tag{1.1}$$

the unknown $f(t,x,v) \geq 0$ being interpreted as the density of particles occupying at time $t \geq 0$, the position $x \in \mathbb{R}^d$ with a velocity $v \in \mathbb{R}^d$. We give a presentation of the problem in any dimension d but the result proved in this paper concerns the dimension 1 The Fokker Planck operator Q is given by

$$Q(f) = \nabla_v \cdot (\frac{1}{\omega} \nabla_v(\omega f)) \tag{1.2}$$

for a fixed function of v, $\omega(v)$, that determines the equilibrium $F \sim \frac{1}{\omega}$.

Recall that the aim of diffusion approximation is to provide a simpler model when the interaction between particles are the dominant phenomena and when the observation time is very large. For that purpose, we introduce a small parameter, ε, the mean free path and we proceed to a rescaling in time and space

$$t = \frac{t'}{\theta(\varepsilon)} \quad x = \frac{x'}{\varepsilon}$$

which leads to the following rescaled equation (without primes)

$$\theta(\varepsilon)\partial_t f^\varepsilon + \varepsilon v \cdot \nabla_x f^\varepsilon = Q(f^\varepsilon). \tag{1.3}$$

Passing formally to the limit, we get that

$$f^\varepsilon \to_{\varepsilon \to 0} f^0 = \rho(t,x)F(v)$$

where $F(v)$ is the equilibrium defined above. It remains to identify the equation satisfied by ρ.

When the equilibrium F is a gaussian, it is classical (see [2],[5],[13],[14],[11] for Boltzmann and [12] for Fokker Planck) that by taking the classical time scaling $\theta(\varepsilon) = \varepsilon^2$, we obtain for ρ a diffusion equation

$$\partial_t \rho - \nabla_x (D \nabla \rho) = 0 \tag{1.4}$$

where

$$D = \int v Q^{-1}(-vF) dv. \tag{1.5}$$

Indeed, the formal expansion $f^\varepsilon = f^0 + \varepsilon f^1 + \varepsilon^2 f^2 ...$ gives

$$\begin{array}{rcl} Q(f^0) & = & 0 \\ Q(f^1) & = & v \cdot \nabla_x f^0 \\ Q(f^2) & = & \partial_t f^0 + v \cdot \nabla f^1 \end{array}$$

and the compatibility equation for the equation giving f^2 gives

$$\partial_t \int f^0 + \int \nabla \cdot (v Q^{-1}(v \cdot \nabla_x f^0)) = 0$$

which is another formulation of (1.4) since $f^0 = \rho(t,x)F(v)$ and F is normalized by $\displaystyle\int F = 1$.

In the present work, we consider for any $\beta > 1$ heavy tail equilibria $F(v) = \frac{C_\beta^2}{\omega}$ with $\omega = (1+|v|^2)^{\frac{\beta}{2}}$ and C_β a normalization constant. In [17], the classical scaling is studied and it is proved in any dimendion d that we obtain a diffusion equation (1.4), (1.5) as soon as $\beta > d+4$. The critical case where $\beta = d+4$ is studied in [7] where the expected result of classical diffusion with an anomalous time scaling is proved.

The aim of this paper is to study the case where $1 < \beta < d+4$, when the diffusion coefficient (1.5) is not defined anymore. We need to operate an ad hoc rescaling in time that we will compute during the proof. Fractional diffusion limit has been already obtained in the case of the linear Boltzmann equation for heavy tail equilibria when the cross section is such that the operator has a spectral gap (see [16] for the pioneer paper in the case of space independent cross section, [15] for a weak convergence result and [3] for a strong convergence result) and when the cross section is degenerated [3]. The main difficulty of our case is due to the fact that the Fokker Planck operator Q has no spectral gap. The idea here is thus to study the whole operator, advection plus collision, at ε fixed, to compute the first eigenvalue and its corresponding eigenvector. The dependency of the first eigenvalue with respect to ε will give us the right time scaling and the power of the limiting fractional diffusion operator. Note that a fractional diffusion has also been obtained for a Fokker-Planck like operator in [8], for which the result is obtained thanks to the spectral gap of the operator. In the present paper, it is not enough to project the solution onto the kernel of the operator. Note that a spectral analysis of the whole operator is much easier in dimension one since we deal with two order differential equations. However, since we conclude by using the moment method, we just need to construct a sequence converging toward the equilibrium thus we don't need a complete spectral study with the correctors. Therefore, the multi dimensional-case should be possible to handle with such a method.

Outline of the paper

In the next subsection, we recall the previous results obtained for this equation with heavy tail equilibria, we quote the main theorem of this present paper and proceed to a change of unknown. It is followed by a section dedicated to the computation of the first eigenfunction and eigenvalue. Finally, in section 3, we apply the moment method to complete the proof of the main theorem.

1.2 Previous results

The functional setting of the study of equation (1.3) has been settled in [17] where we define the functional ad hoc spaces $Y^p_\omega\left(\mathbb{R}^{2d}\right) = L^p\left(\mathbb{R}^d,\ H_p(\mathbb{R}^d)\right)$, where

$$H_p(\mathbb{R}^d) = \left\{ f : \mathbb{R}^d \to \mathbb{R},\ \int_{\mathbb{R}^d} |f|^p\ \omega^{p-1}\ \mathrm{d}v < \infty \right\}, \tag{1.6}$$

where $\omega = (1+||v||^2)^{\frac{\beta}{2}}$ and

$$L^\infty_\omega(\mathbb{R}^d) = \{f : \mathbb{R}^d \to \mathbb{R}, f\omega \in L^\infty(\mathbb{R}^d)\}.$$

Define

$$V = \left\{ f : \mathbb{R}^d \to \mathbb{R},\ \int_{\mathbb{R}^d} |f|^2\ \omega\ \mathrm{d}v < \infty \text{ and } \int_{\mathbb{R}^d} \frac{|\nabla_v(f\ \omega)|^2}{\omega}\ \mathrm{d}v < \infty \right\}, \tag{1.7}$$

V' being its dual.

Operator's properties. We sumerize in the following proposition the main properties of the interaction operator.

Proposition 1.1 *[17] Let f and g be smooth functions in V defined in (1.7). The following assertions hold true:*

1. *The operator Q is conservative, thus equation (1.3) preserves the total mass of the distribution*

$$\int_{\mathbb{R}^d} Q(f)\ dv = 0, \quad \textit{for all } f \in V.$$

2. *The operator Q is self-adjoint with respect to the measure ω dv:*

$$\int_{\mathbb{R}^d} Q(f)\ g\ \omega\ dv = -\int_{\mathbb{R}^d} \frac{\nabla_v(f\ \omega)\cdot\nabla_v(g\ \omega)}{\omega}\ dv = \int_{\mathbb{R}^d} f\ Q(g)\ \omega\ dv. \tag{1.8}$$

3. *The operator Q is dissipative:*

$$\int_{\mathbb{R}^d} Q(f)\ f\ \omega\ dv = -\int_{\mathbb{R}^d} \frac{|\nabla_v(f\ \omega)|^2}{\omega}\ dv \le 0. \tag{1.9}$$

4. *The kernel of Q is one-dimensional and spanned by $\frac{1}{\omega}$.*

5. *The operator Q is continuous from $V \longrightarrow V'$.*

Existence theorem. We recall the following theorem inspired from [10]

Theorem 1.2 *[17]Let ε be fixed. Assume that $f_0 \in Y^2_\omega(\mathbb{R}^d)$, equation (1.3) has a unique solution f in the class of functions Y defined by:*

$$Y = \left\{ f \in L^2\left([0,T]\times\mathbb{R}^d, V\right),\quad \theta(\varepsilon)\partial_t f + \varepsilon v\cdot\nabla_x f \in L^2\left([0,T]\times\mathbb{R}^d, V'\right)\right\}.$$

Classical diffusion approximation. The case where $\beta > d+4$ leads to a diffusion equation as described in the following theorem.

Theorem 1.3 *[17] Assume now that $\beta > d+4$. Assume that f_0 is a nonnegative function in $Y^2_\omega \cap Y^p_\omega$ with $p > 2$. Assume that $\theta(\varepsilon) = \varepsilon^2$, let f^ε be the solution of (1.3) in Y with initial data f_0.*

Then, f^ε converges weakly star in $L^\infty\left([0,T],\ Y^p_\omega(\mathbb{R}^{2d})\right)$ towards $\rho(t,x)\ \frac{C^2_\beta}{\omega}$ where $\rho(t,x)$ is the unique solution of the system

$$\partial_t\rho + \nabla_x\cdot j = 0 \tag{1.10}$$

$$j = -D\ \nabla_x\rho, \tag{1.11}$$

where the initial datum is given by $\rho_0(x) = \int_{\mathbb{R}^d} f_0\ dv$, and the diffusion tensor D is given by

$$D = \int_{\mathbb{R}^d} v\otimes\chi\ dv, \tag{1.12}$$

where χ is the unique solution of the cell equation $Q(\chi) = \frac{-C^2_\beta\ v}{\omega}$ with $\int_{\mathbb{R}^d}\chi\ dv = 0$.

Critical case, $\beta = d+4$.

Theorem 1.4 *[7] Assume that $\beta = d+4$. Then there exists $\kappa > 0$ such that, with $\theta(\varepsilon) = \varepsilon^2\ln(1/\varepsilon)$, for all initial density of probability f_0, the solution f^ε_t of (1.3) weakly converges as $\varepsilon \to 0$ towards*

$$(v,x) \mapsto C^2_\beta\,\omega^{-1}_\beta(v)\,(h_0 * \rho_t)(x)$$

where ρ_t is the density of a centered gaussian random vector with covariance matrix $(2\kappa/3)\,t\,Id$ and $h_0(x) = \int f_0(x,v)\,dv$.

1.3 Main theorem

Assume from now on that the dimension $d = 1$.

Theorem 1.5 *Assume that* $1 < \beta < 5$ *with* $\beta \neq \{2,3,4\}$*. Assume that* $f_0 \in L^1(\mathbb{R}\times\mathbb{R})$ *is a nonnegative function in* Y^2_ω *and* $f_0\omega \in L^\infty(\mathbb{R}\times\mathbb{R})$*. Let* f^ε *be the solution of* (1.3) *in* Y *with initial data* f_0*, when* $\theta(\varepsilon) = \varepsilon^{\frac{\beta+1}{3}}$.
Let $\kappa = 2C^2_\beta(\beta+1)9^{-\frac{\beta+1}{3}}\cos(\frac{\pi}{2}\frac{\beta+1}{3})\frac{\Gamma(1-\frac{\beta+1}{3})}{\Gamma(1+\frac{\beta+1}{3})} > 0$*, where* Γ *is the Euler function.*
Then f^ε *converges weakly star in* $L^\infty\left([0,T],\ Y^2_\omega(\mathbb{R}^2)\right)$ *towards* $\rho(t,x)\ \frac{C^2_\beta}{\omega}$ *where* $\rho(t,x)$ *is the solution to*

$$\partial_t\rho + \kappa(-\Delta)^{\frac{\beta+1}{6}}\rho = 0, \quad \rho(0,x) = \int f_0 dv\ . \tag{1.13}$$

Remark 1.6 *Note that we will work with the Fourier transform of* ρ *and we will prove that* $\hat\rho(t,k) = \int e^{-ixk}\rho(t,x)dx$ *satisfies*

$$\partial_t\hat\rho + \kappa|k|^{\frac{\beta+1}{3}}\hat\rho = 0. \tag{1.14}$$

Remark 1.7 *The hypothesis* $\beta \neq \{2,3,4\}$ *is technical. It avoids to introduce logarithmic terms in the expression of the solution* g *in* (2.4)*. However, the result is the same with the same method.*

Observe that $\alpha = \frac{\beta+1}{3} \in]2/3,2[$*, and that for* $\alpha \in]2/3,2[$*, one has* $\Psi(\alpha) = \cos(\frac{\pi}{2}\alpha)\frac{\Gamma(1-\alpha)}{\Gamma(1+\alpha)} > 0$*,* $\Psi(1) = \pi/2$ *and* $\lim_{\alpha\to 2}\Psi(\alpha) = +\infty$.

As we said, in order to prove this theorem, we compute the first eigenvalue and eigenvector of the whole operator $(-i\varepsilon v\cdot\nabla + Q)$ and for that purpose, to simplify the computation, we proceed to a change of unknown such that the new operator splits into a Schrödinger operator.

Changing the unknown. We start with the Fokker Planck equation

$$\partial_t f + v\cdot\nabla_x f = Q(f) = \nabla_v(F\nabla_v(\frac{f}{F}))$$

with equilibria given by

$$F = \frac{C^2_\beta}{(1+|v|^2)^{\frac{\beta}{2}}} = \frac{C^2_\beta}{(1+|v|^2)^\gamma}.$$

Since we impose $\gamma = \frac{\beta}{2} > \frac{1}{2}$, $F \in L^1(\mathbb{R})$, and we chose C_β such that $\displaystyle\int F dv = 1$. In order to work with a self adjoint operator in $L^2(\mathbb{R})$, we proceed to a change of unknown by writing

$$f = F^{\frac{1}{2}}g$$

and the equation becomes

$$\partial_t g + v\cdot\nabla_x g = F^{-\frac{1}{2}}\nabla_v(F\nabla_v(\frac{g}{F^{\frac{1}{2}}}))$$

that can be written

$$\partial_t g + v\cdot\nabla_x g = \Delta_v g - W(v)g$$

with

$$W(v) = -\frac{1}{2}F^{-\frac{1}{2}}\nabla\cdot(F^{-\frac{1}{2}}\nabla F).$$

We see the equation as

$$\partial_t g = -\mathcal{L} g$$

where $\mathcal{L} = -\Delta_v + W(v) + v \cdot \nabla_x$ is a non negative operator since

$$(\mathcal{L}g|g) = \int_{\mathbb{R}} |\partial_v g|^2 dv + \int_{\mathbb{R}} W(v)|g|^2 dv = \int_{\mathbb{R}} F|\partial_v(\frac{g}{F^{1/2}})|^2 dv \geq 0 \ ,$$

where $(\cdot|\cdot)$ denotes the scalar product in $L^2(\mathbb{R})$. Thus

$$g = e^{-t\mathcal{L}} g_0.$$

Since the operator has coefficient that do not depend on x, we operate a Fourier transform in x and proceed to a second change of unknown by writing

$$g(s, x, v) = (2\pi)^{-1} \int e^{ix\cdot\xi} \tilde{g}(s, \xi, v) d\xi$$

where $\tilde{g}$ satisfies

$$\partial_t \tilde{g} = -\mathcal{L}\tilde{g}$$

where

$$\mathcal{L}\tilde{g} = -\Delta_v \tilde{g} + W(v)\tilde{g} + i(\xi \cdot v)\tilde{g}.$$

Rescaling. We do a rescaling both in space and time

$$t = Ts, \quad \xi = T^{\delta-1} k.$$

Note that the rescaling in ξ corresponds to a rescaling in $x = T^{1-\delta} y$ so that $e^{ix\cdot\xi} = e^{iy\cdot k}$.

The equation becomes

$$\partial_s \tilde{g} = -T\mathcal{L}_\varepsilon(\tilde{g}) \tag{1.15}$$

with $\varepsilon = T^{\delta-1}$ and

$$\mathcal{L}_\varepsilon(\tilde{g}^\varepsilon) = -\Delta_v \tilde{g}^\varepsilon + W(v)\tilde{g}^\varepsilon + i(v \cdot \varepsilon k)\tilde{g}^\varepsilon \ .$$

Classical diffusion corresponds to $\delta = \frac{1}{2}$. When $\beta < 5$, the right scaling will be given by the power of ε of the leading term of the first eigenvalue of the full operator.

2 Spectral study of the operator :

In this section, for $\varepsilon > 0$, we compute the eigenvalue μ^ε with lowest absolute value and the associated eigenfunction M^ε (normalized by $M^\varepsilon(0)$=1) of the unbounded operator $\mathcal{L}_\varepsilon$ acting on L^2:

$$\mathcal{L}_\varepsilon M^\varepsilon = -\Delta_v M^\varepsilon + W(v)M^\varepsilon + i(v \cdot \varepsilon k)M^\varepsilon = \mu^\varepsilon M^\varepsilon. \tag{2.1}$$

In dimension 1, the equation leading to the eigenvalue can be written

$$[-\partial_v^2 + W(v) + i\varepsilon k v - \mu^\varepsilon]M^\varepsilon = 0$$

and W is given by

$$W(v) = \frac{\gamma}{(1 + |v|^2)^2}[v^2(\gamma + 1) - 1]$$

and its asymptotic behavior for high velocities is

$$W(v) \sim_{v\sim\infty} \frac{\gamma(\gamma + 1)}{|v|^2}.$$

The domain of $\mathcal{L}_\varepsilon$ is

$$D(\mathcal{L}_\varepsilon) = \{g \in L^2(\mathbb{R}), \quad \partial_v^2 g \in L^2(\mathbb{R}), \quad vg \in L^2(\mathbb{R})\} .$$

Note that for $\varepsilon > 0$, the domain of $\mathcal{L}_\varepsilon$ is not equal to the domain of the limiting operator.
In dimension 1, the domain is compact, then the operator has a compact resolvent thus the spectrum is discrete [9]. The construction of the eigenvalue turns out to be a connexion problem between

$$E_\mu^\pm = \{g|\mathcal{L}_\varepsilon(g) - \mu g = 0 \quad \text{with } g \in L^2(\pm v \geq 0)\}.$$

So we will first compute the solution for μ and ε fixed of the equation

$$-\partial_v^2 M_\mu^\varepsilon + W(v) M_\mu^\varepsilon + i(v\varepsilon k) M_\mu^\varepsilon = \mu M_\mu^\varepsilon$$

and we denote $b(\lambda, \varepsilon) = (M^\varepsilon)'(0)$, λ being defined by $\lambda = \eta^{-\frac{2}{3}}\mu$.

If we change v into $-v$, the equation remains the same except that we have to change i into $-i$ (note that we use here the parity of the equilibrium M) which means that

$$M_\mu^\varepsilon(v) = \overline{M^\varepsilon}_{\overline{\mu}}(-v).$$

Thus, if we want to reconnect the derivative for $v = 0$ in order to have a $C^1(\mathbb{R})$ function, we get the constraint ${M^\varepsilon}'_\mu(0) = -\overline{M^\varepsilon}'_{\overline{\mu}}(0)$ which is equivalent to

$$b(\lambda, \eta) + \overline{b}(\overline{\lambda}, \eta) = 0.$$

This condition will give us the expression of λ function of η.

The construction of the eigenvector is done via two successive fixed point. First of all, the very large velocity asymptotic is given by a Airy profile solution to

$$(-\partial_s^2 + is - \lambda) g = 0 \tag{2.2}$$

by neglecting the potential. The first fixed point process is performed to obtain a solution to a first approximation of the equation with a simplified potential

$$(-\partial_v^2 + \frac{\gamma(\gamma+1)}{v^2} + i\varepsilon k v) f = \mu f, \quad v \in]0, \infty[.$$

A last fixed point argument leads to the solution of the complete equation with the correct potential.

2.1 Large velocities asymptotic : solution to an approximated equation

Since $W(v) \sim_{|v|\to\infty} \frac{\gamma(\gamma+1)}{v^2}$, we will first consider the approximated differential equation

$$(-\partial_v^2 + \frac{\gamma(\gamma+1)}{v^2} + i\varepsilon k v) f = \mu f, \quad v \in]0, \infty[$$

or

$$-v^2 \partial_v^2 f + \gamma(\gamma+1) f + i\varepsilon k v^3 f = \mu v^2 f.$$

If we want to get rid of the parameter ε, we need to proceed to the following rescaling

$$v = (\varepsilon k)^{-\frac{1}{3}} s, \quad \mu = (\varepsilon k)^{\frac{2}{3}} \lambda$$

that leads to

$$(-s^2 \partial_s^2 + \gamma(\gamma+1) + is^3) f = \lambda s^2 f, \quad s \in]0, \infty[. \tag{2.3}$$

Near $s = 0$, equation (2.3) is a differential equation with regular singular points. We proceed to a change of unkown by writing $f = s^\delta g$, with $\delta(\delta - 1) = \gamma(\gamma + 1)$, i.e $\delta = -\gamma$ or $\delta = \gamma + 1$. Then the new unknown g satisfies

$$-\partial_s^2 g - \frac{2\delta}{s}\partial_s g + (is - \lambda)g = 0. \tag{2.4}$$

Writing $g = \sum_0^\infty g_n s^n$ leads to the following equation for g_n

$$-\sum_{n\geq 0}(n+2)(n+1)g_{n+2}s^n - 2\delta\sum_{n\geq 0}(n+2)g_{n+2}s^n - 2\delta\frac{g_1}{s} - \lambda\sum_{n\geq 0}g_n s^n + i\sum_{n=1}^\infty g_{n-1}s^n = 0\ ,$$

that gives assuming $\gamma \neq \frac{k+1}{2}, k \in \mathbb{N}$,

$$\begin{cases} g_1 &= 0 \\ g_{n+2} &= \frac{1}{(n+2)(n+1+2\delta)}[-\lambda g_n + i g_{n-1}] \quad \forall n \geq 0 \quad (g_{-1} = 0). \end{cases} \tag{2.5}$$

That define a unique solution if $g_0 = 1$. Define

$$\begin{array}{ll} F_{+,\lambda}(s) = \sum_0^\infty g_n s^n, & g_0 = 1, g_1 = 0, g_n \text{ defined as above with } \delta = -\gamma \\ F_{-,\lambda}(s) = \sum_0^\infty g_n s^n, & g_0 = 1, g_1 = 0, g_n \text{ defined as above with } \delta = \gamma + 1. \end{array} \tag{2.6}$$

A basis of the solution space of equation (2.3) is thus given by the two independent solutions

$$F_{+,\lambda}(s)s^{-\gamma} \quad \text{and} \quad F_{-,\lambda}(s)s^{\gamma+1}\ .$$

$F_{\pm,\lambda}$ are normalized by $F_{\pm,\lambda}(0) = 1$ and are entire functions of $s \in \mathbb{C}$.

Proposition 2.1 *Let $F_{+,\lambda}$ and $F_{-,\lambda}$ be defined in (2.6). There exists λ_0 such that for all $\lambda \in \mathbb{C}$, such that $|\lambda| \leq \lambda_0$, equation (2.3) has a unique solution $H_\lambda(s)$ such that*

1. $\int_1^\infty |H_\lambda(s)|^2 ds < \infty$.
2. $H_\lambda(s) = s^{-\gamma}F_{+,\lambda}(s) + d(\lambda)F_{-,\lambda}(s)s^{\gamma+1}$
   where $d(\lambda)$ is an holomorphic function for $|\lambda| \leq \lambda_0$.

Proof. For $s >> 1$, we first consider the approximate equation

$$(-\partial_s^2 + is - \lambda)g = 0 \tag{2.7}$$

that define a unique $L^2(s > R)$ solution (up to a constant) given by

$$G_\lambda(s) = Ai[e^{i\frac{\pi}{6}}(s + i\lambda)]$$

where Ai is the Airy function given by

$$Ai(z) = \frac{1}{2\pi}\int_{-\infty}^\infty e^{i(\frac{t^3}{3}+zt)}dt.$$

Since we know the asymptotic profile of the solution to (2.3), we will look for a solution via the following change of unknown

$$H_\lambda = CAi[e^{i\frac{\pi}{6}}(s + i\lambda)][1 + R_\lambda(s)]$$

and perform a fixed point argument on R_λ that satisfies

$$2G'_\lambda R'_\lambda + G_\lambda R''_\lambda = \frac{\gamma(\gamma+1)}{s^2} G_\lambda(1+R_\lambda)$$

that can be written

$$R''_\lambda + 2\frac{G'\lambda}{G_\lambda} R'_\lambda = \frac{\gamma(\gamma+1)}{s^2}(1+R_\lambda)$$

and that leads to the implicit equation

$$R_\lambda(s) = \int_s^\infty [\int_s^z \frac{G_\lambda^2(z)}{G_\lambda^2(u)} du] \frac{\gamma(\gamma+1)}{z^2}(1+R_\lambda(z))dz.$$

We need to prove the following lemma.

Lemma 2.2 *Define*

$$\mathbb{K}_\lambda(g) = \int_s^\infty [\int_s^z \frac{G_\lambda^2(z)}{G_\lambda^2(u)} du] \frac{\gamma(\gamma+1)}{z^2} g(z)dz \ . \tag{2.8}$$

For $s_0 > 0$ large enough, there exists a unique $R_\lambda(s) \in L^\infty([s_0, \infty[)$ solution to

$$(Id - \mathbb{K}_\lambda)R_\lambda = \mathbb{K}_\lambda(1). \tag{2.9}$$

Moreover, R_λ is holomorphic in $|\lambda| < \lambda_0$ and $R_\lambda(s) = O(s^{-\frac{3}{2}})$ uniformly in $|\lambda| < \lambda_0$.

Proof. As we said, we apply a fixed point theorem. First of all, there exists a constant K such that for all $s \geq 1$, $|\lambda| \leq \lambda_0$ and $z \geq s$, we have

$$|\int_s^z \frac{G_\lambda^2(z)}{G_\lambda^2(u)} du| \leq \frac{K}{(1+|z|)^{\frac{1}{2}}}.$$

Indeed, let us denote $U = \{(x+i\lambda)e^{i\frac{\pi}{6}}, \quad x \geq 0, |\lambda| \leq \lambda_0\}$. For $z \in U, |z| \geq \frac{1}{2}$, we have [18]

$$Ai(z) = e^{-\frac{2}{3}z^{\frac{3}{2}}}\tau(z), \text{ with } \quad \frac{c_0}{(1+|z|)^{\frac{1}{4}}} \leq |\tau(z)| \leq \frac{c_1}{(1+|z|)^{\frac{1}{4}}}. \tag{2.10}$$

Then

$$\begin{aligned} |\int_s^z \frac{G_\lambda^2(z)}{G_\lambda^2(u)} du| &\leq C \int_s^z e^{-\frac{4}{3}\mathrm{Re}([(z+i\lambda)^{\frac{3}{2}} - (u+i\lambda)^{\frac{3}{2}}]e^{i\frac{\pi}{4}})} du \\ &= Cz \int_{\frac{s}{z}}^1 e^{-\frac{4}{3}z^{\frac{3}{2}}\mathrm{Re}([(1+i\frac{\lambda}{z})^{\frac{3}{2}} - (t+i\frac{\lambda}{z})^{\frac{3}{2}}]e^{i\frac{\pi}{4}})} dt \\ &\leq Cz \int_0^\infty e^{-tz^{\frac{3}{2}}} dt \sim \frac{C}{z^{\frac{1}{2}}} \quad \text{if } z \geq s \geq 1. \end{aligned}$$

Thus for $|\lambda| \leq \lambda_0$

$$|s^{n+\frac{3}{2}}\mathbb{K}_\lambda(g)(s)| \leq K\gamma(\gamma+1)\int_s^\infty \frac{s^{n+\frac{3}{2}}}{z^{n+\frac{3}{2}+1}}|g(z)z^n|dz.$$

Then

$$||s^{n+\frac{3}{2}}\mathbb{K}_\lambda(g)||_{L^\infty([1,\infty[)} \leq \frac{K\gamma(\gamma+1)}{(n+\frac{3}{2})}||s^n g||_{L^\infty([1,\infty[)}.$$

Finally, $\mathbb{K}_\lambda$ is bounded in $L^\infty([s_0, \infty[)$ with

$$||\mathbb{K}_\lambda||_{L^\infty([s_0,\infty[)} \leq \frac{2K}{3}\gamma(\gamma+1)s_0^{-\frac{3}{2}} \leq \frac{1}{2} \quad \text{if } s_0 \text{ big enough.}$$

Then (2.9) has a unique solution in $L^\infty([s_0,\infty[)$, $R_\lambda(s)$, holomorphic in $|\lambda| \le \lambda_0$ and, since $\mathbb{K}_\lambda(1) = O(s^{-\frac{3}{2}})$, we get the following asymptotics $R_\lambda(s) = O(s^{-\frac{3}{2}})$.

Moreover, since $\mathbb{K}_\lambda^{(n+1)}(1) = O(s^{-\frac{3(n+1)}{2}})$, the sum $\sum_0^\infty \mathbb{K}_\lambda^{(n+1)}(1)$ converges and we can write the asymptotic expansion

$$R_\lambda = \sum_0^\infty \mathbb{K}_\lambda^{(n+1)}(1).$$

□

Let us resume the proof of Proposition 2.1. Since $Ai(e^{i\frac{\pi}{6}}(s+i\lambda))(1+R_\lambda(s))$ is solution on $[s_0,\infty[$, by (2.10) and $R_\lambda(s) = O(s^{-\frac{3}{2}})$, we get that $H_\lambda \in L^2([1,\infty))$.

Moreover, it may be extended on $]0,\infty[$ by $\tilde{H}_\lambda(s)$ in an holomorphic way for $|\lambda| < \lambda_0$ that can thus be written

$$\tilde{H}_\lambda = a(\lambda)s^{-\gamma}F_{+,\lambda}(s) + b(\lambda)s^{\gamma+1}F_{-,\lambda}(s)$$

where $a(\lambda)$ and $b(\lambda)$ are holomorphic for $|\lambda| < \lambda_0$. It remains now to prove that $a(0) \ne 0$. For that purpose, assume that $a(0) = 0$. Since $\gamma > \frac{1}{2}$, we get that the solution of (2.3) $\tilde{H}_0(s) \in L^2(\mathbb{R}^+)$, and $\tilde{H}_0(s) = O(s^{-\infty})$, then by integration by parts, since $\tilde{H}_0(0) = 0$ and since, because of the asymptotic behavior of the chosen Airy function, $|\tilde{H}_0(s)| \le_{s\sim 0} Cs^{\gamma+1}$, we write

$$\int_0^\infty |\tilde{H}_0'(s)|^2 + \frac{\gamma(\gamma+1)}{s^2}|\tilde{H}_0(s)|^2 + is|\tilde{H}_0(s)|^2 ds = 0$$

that leads to $\tilde{H}_0 = 0$ which leads to a contradiction. To end the proof, we just define $H_\lambda(s) = \frac{1}{a(\lambda)}\tilde{H}_\lambda(s)$. □

Lemma 2.3 *For all $\lambda \in \mathbb{C}$, $|\lambda| \le \lambda_0$ and $s \in]0,\infty[$, we have $H_\lambda(s) \ne 0$.*

Proof. If $H_\lambda(s_0) = 0$, since $H_\lambda^{(k)} =_{s\sim\infty} O(s^{-\infty})$ for any derivative of order $k \in \mathbb{N}$, as above, by an integration by parts, we get

$$\int_{s_0}^\infty |H_\lambda'(s)|^2 + [is + \frac{\gamma(\gamma+1)}{s^2}]|H_\lambda(s)|^2 ds = \lambda \int_{s_0}^\infty |H_\lambda(s)|^2 ds.$$

It leads to

$$\int_{s_0}^\infty s|H_\lambda(s)|^2 ds = \mathrm{Im}\lambda \int_{s_0}^\infty |H_\lambda(s)|^2 ds \quad \Rightarrow \mathrm{Im}\lambda > 0\ ,$$

$$\int_{s_0}^\infty \frac{\gamma(\gamma+1)}{s^2}|H_\lambda(s)|^2 ds \le \mathrm{Re}\lambda \int_{s_0}^\infty |H_\lambda(s)|^2 ds \quad \Rightarrow \mathrm{Re}\lambda > 0.$$

More precisely, by summing those two equations, we get

$$\mathrm{Im}\lambda + \mathrm{Re}\lambda \ge c_0 = min_{s\ge 0}(s + \frac{\gamma(\gamma+1)}{s^2})$$

which contradicts the fact that $|\lambda| \le \lambda_0$. Indeed, c_0 does not depend on λ_0, and we can choose λ_0 as small as we want which leads to a contradiction. □

In the remaining part of this section, we prove the fact that $d(0) \ne 0$. For that purpose, we prove that

Lemma 2.4 *Let $F_{+,\lambda}$ be defined in (2.6), we have*

$$s^{-\gamma}F_{+,0}(s) \notin L^2([1,\infty[).$$

Proof. **Step1 : Changing the unknown.**
Following (2.5), we write

$$F_{+,0} = \sum_0^\infty d_n s^{3n}, \quad \text{with } d_0 = 1, \quad d_{n+1} = \frac{i}{9(n+1)(n+1-\alpha)} d_n, \text{ where } \alpha = \frac{2\gamma+1}{3} \notin \mathbb{N}$$

that can also be written

$$F_{+,0} = D_\alpha(s^3), \quad \text{where } D_\alpha = \sum_0^\infty d_n x^n.$$

By introducing the sequence h_n writing $d_n = (\frac{i}{9})^n \frac{h_n}{n!}$ we get the following recurrence formula

$$h_0 = 1 \quad \text{and } h_{n+1} = \frac{h_n}{n+1-\alpha}. \tag{2.11}$$

Note that it implies that $|d_n| \leq 9^{-n}(\frac{1}{n!})^2$ which implies that $|D_\alpha(x)| \leq Ce^{C\sqrt{x}}$.
Define now $F_\alpha(z) = \sum_0^\infty h_n z^n$, we have for all $t \in \mathbb{C}$, $\text{Re}t > 0$,

$$\int_0^\infty e^{-\frac{x}{t}} D_\alpha(x) dx = tF_\alpha(\frac{it}{9}). \tag{2.12}$$

Let us now study F_α.
Step2 : Study of F_α

Lemma 2.5 *Let $F_\alpha = \sum_0^\infty h_n z^n$ with h_n satisfying (2.11). One has*

$$F_\alpha(z) = \Gamma(1-\alpha) z^\alpha e^z + \alpha \int_0^\infty \frac{e^{-zv}}{(1+v)^{\alpha+1}} dv \quad \forall z \in \mathbb{C} \textit{ such that } Rez > 0. \tag{2.13}$$

Proof. Since the sequence h_n is defined by (2.11), the function F_α satisfies the differential equation

$$F'_\alpha - \frac{\alpha}{z}(F_\alpha - F_\alpha(0)) = F_\alpha, \quad F_\alpha(0) = 1.$$

By integrating this equation, we get for $z > 0$

$$F_\alpha(z) = C(\alpha) z^\alpha e^z + z^\alpha e^z \int_z^\infty \frac{\alpha e^{-s}}{s^{\alpha+1}} ds.$$

But by integration by part, we write

$$\int_z^\infty \frac{\alpha e^{-s}}{s^{\alpha+1}} ds = \frac{1}{z^\alpha} e^{-z} + \frac{1}{1-\alpha} \int_z^\infty \frac{(\alpha-1)e^{-s}}{s^\alpha} ds.$$

By iterating this process, we obtain

$$\begin{aligned} F_\alpha(z) &= 1 + \frac{z}{1-\alpha} + \cdots + \frac{z^n}{(1-\alpha)\cdots(n-\alpha)} + z^\alpha e^z [C(\alpha) + \frac{1}{(1-\alpha)\cdots(n-\alpha)} \int_z^\infty e^{-s} s^{n-\alpha} ds] \\ &= 1 + \frac{z}{1-\alpha} + \cdots + \frac{z^n}{(1-\alpha)\cdots(n-\alpha)} \\ &\quad + z^\alpha e^z [C(\alpha) - \Gamma(1-\alpha) + \frac{\Gamma(1-\alpha)}{\Gamma(n-\alpha+1)} \int_0^z e^{-s} s^{n-\alpha} ds]. \end{aligned}$$

By letting $n \to \infty$, we get that

$$C(\alpha) = \Gamma(1-\alpha)$$

where $\Gamma(z) = \displaystyle\int_0^\infty x^{z-1} e^{-x} dx$. Which concludes the proof of the lemma. $\square$

Step3 : Proof that D_α is not a tempered distribution
Going back to (2.12), we obtain

$$\int_0^\infty e^{-\frac{x}{t}} D_\alpha(x)dx = tF_\alpha(i\frac{t}{9}) = t[\Gamma(1-\alpha)(i\frac{t}{9})^\alpha e^{i\frac{t}{9}} + \alpha \int_0^\infty \frac{e^{-i\frac{t}{9v}}}{(1+v)^{\alpha+1}} dv]$$

then after the change of variable $w = \frac{1}{9\lambda v}$, the Laplace transform of $1_{x>0} D_\alpha(x)$ is given by

$$\int_0^\infty e^{-\lambda x} D_\alpha(x)dx = [\frac{\Gamma(1-\alpha)}{\lambda}(\frac{i}{9\lambda})^\alpha e^{i\frac{1}{9\lambda}} + 9\alpha \int_0^\infty \frac{e^{-iw}}{(1+9\lambda w)^{\alpha+1}} dw]. \tag{2.14}$$

Since $1_{x>0} D_\alpha(x) \leq Ce^{C\sqrt{x}}$, the Fourier transform

$$\int_0^\infty e^{-ix\xi} 1_{x>0} D_\alpha(x)dx$$

exists and is holomorphic in $\mathrm{Im}\xi < 0$ and from (2.14), we get

$$\int_0^\infty e^{-ix\xi} D_\alpha(x)dx = \frac{\Gamma(1-\alpha)}{i\xi}(\frac{1}{9\xi})^\alpha e^{\frac{1}{9\xi}} + 9\alpha \int_0^\infty \frac{e^{-iw}}{(1+9iw\xi)^{\alpha+1}} dw. \tag{2.15}$$

In (2.15), the term $\frac{\Gamma(1-\alpha)}{i\xi}(\frac{1}{9\xi})^\alpha e^{\frac{1}{9\xi}}$ is not the Fourier transform of a tempered distribution.
Let us introduce W_α, the non tempered part of D_α, *i.e.*

$$W_\alpha(x) = \frac{\Gamma(1-\alpha)}{2\pi} \int_{\mathrm{Im}\xi<0} e^{ix\xi + \frac{1}{9\xi}} \left(\frac{1}{9\xi}\right)^\alpha \frac{d\xi}{i\xi}.$$

To compute its asymptotic, we use the stationary phase method with the phase $\phi(\xi) = x\xi - i\frac{1}{9\xi}$. The critical point corresponding to the point where ϕ' equals zero is given by $3\xi_c = e^{-i\frac{\pi}{4}} x^{-\frac{1}{2}}$ and we get

$$W_\alpha(x) \sim_{x\sim\infty} c_0 \Gamma(1-\alpha) 3^{-\alpha} x^{\frac{\alpha}{2}-1} e^{i\frac{\pi}{4}\alpha} e^{\tau x^{\frac{1}{2}}},$$

where $\tau = e^{i\frac{\pi}{4}} + \frac{1}{3} e^{-i\frac{\pi}{4}}$. So

$$F_\alpha(s) \sim_{s\sim\infty} c_0 \Gamma(1-\alpha) 3^{-\alpha} s^{\frac{3\alpha}{2}-3} e^{i\frac{\pi}{4}\alpha} e^{\tau s^{\frac{3}{2}}}$$

which is not tempered.
On the other hand, the second term is a Fourier transform of a tempered distribution, indeed

$$9\alpha \int_0^\infty \frac{e^{-iw}}{(1+9iw\xi)^{\alpha+1}} dw = \mathcal{F}(1_{x>0} \frac{\kappa(\alpha)}{2i\pi} \int_0^\infty e^{-i\frac{9x}{t} - t} t^{\alpha-1} dt).$$

with $\kappa(\alpha) = \int_{\gamma_0} e^z \frac{dz}{z^\alpha}$, where γ_0 is the contour in $\mathbb{C}$ connecting $-\infty$ to $-\infty$ with on loop conterclockwise around $z = 0$. Thus D_α is not tempered and $F_{+,0}(s) = D_{\frac{(1+2\gamma)}{3}}(s^3)$ is not tempered either. $\square$

Lemma 2.6 *The $L^2([1,\infty))$ solution of (2.3) is given by*

$$H_\lambda(s) = s^{-\gamma}F_{+,\lambda}(s) + d(\lambda)F_{-,\lambda}(s)s^{\gamma+1}$$

where $d(\lambda)$ is an holomorphic function for $|\lambda| \le \lambda_0$ and $d(0) \neq 0$. Moreover

$$d(0) = -e^{i\frac{\pi}{2}\frac{2\gamma+1}{3}}9^{-\frac{2\gamma+1}{3}}\frac{\Gamma(1-\frac{2\gamma+1}{3})}{\Gamma(1+\frac{2\gamma+1}{3})}. \tag{2.16}$$

Proof. If $d(0) = 0$, then $s^{-\gamma}F_{+,0} = H_0 \in L^2([1,\infty))$ which is a contradiction by Lemma 2.4. We need to compute the asymptotic of the non tempered part of $F_{+,\lambda}$ and $F_{-,\lambda}$ in order to compute the value of $d(0)$, the unique value such that this non tempered part vanishes. Since

$$F_\alpha(s) \sim_{s\sim\infty} c_0\Gamma(1-\alpha)3^{-\alpha}s^{\frac{3\alpha}{2}-3}e^{i\frac{\pi}{4}\alpha}e^{\tau s^{\frac{3}{2}}}$$

we get

$$H_0(s) \sim_{s\sim\infty} c_0 e^{\tau s^{\frac{1}{2}}}s^{-3}[\Gamma(1-\alpha)3^{-\alpha}s^{3\frac{\alpha}{2}}s^{-\gamma}e^{i\frac{\pi}{4}\alpha} + d(0)\Gamma(1+\alpha)3^{\alpha}s^{3\frac{-\alpha}{2}}s^{1+\gamma}e^{-i\frac{\pi}{4}\alpha}]$$

which, since $s^{3\frac{\alpha}{2}}s^{-\gamma} = s^{\frac{-3\alpha}{2}}s^{1+\gamma}$ implies that the only value of $d(0)$ such that $H_0 \in L^2([1,\infty))$ is given by

$$d(0) = -e^{i\frac{\pi}{2}\frac{2\gamma+1}{3}}9^{-\frac{2\gamma+1}{3}}\frac{\Gamma(1-\frac{2\gamma+1}{3})}{\Gamma(1+\frac{2\gamma+1}{3})}.$$

□

Going back to the starting variable, introducing $\eta = \varepsilon k$, $0 < \eta \le \eta_0$, let us denote

$$\mathbb{L}^0_\eta = -\partial_v^2 + \frac{\gamma(\gamma+1)}{v^2} + i\eta v.$$

We get the following proposition

Proposition 2.7 *For any $\mu \in \mathbb{C}$, $|\mu| \le \eta^{\frac{2}{3}}\lambda_0$, the function*

$$\begin{aligned}\Theta_{\lambda,\eta}(v) &= v^{-\gamma}F_{+,\lambda}(\eta^{\frac{1}{3}}v) + d(\lambda)F_{-,\lambda}(\eta^{\frac{1}{3}}v)v^{\gamma+1}\eta^{\frac{2\gamma+1}{3}} \\ &= \eta^{\frac{\gamma}{3}}H_\lambda(\eta^{\frac{1}{3}}v)\end{aligned} \tag{2.17}$$

spans the space of solution defined on $\mathbb{R}^+$, belonging to $L^2([1,\infty[)$ of the equation

$$(\mathbb{L}^0_\eta - \mu)g = 0, \quad \textit{with } \mu = \lambda\eta^{\frac{2}{3}}. \tag{2.18}$$

Proof. Define $g \in L^2(v_0,\infty)$ solution of (2.18), defining $v = \eta^{-\frac{1}{3}}s$ and $\mu = \eta^{\frac{2}{3}}\lambda$, the function $\tilde{g}(s) = g(\eta^{-\frac{1}{3}}s)$ satisfies $\tilde{g} \in L^2(s_0,\infty)$ and

$$(-\partial_s^2 + \frac{\gamma(\gamma+1)}{s^2} + is - \lambda)\tilde{g} = 0$$

which ends the proof. □

2.2 Back to the real equation : existence and properties of the solution

We consider now the complete operator,

$$\mathbb{L}_\eta = \mathbb{L}^0_\eta + N(v) = \mathcal{L}_\varepsilon$$

where $N(v) = W(v) - \frac{\gamma(\gamma+1)}{v^2} \in O(\frac{1}{v^4})$.

The goal of this section is to prove the following proposition.

Proposition 2.8 *There exists λ_0, η_0, such that the equation*

$$\begin{cases} (\mathbb{L}_\eta - \lambda\eta^{\frac{2}{3}})J_{\lambda,\eta}(v) &= 0, \quad v \in [0,\infty[\\ J_{\lambda,\eta}(0) &= 1 \end{cases}$$

has a continuous solution in $(\lambda,\eta,v) \in \{|\lambda| \le \lambda_0\} \times \{0 \le \eta \le \eta_0\} \times [0,\infty[$, holomorphic in $\lambda \in \{|\lambda| < \lambda_0\}$ and satisfying $\int_0^\infty |J_{\lambda,\eta}(v)|^2 dv < \infty$. Moreover this solution is unique.

As in the previous section, we will look for solutions in $L^2([v_0,\infty[)$, close to $\Theta_{\lambda,\eta}$ when $v \to \infty$ by writing

$$G_{\lambda,\eta} = \Theta_{\lambda,\eta}(1 + R_{\lambda,\eta}), \quad \text{where } R_{\lambda,\eta}(v) \to_{v\to\infty} 0.$$

This change of unknown leads to the following equation for $R_{\lambda,\eta}$

$$\begin{cases} (\mathrm{Id} - \mathbb{K}_{\lambda,\eta})R_{\lambda,\eta} &= \mathbb{K}_{\lambda,\eta}(1) \\ \mathbb{K}_{\lambda,\eta}(g)(v) &= \displaystyle\int_v^\infty (\int_v^w \frac{\Theta^2_{\lambda,\eta}(w)}{\Theta^2_{\lambda,\eta}(u)} du) N(w) g(w) dw. \end{cases} \tag{2.19}$$

Note that by Lemma 2.3, we are allowed to divide by $\Theta^2_{\lambda,\eta}(u)$.

Before proving the proposition, we start with a series of lemma in order to proceed to a fixed point argument.

Lemma 2.9 *There exists C_0 such that for all $0 < v < w$, we have*

$$|\int_v^w \frac{\Theta^2_{\lambda,\eta}(w)}{\Theta^2_{\lambda,\eta}(u)} du| \le C_0 w \quad \forall |\lambda| \le \lambda_0, \quad \forall 0 < \eta \le \eta_0. \tag{2.20}$$

Proof. Back to the definition (2.17) of Θ and by writing $v = \eta^{-\frac{1}{3}}a$, $w = \eta^{-\frac{1}{3}}b$ and $u = \eta^{-\frac{1}{3}}t$, (2.20) is true if and only if

$$|\int_a^b \frac{H^2_\lambda(b)}{H^2_\lambda(t)} dt| \le C_0 b$$

then it is sufficient to prove that

$$|\int_0^b \frac{H^2_\lambda(b)}{H^2_\lambda(t)} dt| \le C_0 b. \tag{2.21}$$

It is true if b is small since $t \to |H_\lambda(t)|$ is decreasing near 0. It is true for $b \in [b_0, B_0]$ compact set included in $]0,\infty[$. Finally, for $b > B_0$, we use the asymptotic coming from the Airy function

$$|H_\lambda(s)| \sim_{s\sim\infty} \frac{C}{s^{\frac{1}{4}}} e^{-\frac{\sqrt{2}}{3} s^{\frac{3}{2}}}. \tag{2.22}$$

□

Remark 2.10 *Note that for b small, (2.21) is sharp, but for b large, one can get the better estimate $C_0 b^{-1/2}$.*

Lemma 2.11 *There exists a function $G_{\lambda,\eta}(v)$ solution to $(\mathbb{L}_\eta - \lambda\eta^{\frac{2}{3}})G_{\lambda,\eta}(v) = 0$ for $v \in [0,\infty[$ and $G_{\lambda,\eta}$ is continuous in $\eta \in [0,\eta_0]$, holomorphic in $\lambda \in \mathbb{C}$, $|\lambda| < \lambda_0$, continuous in $(\lambda,\eta,v) \in \{|\lambda| \le \lambda_0\} \times \{0 \le \eta \le \eta_0\} \times [0,\infty[$ and there exists $v_0 > 0$ such that*

$$G_{\lambda,\eta} = \Theta_{\lambda,\eta}(1 + R_{\lambda,\eta}), \quad \text{with } |R_{\lambda,\eta}(v)| \le \frac{C}{v^2}, \quad \text{for all } v \ge v_0$$

where C does not depend on $(\lambda,\eta) \in \{|\lambda| \le \lambda_0\} \times \{0 \le \eta \le \eta_0\}$.
Moreover, $G_{\lambda,0}$ does not depend on λ.

Proof. Since $N(w) =_{w\to\infty} O(\frac{1}{w^4})$, we get as in the proof of Lemma 2.2

$$||v^{n+2}\mathbb{K}_{\lambda,\eta}(g)||_{L^\infty([1,\infty[)} \le \frac{C_0}{n+2}||v^n g||_{L^\infty([1,\infty[)}.$$

Then, there exists $v_0 >> 1$ that does not depend on $|\lambda| \le \lambda_0$ and $0 < \eta \le \eta_0$ such that (2.19) has a unique solution $R_{\lambda,\eta}(v) \in L^\infty([v_0,\infty[)$ and we have

$$R_{\lambda,\eta} = O(\frac{1}{v^2}) \quad \text{when } v \to \infty.$$

Moreover, thanks to (2.19), $R_{\lambda,\eta}(v)$ is an holomorphic function in $\{\lambda \in \mathbb{C}, |\lambda| \le \lambda_0\}$ for all $0 < \eta \le \eta_0$ as well as $G_{\lambda,\eta}$ for $v \in [0,\infty[$ since

$$G_{\lambda,\eta} = \Theta_{\lambda,\eta}(1 + R_{\lambda,\eta}) \quad \text{for all } v \ge v_0$$

and $G_{\lambda,\eta}$ satisfies the differential equation

$$(\mathbb{L}_\eta - \lambda\eta^{\frac{2}{3}})G_{\lambda,\eta} = 0, \quad \forall v \in \mathbb{R}.$$

Note also that $G_{\lambda,\eta}$ may be extended to $\eta = 0$ and $G_{\lambda,0}$ does not depend on λ since $\Theta_{\lambda,0}(v) = v^{-\gamma}$ and $\mathbb{K}_{\lambda,0}(g) = \int_v^\infty \frac{w}{2\gamma+1}(1 - (\frac{v}{w})^{2\gamma+1})N(w)g(w)dw$ do not depend on λ. Thus we get $G_{\lambda,0} = G_{0,0}(v) = v^{-\gamma}(1 + O(v^{-2}))$. Continuity follows from the fact that thanks to (2.17) and (2.20), the function $0 < v < w$, $\lambda \in \mathbb{C}$, $|\lambda| \le \lambda_0$, $0 \le \eta \le \eta_0$

$$\int_v^w \frac{\Theta^2_{\lambda,\eta}(w)}{\Theta^2_{\lambda,\eta}(u)}du$$

is holomorphic in λ, continuous in $\eta \in [0,\eta_0]$ and bounded by $C_0 w$ which implies that we can apply the Lebesgue Theorem. □

Proof. **Proof of Proposition2.8** First of all, since $(-\partial_v^2 + W)G_{0,0} = 0$, we have $\partial_v^2 G_{0,0} = O(v^{-(\gamma+2)})$ then $\partial_v G_{0,0} = O(v^{-(\gamma+1)})$ (since $G_{0,0} = O(v^{-\gamma})$). Assume that $G_{0,0}(0) = 0$, then by integration by parts of the collision operator written as in (1.2), we get

$$\int_0^\infty F[(\frac{G_{0,0}}{F^{\frac{1}{2}}})']^2 = 0$$

then $G_{0,0} = CF^{\frac{1}{2}}$ and since $F(0) \ne 0$, we get $C = 0$, then $G_{0,0} = 0$ that contradicts the fact that $G_{0,0} \sim_{v\sim\infty} v^{-\gamma}$. Therefore, $G_{0,0}(0) \ne 0$.

Then, for λ_0, η_0 small, and $|\lambda| \le \lambda_0$, $0 \le \eta \le \eta_0$, we have for s sufficiently small, $G_{\lambda,\eta}(s) \ne 0$ since $G_{0,0}(0) \ne 0$ and $G_{\lambda,\eta}(0)$ is continuous with respect to λ, η. Then $J_{\lambda,\eta} = \frac{G_{\lambda,\eta}}{G_{\lambda,\eta}(0)}$ is well defined. Uniqueness comes from the results above for $\eta > 0$. When $\eta = 0$, we also have uniqueness since the only solution of

$$[-\partial_v^2 + W]f = 0, \quad f(0) = 0, \quad f \in L^2$$

is $f = 0$.

□

Remark 2.12 *The function $M(v) = \frac{1}{(1+v^2)^{\gamma/2}}$ is the unique solution in $L^2([0,\infty[)$ of the equation $(-\partial_v^2 + W)f = 0$ which satisfies $f \simeq v^{-\gamma}$ for $v \to \infty$. Since in the proof of Lemma 2.11 we have shown $G_{\lambda,0}(v) = G_{0,0}(v) = v^{-\gamma}(1 + O(v^{-2}))$, we get*

$$G_{\lambda,0}(v) = G_{0,0}(v) = M(v) \ . \tag{2.23}$$

Proposition 2.13 *Properties of $G_{\lambda,\eta}$.*

- *There exists a constant C_0 such that $\forall v \geq 0$, $|\lambda| < \lambda_0$, $\eta \in [0, \eta_0]$*

$$|G_{\lambda,\eta}(v)| \leq C_0 M(v). \tag{2.24}$$

- *We have the following limit*

$$\lim_{\eta \to 0^+} \int_0^\infty \eta^{\frac{1}{3}} v G_{\lambda,\eta}(v) M(v) dv = 0. \tag{2.25}$$

- *For all λ, for all v,*

$$\lim_{\eta \to 0} G_{\lambda,\eta}(v) = M(v). \tag{2.26}$$

Proof. Concerning the first point, for $v \geq v_0$, we use the fact that the function $s^\gamma H_\lambda(s)$ is bounded on $[0,\infty[$, uniformly in $|\lambda| \leq \lambda_0$, and we write, with $s = \eta^{1/3} v$,

$$|G_{\lambda,\eta}(v)| = |\Theta_{\lambda,\eta}(v)(1 + R_{\lambda,\eta})(v)| \leq C|\Theta_{\lambda,\eta}(v)| = Cv^{-\gamma}|s^\gamma H_\lambda(s)| \leq C' v^{-\gamma} \sim C' M(v) \ .$$

For $v \in [0, v_0]$, it follows from the continuity of $G_{\lambda,\eta}$, $G_{0,0} = M$ and $\min_{v \in [0,v_0]} M(v) > 0$.
To prove the limit of the second point, we cut the expression in the following way

$$\begin{aligned} |\int_0^\infty \eta^{\frac{1}{3}} v G M| &\leq |\int_0^{s_0 \eta^{-\frac{1}{3}}} \eta^{\frac{1}{3}} v G M dv| + \eta^{\frac{2\gamma-1}{3}} \int_{s_0}^\infty s^{1-\gamma} |H_\lambda(s)| ds \\ &\leq C_0 s_0 \int_0^\infty M^2(v) dv + \eta^{\frac{2\gamma-1}{3}} \int_{s_0}^\infty s^{1-\gamma} |H_\lambda(s)| ds \end{aligned}$$

and we pass to the limit in η noticing $\gamma > 1/2$ which leads to

$$\lim_{\eta \to 0} \eta^{\frac{2\gamma-1}{3}} \int_{s_0}^\infty s^{1-\gamma} |H_\lambda(s)| ds = 0 \quad \forall s_0 > 0.$$

Then, we pass to the limit when $s_0 \to 0$.
The third point follows from (2.23) and the continuity with respect to η of $G_{\lambda,\eta}$. □

2.3 Computation of the eigenvalue

In this subsection, we proceed to a reconnection of the two parts of the eigenvector, the positive velocity part and the negative velocity part. In order to be able to do the reconnection, we need to compute the derivative of the eigenvector at $v = 0$.

Let $G_{\lambda,\mu}$ defined above and introduce the notations

$$a(\lambda, \eta) \text{ satisfying } a(\lambda, \eta) G_{\lambda,\eta|_{v=0}} = 1, \quad \text{and} \quad b(\lambda, \eta) \text{ defined by } a(\lambda, \eta) G'_{\lambda,\eta|_{v=0}} = b(\lambda, \eta).$$

Observe that the functions $a(\lambda, \eta), b(\lambda, \eta)$ are holomorphic in $\lambda \in \mathbb{C}, |\lambda| < \lambda_0$, and since $G_{\lambda,0} = M$, one has $a(\lambda, 0) = 1, b(\lambda, 0) = 0$.
As we said at the begining of the section, due to symmetries in particular due to the parity of M, the connection condition reads $b(\lambda, \eta) + \overline{b}(\overline{\lambda}, \eta) = 0$. We thus need to compute $\mathrm{Re}b(0, \eta)$ and the begining of the expansion of $b(\lambda, \eta)$ with respect to λ. We gather all the needed results in the following proposition

Proposition 2.14

- *The expression of $b(\lambda, \eta)$ is given by*

$$b(\lambda, \eta) = a(\lambda, \eta) \eta^{\frac{2}{3}} \int_0^{\infty} (\lambda - i\eta^{\frac{1}{3}} v) G_{\lambda,\eta}(v) M(v) dv. \tag{2.27}$$

- *The η order of the coefficient in front of λ in the expansion on λ of $b(\lambda, \eta)$ is given by*

$$\lim_{\eta \to 0^+} b(\lambda, \eta) \eta^{-\frac{2}{3}} = \lambda \int_0^{\infty} M^2(v) dv. \tag{2.28}$$

- *concerning the real part of $b(0, \eta)$, we get*

$$\lim_{\eta \to 0^+} \eta^{-\frac{2\gamma+1}{3}} Reb(0, \eta) = \int_0^{\infty} s^{1-\gamma} Im(H_0(s)) ds = (2\gamma + 1) Re(d(0)) \tag{2.29}$$

where

$$d(0) = -e^{i\frac{\pi}{2}\frac{2\gamma+1}{3}} 9^{-\frac{2\gamma+1}{3}} \frac{\Gamma(1 - \frac{2\gamma+1}{3})}{\Gamma(1 + \frac{2\gamma+1}{3})}.$$

Proof. The first point is obtained by integrating the equation satisfied by $G_{\lambda,\eta}$ by part.
To get the second point, we use $\lim_{\eta \to 0^+} G_{\lambda,\eta} = M$ which implies $\lim_{\eta \to 0^+} a(\lambda, \eta) = 1$, and we conclude by using 2.25.

The computation of $\mathrm{Re}b(0, \eta)$ will be split into three steps. By equation (2.27),

$$b(0, \eta) = -i\eta a(0, \eta) \int_0^{\infty} w G_{0,\eta}(w) M(w) dw.$$

In order to get the result, we prove the three following lemmas.

Lemma 2.15 *The small velocities don't participate to the limit of the coefficient $b(0, \eta)$,*

$$\lim_{\eta \to 0^+} \eta^{-2\frac{(\gamma-1)}{3}} \int_0^{v_0} w Im[a(0, \eta) G_{0,\eta}] M dw = 0. \tag{2.30}$$

Lemma 2.16 *We have*

$$\lim_{\eta \to 0^+} \eta^{-2\frac{(\gamma-1)}{3}} \int_{v_0}^{\infty} w Im[a(0, \eta) G_{0,\eta}] M(w) dw = \int_0^{\infty} s^{1-\gamma} Im H_0(s) ds. \tag{2.31}$$

In order to prove those results, we need the following lemma

Lemma 2.17 *For all $\gamma > 1$, we have*

$$|Re(a(0,\eta)G_{0,\eta}) - M| \leq C\eta, \tag{2.32}$$

$$|Im(a(0,\eta)G_{0,\eta})| \leq C\eta. \tag{2.33}$$

Moreover, for large velocities,

$$|Re(a(0,\eta)G_{0,\eta}) - M| \leq C\eta\langle v\rangle^{3-\gamma}, \quad \forall v \in [v_0, s_0\eta^{-\frac{1}{3}}] \tag{2.34}$$

$$|Im(a(0,\eta)G_{0,\eta})| \leq C\eta\langle v\rangle^{3-\gamma}, \quad \forall v \in [v_0, s_0\eta^{-\frac{1}{3}}], \tag{2.35}$$

where $\langle v\rangle = \sqrt{1+v^2}$.

Proof. Proof of lemma 2.17

In order to compute $\mathrm{Re}(a(0,\eta)G_{0,\eta})$ and $\mathrm{Im}(a(0,\eta)G_{0,\eta})$, we introduce the second fundamental solution of $Q[f] = 0$. We already introduced M solution of $Q[M] = 0, \quad M(0) = 1 \quad M'(0) = 0$. We need now the second one, that we denote Z satisfying $Q[Z] = 0, \quad Z(0) = 0 \quad Z'(0) = 1$.

Then, the solution of $Q(f) = g, \quad f(0) = a$ and $f'(0) = b$ is given by

$$f = -\int_0^v g(w)M(w)dwZ(v) + \int_0^v g(w)Z(w)dwM(v) + aM(v) + bZ(v)$$

where

$$M(v) \sim_{v\sim\infty} v^{-\gamma} \quad \text{and} \quad Z(v) = M(v)\int_0^v \frac{1}{M^2(w)}dw \sim_{v\sim\infty} v^{\gamma+1}.$$

Set $f_\eta = \mathrm{Re}(a(0,\eta)G)$, and $\eta l_\eta = \mathrm{Im}(a(0,\eta)G)$. They satisfy the following equations, with $Q = -\partial^2 + W$

$$Q[f_\eta] - \eta^2 v l_\eta = 0, \quad f_\eta(0) = 1 \tag{2.36}$$

$$Q[l_\eta] + vf_\eta = 0, \quad l_\eta(0) = 0. \tag{2.37}$$

By multiplying the equation by M and integrating by parts, we compute their derivatives

$$f_\eta'(0) = \eta^2\int_0^\infty wl_\eta M dw \quad \text{and} \quad l_\eta'(0) = -\int_0^\infty wf_\eta M dw\ .$$

Lemma 2.17 can be reformulated as follows

$$f_\eta(v) = M(v) + \tilde{f}_\eta \text{ with } |\tilde{f}_\eta| \leq C\eta\langle v\rangle^{3-\gamma}\ , \tag{2.38}$$

and

$$|l_\eta| \leq C\langle v\rangle^{3-\gamma}. \tag{2.39}$$

Since the function f_η satisfies

$$Q[f_\eta] = \eta^2 v l_\eta, \quad f_\eta(0) = 1, \quad f_\eta'(0) = \eta^2\int_0^\infty wl_\eta M dw.$$

We get

$$f_\eta(v) = M(v) + (\int_0^\infty \eta^2 wl_\eta M dw)Z(v) + (\int_0^v \eta^2 wl_\eta Z dw)M(v) - (\int_0^v \eta^2 wl_\eta M dw)Z(v)$$

which can be rewritten

$$f_\eta = M(v) + \tilde{f}_\eta(v)$$

where

$$\tilde{f}_\eta(v) = (\int_0^v \eta^2 w l_\eta Z dw)M(v) + (\int_v^\infty \eta^2 w l_\eta M dw)Z(v).$$

Since $|a(0,\eta)G_{0,\eta}| \le CM$, we get both $|f_\eta| \le CM$ and $|\eta l_\eta| \le CM$. Since $\gamma > 1$, vM^2 is integrable at infinity and we write $\int_v^\infty wM^2 dw \le C\langle v\rangle^{2-2\gamma}$ and we finally get (2.38).

Concerning l_η, it satisfies the equation

$$Q[l_\eta] = -vf_\eta, \quad l_\eta(0) = 0, \quad l'_\eta(0) = -\int_0^\infty wf_\eta M dw$$

which leads to the following formula

$$\begin{aligned} l_\eta(v) &= -(\int_0^\infty wf_\eta M dw)Z(v) - (\int_0^v wf_\eta Z dw)M(v) + (\int_0^v wf_\eta M dw)Z(v) \\ &= -(\int_v^\infty wf_\eta M dw)Z(v) - (\int_0^v wf_\eta Z dw)M(v). \end{aligned}$$

As before, since $\gamma > 1$ and $f_\eta \le CM$, we get (2.39).

□

Proof. Proof of Lemma 2.15
Case 1 : $\gamma \in]1, \frac{5}{2}]$.
First of all, since $2(\gamma-1)/3 < 1$, and $|\mathrm{Im}[a(0,\eta)G_{0,\eta}]| = |\eta l_\eta| \le C\eta$, we get

$$\eta^{-2\frac{(\gamma-1)}{3}} w\mathrm{Im}[a(0,\eta)G_{0,\eta}(w)]M(w) \to_{\eta\to 0} 0 \quad \text{for all } w.$$

But since $|a(0,\eta)G_{0,\eta}| \le CM$, when $\gamma > 1$, one has $w|a(0,\eta)G_{0,\eta}|M \le CwM^2 \in L^1$ and we conclude by the Lebesgue theorem.
Case 2: $\gamma \in [\frac{1}{2}, 1]$.
Since $2(\gamma-1)/3 \le 0$, we obtain directly the result by using the Lebesgue theorem and the third point of Proposition 2.13 that gives

$$\int_0^{v_0} wa(0,\eta)G_{0,\eta} M dw \to_{\eta\to 0} \int_0^{v_0} wM^2 dw$$

thus the imaginary part goes to zero.

□

Proof. Proof of Lemma 2.16

In order to prove (2.31), we proceed to a change of variable $w = \eta^{-\frac{1}{3}}s$, which means that we need to compute

$$\lim_{\eta\to 0^+} \int_{\eta^{\frac{1}{3}} v_0}^\infty \mathrm{Im}[a(0,\eta)\eta^{-\frac{\gamma}{3}} G_{0,\eta}(\eta^{-\frac{1}{3}}s)]s\eta^{-\frac{\gamma}{3}} M(\eta^{-\frac{1}{3}}s)ds$$

where $\eta^{-\frac{\gamma}{3}} G_{0,\eta}(\eta^{-\frac{1}{3}}s) = H_0(s)[1+R_{0,\eta}(\eta^{-\frac{1}{3}}s)]$. For that purpose, we use the Lebesgue Theorem, by writing that

$$\forall s > 0 \quad , \lim_{\eta\to 0^+} \mathrm{Im}[a(0,\eta)\eta^{-\frac{\gamma}{3}} G_{0,\eta}(\eta^{-\frac{1}{3}}s)]s\eta^{-\frac{\gamma}{3}} M(\eta^{-\frac{1}{3}}s) = s^{1-\gamma}\mathrm{Im}(H_0(s)).$$

To obtain the domination, we use $\beta = 2\gamma$ with $\beta \in]1,5[\setminus\{2,3,4\}$. Therefore, one has $\gamma \in]1/2,1[\cup]1,5/2[$. When $\gamma \in]1,5/2[$, we use $M(w) = \frac{1}{(1+w^2)^{\frac{\gamma}{2}}} \leq |w|^{-\gamma}$, which leads for $|s| > \eta^{\frac{1}{3}}v_0$,

$$|\mathrm{Im}[a(0,\eta)\eta^{-\frac{\gamma}{3}}G_{0,\eta}(\eta^{-\frac{1}{3}}s)]s\eta^{-\frac{\gamma}{3}}M(\eta^{-\frac{1}{3}}s)| \leq C|\mathrm{Im}[a(0,\eta)\eta^{-\frac{\gamma}{3}}G_{0,\eta}(\eta^{-\frac{1}{3}}s)]s^{1-\gamma}\ .$$

Moreover, $\mathrm{Im}[a(0,\eta)G_{0,\eta}] = \eta l_\eta$ and since $\gamma > 1$, we have for any $v \in [v_0, s_0\eta^{-1/3}]$, $|l_\eta| \leq C|v|^{3-\gamma}$. So for $s \leq s_0$, we get

$$|\mathrm{Im}[a(0,\eta)\eta^{-\frac{\gamma}{3}}G_{0,\eta}(\eta^{-\frac{1}{3}}s)]| \leq Cs^{3-\gamma}$$

and since $\gamma < \dfrac{5}{2}$

$$|\mathrm{Im}[a(0,\eta)\eta^{-\frac{\gamma}{3}}G_{0,\eta}(\eta^{-\frac{1}{3}}s)]s\eta^{-\frac{\gamma}{3}}M(\eta^{-\frac{1}{3}}s)| \leq Cs^{4-2\gamma} \in L^2(]0,1])\ .$$

For $s \geq s_0$, we use the fact that $|a(0,\eta)] \leq C$ and $|R_{0,\eta}| \leq C$ and we write

$$|\mathrm{Im}[a(0,\eta)\eta^{-\frac{\gamma}{3}}G_{0,\eta}(\eta^{-\frac{1}{3}}s)]s\eta^{-\frac{\gamma}{3}}M(\eta^{-\frac{1}{3}}s)| \leq C|H_0(s)|s^{1-\gamma} \in L^1[1,\infty[)$$

since $H_0(s) \sim_\infty s^{-\frac{1}{4}}e^{-\frac{\sqrt{2}}{3}s^{\frac{3}{2}}}$. When $\gamma \in]1/2,1[$, we just use $H_0(s) \sim_0 s^{-\gamma}$, and we write

$$|\mathrm{Im}[a(0,\eta)\eta^{-\frac{\gamma}{3}}G_{0,\eta}(\eta^{-\frac{1}{3}}s)]s\eta^{-\frac{\gamma}{3}}M(\eta^{-\frac{1}{3}}s)| \leq C|H_0(s)|s^{1-\gamma} \in L^1]0,\infty[)\ .$$

Then since the function is dominated by an integrable function, we can pass to the limit and we conclude that (2.31) holds true. □

Lemma 2.18 (Computation of the coefficient) *The coefficient of the leading power in* η *of the real part of* $b(0,\eta)$ *given in Lemma 2.16 is equal to*

$$\int_0^\infty s^{-\gamma} s Im H_0 ds = (1+2\gamma) Red(0). \tag{2.40}$$

Proof. Recall that H_0 satisfies

$$P(H_0) = -isH_0, \quad P(f) = (-\partial_s^2 + \frac{\gamma(\gamma+1)}{s^2})f$$

that implies $s\mathrm{Im}H_0 = \mathrm{Re}(PH_0) = P(\mathrm{Re}H_0)$. In another hand,

$$H_0(s) = s^{-\gamma}F_{+,0}(s) + d(0)s^{\gamma+1}F_{-,0}(s) = s^{-\gamma}\big(1 + \frac{is^3}{6(1-\gamma)} + O(s^6) + d(0)s^{2\gamma+1}(1+O(s^3))\big)\ .$$

Since $\gamma \in]1/2,1[\cup]1,5/2[$, this implies that $s^{-\gamma}s\mathrm{Im}H_0$ is integrable when $s \sim 0$. Moreover, we can proceed to a double integration by part that leads only to the two boundary terms since $P(s^{-\gamma}) = 0$, by writing

$$\int_0^\infty s^{-\gamma}s\mathrm{Im}H_0 ds = \lim_{s_0\to 0}\int_{s_0}^\infty s^{-\gamma}P(\mathrm{Re}H_0)ds = \lim_{s_0\to 0}[\partial_s\mathrm{Re}H_0 s_0^{-\gamma} - \mathrm{Re}H_0\partial_s(s^{-\gamma})] = (2\gamma+1)\mathrm{Re}d(0).$$

Recall that $\mathrm{Re}d(0) \neq 0$ and the computation of $d(0)$ has been done in Lemma2.6. □
The proof of Proposition 2.14 is complete. □

2.4 Extension to the negative velocities and computation of the eigenvalue with lowest absolute values .

Until this subsection, all the computations have been done for non negative velocities. We now need to extend this solution to negative velocities. For that purpose, we need to make a C^1 connection by connecting the value and the derivative at $v = 0$.

Proposition 2.19 *Let $\eta_0 > 0$ and $\lambda_0 > 0$ small enough. For all $\eta \in [0, \eta_0]$, there exists in the complex disc $\{\mu \in \mathbb{C}, |\mu| \leq \eta^{2/3}\lambda_0\}$ a unique $\mu(\eta)$ such that the equation (2.1) (with $\eta = \varepsilon k$) admits a solution M^η in $L^2(\mathbb{R})$. Moreover, this solution is unique, and one has*

$$\begin{aligned} &\mu(\eta) = \kappa\eta^{\frac{2\gamma+1}{3}}(1 + O(\eta^{\frac{2\gamma+1}{3}})) \\ &\kappa = 2C_\beta^2(2\gamma+1)9^{-\frac{2\gamma+1}{3}}\cos(\frac{\pi}{2}\frac{2\gamma+1}{3})\frac{\Gamma(1-\frac{2\gamma+1}{3})}{\Gamma(1+\frac{2\gamma+1}{3})} > 0\ . \end{aligned} \tag{2.41}$$

For $\eta \in [-\eta_0, 0]$, by complex conjugaison on the equation, we get

$$\mu(\eta) = \overline{\mu(-\eta)} = \kappa|\eta|^{\frac{2\gamma+1}{3}}(1 + O(|\eta|^{\frac{2\gamma+1}{3}})).$$

Proof. Recall that the equation we consider is given by

$$-\partial_v^2 + \frac{\gamma}{(1+|v|^2)^2}[|v|^2(\gamma+1) - 1] + i\eta v)M^\eta = \mu M^\eta.$$

and that to reconnect the two parts of the equation (defined for $v > 0$ and for $v < 0$) to have a $C^1(\mathbb{R})$ function, M^ε must satisfy the constraint ${M^{\varepsilon}}'_{\mu^\varepsilon}(0) = -\overline{M^\varepsilon}'_{\overline{\mu^\varepsilon}}(0)$ which is equivalent to

$$b(\lambda, \eta) + \bar{b}(\overline{\lambda}, \eta) = 0.$$

By Proposition 2.14 and the normalization $C_\beta^2 \int M^2 dv = 1$, one has

$$\eta^{-2/3}b(\lambda, \eta) = \eta^{-2/3}b(0, \eta) + \frac{\lambda}{2C_\beta^2}(1 + o_\eta(1)) + O(\lambda^2),$$

thus the connection equation reads $\eta^{-2/3}\mathrm{Re}(b(0, \eta)) + \frac{\lambda}{2C_\beta^2}(1 + o_\eta(1)) + O(\lambda^2) = 0$. Since $\mu = \eta^{\frac{2}{3}}\lambda$, this implies $\mu = -2C_\beta^2\mathrm{Re}(b(0, \eta)) + o((\mathrm{Re}(b(0, \eta))))$. Then the result follows by the third point in Proposition 2.14, by Lemma 2.6, formula (2.16), one has $2C_\beta^2(1 + 2\gamma)\mathrm{Re}(d(0)) = -\kappa$. □

3 Proof of Theorem 1.5 : Moment method

3.1 A priori estimates

We start with a compactness Lemma.

Lemma 3.1 *[17] For initial datum $f_0 \in Y_\omega^p$ where $p \geq 2$ and a positive time T.*

1. *The solution f^ε of (1.3) is bounded in $L^\infty([0, T];\ Y_\omega^p))$ uniformly with respect to ε since it satisfies*

$$||f^\varepsilon(T)||_{Y_\omega^p}^p + \frac{p\ (p-1)}{\theta(\varepsilon)}\int_0^T \int_{\mathbb{R}^{2d}} \frac{|\nabla_v(f^\varepsilon\ \omega)|^2}{\omega}\ (f^\varepsilon)^{p-2}\ \omega^{p-2}\ \mathrm{dvdxdt} \leq ||f_0||_{Y_\omega^p}^p. \tag{3.1}$$

2. *The density $\rho^\varepsilon(t,x) = \int_{\mathbb{R}^d} f^\varepsilon \, \mathrm{d}v$ is such that*

$$||\rho^\varepsilon(t)||_p^p \le C_\beta^{-2(p-1)} ||f_0||_{Y_\omega^p}^p \quad \text{for all} \quad t \in [0,T]. \tag{3.2}$$

3. *Up to a subsequence, the density ρ^ε converges weakly star in $L^\infty([0,T]; L^p(\mathbb{R}^d))$ to ρ.*

4. *Up to a subsequence, the sequence f^ε converges weakly star in $L^\infty([0,T]; Y_\omega^p(\mathbb{R}^{2d}))$ to $f = \rho(t,x)\frac{C_\beta^2}{\omega}$.*

Corollary 3.2 *Let $F = \frac{C_\beta^2}{\omega} = C_\beta^2 M^2$, $M = \frac{1}{(1+v^2)^{\frac{\gamma}{2}}}$, $\beta = 2\gamma \in]1,5[$. Let f^ε solution to (1.3) with $\theta(\varepsilon) = \varepsilon^{\frac{2\gamma+1}{3}}$. Assume that $||f_0\omega||_\infty \le C$. Then $g^\varepsilon = f^\varepsilon F^{-1/2}$ satifies the following estimate*

$$\int_0^T \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |g^\varepsilon - \rho^\varepsilon F^{1/2}|^2 dv \right)^{\frac{2\gamma+1}{2\gamma-1}} dsdx \le C\varepsilon^{\frac{2\gamma+1}{3}}. \tag{3.3}$$

Proof. Recall the Nash type inequality [6][19] [1]: for any h such that $\int hF dv = 0$, we have

$$\int_{\mathbb{R}} h^2 F dv \le C \left(\int_{\mathbb{R}} |\partial_v h|^2 F dv \right)^{\frac{2\gamma-1}{2\gamma+1}} (||h||_\infty^2)^{\frac{2}{2\gamma+1}} \ . \tag{3.4}$$

Define $h = g^\varepsilon F^{-1/2} - \rho^\varepsilon = \frac{f^\varepsilon}{F} - \rho^\varepsilon$, define $\alpha = \frac{2\gamma+1}{3}$. Observe that from $||f||_{Y_\omega^p} = ||\omega f||_{L^p(\frac{dxdv}{\omega})}$ and Lemma 3.1, formula (3.1), we have

$$||h_0||_{L^\infty} = \lim_{p\to\infty} ||h_0||_{Y_\omega^p} \ge \lim_{p\to\infty} ||h||_{Y_\omega^p} \ge ||h||_{L^\infty} \ .$$

Thus by Lemma 3.1, formula (3.1) taking $p = 2$, we get

$$\begin{aligned}
\int_0^T \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |g^\varepsilon - \rho^\varepsilon F^{1/2}|^2 dv \right)^{\frac{2\gamma+1}{2\gamma-1}} dsdy &= \int_0^T \int_{\mathbb{R}} \left(\int_{\mathbb{R}} h^2 F dv \right)^{\frac{2\gamma+1}{2\gamma-1}} dsdy \\
&\le C \int_0^T \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |\partial_v h|^2 F dv \right) (||h||_\infty^2)^{\frac{2}{2\gamma-1}} dsdy \\
&\le C \int_0^T \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \frac{|\partial_v (f^\varepsilon \omega)|^2}{\omega} dv \right) dsdy \le C\varepsilon^\alpha.
\end{aligned}$$

□

3.2 Weak limit

Recall $T = \varepsilon^{-\alpha}, \alpha = \frac{2\gamma+1}{3}$. By solving equation (1.15), we write

$$\tilde{g}^\varepsilon(s,v,k) = e^{-sT\mathcal{L}_\varepsilon} \tilde{g}(0,v,k)$$

which gives going back to the rescaled space variable y

$$g^\varepsilon(s,v,y) = \frac{1}{2\pi} \int e^{iyk} \tilde{g}^\varepsilon(s,v,k) dk \ .$$

Our purpose is to pass to the limit when $\varepsilon \to 0$, or $T \to \infty$.
Recall $f^\varepsilon(s,y,v) \ge 0$ and $\int f^\varepsilon(s,y,v) dxdv = \int f_0(x,v) dxdv$ for all $s \ge 0$.
Let $\hat{\rho}^\varepsilon(s,k) = \int e^{-iyk} \rho^\varepsilon(s,y) dy$ be the Fourier transform in y of $\rho^\varepsilon = \int f^\varepsilon dv = \int g^\varepsilon F^{1/2} dv$.

Proposition 3.3 *For all $k \in \mathbb{R}$, $\hat\rho^\varepsilon(.,k)$ converges to $\hat\rho(.,k)$, unique solution to the ode*

$$\partial_s \hat\rho + \kappa |k|^\alpha \hat\rho = 0, \quad \hat\rho_0 = \int_{\mathbb{R}} \hat f_0 dv \ . \tag{3.5}$$

Proof. Recall that $\mathcal{L}_\varepsilon$ is given by $\mathcal{L}_\varepsilon = Q + i\varepsilon k v$, $Q = -\partial_v^2 + W$ and thus is self adjoint in $L^2(\mathbb{C})$. Let $k \in \mathbb{R}$, $\eta = \varepsilon k$, and let $M^\eta(v)$ be the unique solution in $L^2(\mathbb{R})$ of $\mathcal{L}_\varepsilon(M^\eta) = \mu(\eta) M^\eta$ given in Proposition 2.19. One has

$$\begin{aligned} \frac{d}{ds}\int \tilde g^\varepsilon(s,v,k) M^\eta dv &= \int \partial_s \tilde g^\varepsilon M^\eta dv = -\varepsilon^{-\alpha} \int \mathcal{L}_\varepsilon(\tilde g^\varepsilon) M^\eta dv \\ &= -\varepsilon^{-\alpha} \int \tilde g^\varepsilon \mathcal{L}_\varepsilon(M^\eta) dv = -\varepsilon^{-\alpha}\mu(\eta) \int \tilde g^\varepsilon M^\eta dv \ . \end{aligned}$$

Therefore one has, with $F^\varepsilon(s,y) = C_\beta \int g^\varepsilon(s,v,y) M^\eta dv$,

$$\hat F^\varepsilon(s,k) = e^{-s\varepsilon^{-\alpha}\mu(\varepsilon k)} \hat F^\varepsilon(0,k) \quad \forall s \geq 0. \tag{3.6}$$

By Proposition 2.19, we have $\varepsilon^{-\alpha}\mu(\varepsilon k) \to \kappa |k|^\alpha$. Moreover, the following limit holds true:

$$\forall k \in \mathbb{R}, \quad \hat F^\varepsilon(0,k) = C_\beta \int \tilde g^\varepsilon(0,v,k) M^\eta dv \to \hat\rho_0(k) \ . \tag{3.7}$$

The verification of (3.7) is easy. One has $\tilde g^\varepsilon(0,v,k) = \hat f_0(v,k) F^{-1/2}$ and $C_\beta F^{-1/2} M^\eta(v) = \frac{M^\eta}{M}(v) \to 1$ for all $v \in \mathbb{R}$ since our construction gives $M^\eta(v) = a(\lambda,\eta) G_{\lambda,\eta}(v)$ with $a(\lambda,0) = 1$, $G_{\lambda,0} = M$. Moreover, one has by (2.24) the domination $|M^\eta(v)| \leq CM(v)$. Thus (3.7) holds true by Lebesgue Theorem.

Remark 3.4 *Observe that it is only in the verification of (3.7) (initial data at time $s = 0$) that we use the fact that M^η is associated to the eigenvalue of smallest absolute value of the operator $\mathcal{L}_\varepsilon$, since it is the only eigenfunction which satisfy $\lim_{\eta\to 0} M^\eta = M$.*

It remains to verify

$$\forall k \in \mathbb{R}, \quad C_\beta \int \tilde g^\varepsilon(s,v,k) M^\eta dv \to \hat\rho(s,k) \quad \text{in } \mathcal{D}'(]0,\infty[). \tag{3.8}$$

By (3.6) and (3.7), for all $k \in \mathbb{R}$ and $s \geq 0$, one has $\lim_{\varepsilon\to 0} \hat F^\varepsilon(s,k) = e^{-s\kappa|k|^\alpha} \hat\rho_0(k)$, thus (3.8) will be consequence of the weaker

$$C_\beta \int g^\varepsilon(s,y,v) M^\eta dv \to \rho(s,y) \quad \text{in } \mathcal{D}'(]0,\infty[\times\mathbb{R}) \ . \tag{3.9}$$

Let us now verify (3.9). For that purpose, we write

$$C_\beta \int g^\varepsilon M^\eta dv - \rho = C_\beta \int (g^\varepsilon - \rho^\varepsilon F^{1/2}) M^\eta dv + \rho^\varepsilon \int (C_\beta M^\eta - F^{1/2}) F^{1/2} dv + \rho^\varepsilon - \rho \ .$$

By using (3.3), (2.24) and (2.25), and the Lebesgue theorem we pass to the limit. The proof of Proposition 3.3 is complete. □

Proof of The main result: Theorem 1.5. From the two last items in Lemma 3.1, we have just to prove that for any given k, the Fourier transform $\hat\rho(s,k)$ of the weak limit $\rho(s,y)$, is solution of the equation (1.14), which is precisely Proposition 3.3.

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